研究の題目：進体の指数の最大アーベル拡大列の分岐定数について

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ON THE RAMIFICATION NUMBERS OF A TOWER OF
THE MAXIMAL ABELIAN EXTENSION OF \( p \)-ADIC
NUMBER FIELDS WITH EXPONENT \( p^m \)

By

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1. Introduction.

Let \( m \) be a natural number which we fix throughout this note, and let \( k \) be a \( p \)-adic number field and \( p \) the characteristic of the residue class field of \( k \). We define a chain of fields \( K_0=k, K_1, K_2, \cdots \), which has the property such that \( K_i \) is the maximal abelian extension of \( K_{i-1} \) with exponent \( p^m \) for each \( i \geq 1 \).

I.R. Šafarevič [6] has given the detailed structure of such fields, when \( k \) does not contain the \( p \)-th roots of unity and \( m=1 \). For this case, E. Maus [5] has given the upper ramification numbers and J. Idt [3] has given the explicit values of the ramification numbers.

In this note, we compute the ramification numbers and the orders of the ramification groups of \( K_i/k \) for general \( m \).

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2. Preliminaries.

Throughout this note, \( K \) denotes a \( p \)-adic number field, \( k^* \) the multiplicative group of \( k \), \( p \) the characteristic of the residue class field of \( k \), and \( \mu_k \) the order of the group of \( p \) power roots of unity in \( k \). Let \( K_0=k \) and let \( K_i \) be the maximal abelian extension of \( K_{i-1} \) with exponent \( p^m \) for each positive integer \( i \). Let \( \varepsilon, \delta, \) and \( \eta \) denote the absolute ramification index, the absolute residue class degree, and the absolute degree of \( K_i \), respectively. Then we have a following

**Lemma 1.** Suppose that the notations are the same as in the above.

(1) If \( \mu_k = 1 \), then

\[ \eta_i = \eta_{i-1} p^{\varepsilon_i (\delta_i-1)} \delta_i, \quad \delta_i = p^{\eta_i (\varepsilon_i-1)} \eta_i \quad \text{and} \quad \varepsilon_i = \varepsilon_{i-1} p^{\eta_i \delta_i-1} \delta_i \]

(2) If \( \mu_k \geq p^m \), then

\[ \eta_i = \eta_{i-1} p^{\varepsilon_i (\delta_i-1)} \delta_i, \quad \delta_i = p^{\eta_i (\varepsilon_i-1)} \eta_i \quad \text{and} \quad \varepsilon_i = \varepsilon_{i-1} p^{\eta_i \delta_i-1} \delta_i \]

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This follows easily from the local class field theory and the structure of the multiplicative group of \( p \)-adic number fields.

Let \( \pi \) be a prime element in \( k \) and let \( e \) be the absolute ramification index of \( k \). Let \( U_k \) be the group of units of \( k \). We define the usual filtration by \( U_0=U_k, U_i=\{x \in k^*; \text{ord}_x(x-1) \geq i\} \) for \( i \geq 0 \), where \( \text{ord}_x \) is the normalized additive valuation of \( k \). Let \( \lambda \) denote the function defined by

\[
\lambda(n) = \min \{pn, n+e\}, n \in \mathbb{Z}, n > 0.
\]

For each positive integer \( i \), let \( R_i \subseteq U_i \) denote a complete set of representatives for the factor group \( U_i/\mathfrak{U}_{i+1} \). It is shown by Hasse [2] that these representative sets can be chosen in such a manner that

\[
R_i = R_j \quad \text{whenever } i = \lambda(j)
\]

except in case \( \mu_k > 1 \) and \( i = \frac{pe}{p-1} \). It is shown also that, even in the exceptional case, \( R_i \) (where \( j = \frac{e}{p-1} \)) is a subgroup mod \( \pi \) of \( R_i \) with index \( p \).

**Lemma 2.** Let \( t \) be a natural number \( \leq \frac{e}{m+\frac{1}{p-1}} \) with

\[
t \equiv 0 \mod p^m
\]

for \( 0 < t \leq \frac{e}{p-1} + e \)

\[
t \equiv le \mod p^{m-1}
\]

for \( \frac{e}{p-1} + le < t \leq \frac{e}{p-1} + (l+1)e \), \( l = 1, \ldots, m-1 \).

Then

\[
U_i \cap k^{*p^m} \subseteq U_{i+1}.
\]

**Proof.** Suppose that there is an element \( \eta_i \) in \( U_i \cap k^{*p^m} \) but not in \( U_{i+1} \). Then we have \( \eta_i = a^\beta \) where \( a \) is a principal unit. Assume \( a = a_n \in U_n - U_{n+1} \). Then we may assume \( a_n \in R_n \). If \( 0 < \nu < \frac{e}{p^{m-1}(p-1)} \) then \( a_n^\nu \in U_{\nu m} - U_{\nu m+1} \) from the above result of Hasse. Therefore we have \( t = p^{m-1} \nu < \frac{e}{p-1} + e \), which is a contradiction.

If \( \frac{e}{p^{m-1}(p-1)} < \nu < \frac{e}{p^{m-1}(p-1)}, l = 1, \ldots, m-1 \), then \( p^{m-l-1} \nu < \frac{e}{p-1} \) and

\[
\frac{e}{p-1} < p^{m-l-1} \nu.
\]

Thus, by the above result of Hasse, we have

\[
a_n^\beta \in U_{\beta m-l, \ell, \nu} - U_{\beta m-\nu, \ell, \nu+1}.
\]

Therefore we have

\[
\frac{e}{p-1} + le < t = p^{m-l} \nu + le < \frac{e}{p-1} + (l+1)e,
\]

which is a contradiction.

If \( p^{m-l-1}(p-1)|e, l = 0,1, \ldots, m-1 \), then we see \( \frac{e}{p-1} + (l+1)e \equiv le \mod p^{m-l} \). Thus
the consideration for the case \( \nu = \frac{\epsilon}{p^{m-l}(p-1)}, l = 1, \ldots, m \) is not necessary. The proof is complete.

3. The upper ramification numbers and the ramification numbers of \( K_\nu/K_{\nu-1} \).

For a finite Galois extension \( L/k \) with Galois group \( G \), let \( T(L/k) \) denote the set of the upper ramification numbers, i.e. the set of jumps in the upper numbering of the ramification groups of \( L/k \), and let \( v_i = v_i(L/k) \) denote the ramification number, i.e. the jump in the usual numbering of the ramification groups of \( L/k \). For real \( x \geq 0 \), the symbol \( \lfloor x \rfloor \) will denote the least integer \( \geq x \). The next theorem was given by Maus \[4\], \[5\]. We set for brevity \( \bar{v}_0 = 0 \) and \( \bar{v}_l = k + \left[ \frac{\epsilon}{p-1} \right] - \left[ \frac{\epsilon}{p^{m-l}(p-1)} \right] \) for \( l = 1, \ldots, m-1 \).

**Theorem 3 (E. Maus).** Let \( L \) be the maximal abelian extension of \( k \) with exponent \( p^m \), and let \( \epsilon \) be the absolute ramification index of \( k \).

(1) If \( \mu_1 = 1 \), then \( T(L/k) = \{ t_\nu; \nu = 0, 1, \ldots, m \} \), where

\[
t_\nu = \nu + \left[ \frac{\nu - k + 1}{p^{m-l} - 1} \right]
\]

for \( \bar{v}_l \leq \nu < \bar{v}_{l+1}, l = 0, 1, \ldots, m-1 \).

(2) If \( \mu_1 \geq p^m \), then \( T(L/k) = \{ t_\nu; \nu = 0, 1, \ldots, m \} \)

where

\[
t_\nu = \begin{cases} 
\nu + \left[ \frac{\nu - k + 1}{p^{m-l} - 1} \right] & \text{for } \bar{v}_l + l \leq \nu < \bar{v}_{l+1} + l, \ l = 0, 1, \ldots, m-1 \\
k + \frac{\epsilon}{p-1} & \text{for } \nu = \bar{v}_l + l - 1, \ l = 1, 2, \ldots, m \ .
\end{cases}
\]

**Proof.** Let \( \omega: k^* \to G = \text{Gal}(L/k) \) be the reciprocity law map of the local class field theory corresponding to the extension \( L/k \). Then, it is known that \( \omega(U_n) = G^n \) for all \( n \geq 0 \) (cf. Serre \[7\], Theorem 2, p. 235). Therefore, for each integer \( n \geq 0 \), we have

\[
G^n/G^{n+1} \cong U_n/U_{n+1}(U_n \cap k^*)
\]

because \( L/k \) is the maximal abelian extension of exponent \( p^m \).

(1) First, if \( t \) satisfies the hypotheses in Lemma 2, then, \( G^t/G^{t+1} \cong U_t/U_{t+1} \cong \overline{k} \), where \( \overline{k} \) is the residue class field of \( k \). Therefore such \( t \) is an upper ramification number of \( L/k \). Next, let \( t \equiv 0 \ mod \ p^m \) and \( 0 < t \leq \frac{p \epsilon}{p-1} \). We put \( t = p^m \nu \). If \( \eta_i \in U_t \), then \( \eta_i \) is uniquely written as

\[
\eta_i = (1 + a_i \pi^t) \eta_{i+1} \ (a_i \mod \nu, \ \eta_{i+1} \in U_{i+1})
\]

where \( \pi, \nu \) is a prime element and the prime ideal in \( k \) (cf. Hasse \[2\], p. 206). We may assume \( 1 + a_i \pi^t \in R \). From the result of Hasse in \[2\], there is a suitable element \( \eta_v \in U_v \) such that \( \eta_v^{p^m} = (1 + a_i \pi^t) \eta_{i+1} \) where \( \eta_{i+1} \) is some element in \( U_{i+1} \). Hence
\[ U_{i} \equiv U_{i+1}(U_{i} \cap k^{*p^{m}}). \] Therefore \( G' = G'^{+1}. \)

Finally, let \( t \equiv k \mod p^{m+1} \) and \( \frac{\varepsilon}{p-1} + l \leq t \leq \frac{\varepsilon}{p-1} + (l+1)\varepsilon, \) \( l = 1, \ldots, m-1. \) We put \( t = l + p^{m+1}. \) Similarly, for any \( \eta_{x} = (1 + \alpha \pi^{l})\eta_{l+1} \in U_{x}, \) there exists a suitable \( \eta_{x} \in U_{x} \) such that \( \eta_{x}^{\varepsilon m} = (1 + \alpha \pi^{l})\eta_{l+1}. \) Therefore \( U_{l+1}(U_{i} \cap k^{*p^{m}}) = U_{i}, \) so \( G' = G'^{+1}. \) This completes the proof of (1).

(2) Since \( \mu_{k} \geq p^{m}, p^{m+1}(p-1)|\varepsilon. \) Put \( \nu_{l} = \frac{\varepsilon}{p^{m-1}(p-1)} \) for \( l = 1, \ldots, m. \) Let \( \eta_{x} \in U_{x}, \) \( l = 1, \ldots, m. \) Then

\[ \eta_{x} \in U_{x} \] for \( l = 1, \ldots, m. \) From the result of Hasse in 2, there exist some \( \eta_{x} \in U_{x} \) and a suitable \( \eta_{x} \in U_{x} \) such that

\[ \eta_{x}^{\varepsilon m} \eta_{x}^{\varepsilon^{l-1}} = \left( 1 + \alpha \pi^{l} \right) \eta_{x}^{\varepsilon^{l-1}}. \]

Therefore \( U_{x} U_{x}^{+1} \) \( U_{x} U_{x}^{+1} \cap k^{*p^{m}} \) is a cyclic group of order \( p. \) The proof is complete.

In the proof of Theorem 3, we obtain the orders of the ramification groups of \( L/k. \) These give the ramification numbers together with the Hasse's function \( \psi_{L/k}, \) i.e. the function defined by \( \psi_{L/k}(x) = \int_{0}^{x} (G^{0} : G^{1}) dt \) for real \( x \geq 0. \)

Let \( \varepsilon = p^{m-1}(p-1)q_{m-1}, \) \( 0 \leq q_{m-1} < p^{m-1}(p-1), \) and let \( q_{m-1} = (p-1)q_{m-1} + r_{m-1}, \) \( 0 \leq r_{m-1} < p-1, \) for \( l = 1, \ldots, m. \)

**Corollary.** Suppose \( L/k \) is the same as in Theorem 3 and \( \varepsilon \) is the absolute residue class degree of \( k. \) Let \( \mu_{k} = 1 \) and let \( v_{i}, 0 \leq i \leq m-1, \) be the ramification number of \( L/k. \)

(1) If \( 0 \leq j < v_{1}, \) then

\[ v_{j} = \frac{(1 - p^{m+1})(1 - p^{m+1})}{(1 - p^{m+1}) + p^{m+1}} \frac{1 - p^{m+1}}{1 - p^{m+1}}, \]

where \( j = (p^{m-1})q_{m-1} + r_{m-1}, 0 \leq r_{m-1} < p^{m-1}. \)

(2) If \( v_{1} \leq j < v_{m-1}, \) \( 1 \leq l \leq m-1, \) then

\[ v_{j} = v_{j-1} + p^{r_{m-1}} \left( \delta_{m-1} + \frac{1 - p^{m-1}}{1 - p^{m-1}} + p^{m-1} \frac{1 - p^{m-1}}{1 - p^{m-1}} \right) \]

where \( q_{m-1} = \left[ \frac{j - v_{1} + q_{m-1}}{p^{m-1}} \right] \) and \( \delta_{m-1} = \begin{cases} 1 & \text{if } p^{m-1}(p-1)|\varepsilon \\ 0 & \text{otherwise.} \end{cases} \)
On the Ramification Numbers of a Tower of the Maximal Abelian Extension

Proof. Let \( t_{-1} = 0 \) and \( t_j \) be the upper ramification number of \( L/k \). Then we have

\[
v_j = \psi_{L/k}(t_j) = \frac{1}{2} \cdot \sum_{i=0}^{j} \frac{g_0}{g_i} (t_i - t_{i-1})
\]

where \( g_i = (G_i, 1) \). Therefore the corollary follows easily from Theorem 3.

Remark. For the case such that \( \mu_i \geq p^m \), we obtain the similar results. In fact, if \( \nu_i + l \leq j < \nu_{i+1} + l \) then we have

\[
v_j = \sum_{i=0}^{\nu_i} p^{i+1}(t_i - t_{i-1}) + \sum_{i=\nu_i+1}^{\nu_{i+1}} p^{i+1}(t_i - t_{i-1}) + \ldots + \sum_{i=\nu_{i+1}+1}^{j} p^{i+1}(t_i - t_{i-1})
\]

4. The ramification numbers of \( K_n/k \).

In this and the next sections, suppose that the notations are the same as in 2. Furthermore, let \( \psi_j, \psi^{(2)}_j \) be the ramification number of \( K_n/k, K_n/K_{n-1} \), respectively. Let \( \varphi_{s+1} = \varphi_{K_n/K_{n-1}} \) be the inverse function of \( \psi_{s+1} = \psi_{K_n/K_{n-1}} \).

Theorem 4. For each \( j, \psi_j = \psi^{(2)}_j \).

Proof. We prove the theorem by induction on \( s \). Let \( \text{Gal}(K_n/k) = G^{(s)} \) and \( \text{Gal}(K_n/k_{s-1}) = N^{(s)} \). Then we have, for all \( j, N_j^{(s)} = G_j^{(s)} \cap N^{(s)} \) and

\[
G_j^{(s)} N^{(s)} = (G_j^{(s)} N^{(s)}) \varphi_{s-1}(j) = G_{\varphi_{s-1}(j)}^{(s-1)}
\]

from the Herbrand's theorem (cf. Serre [7], Ch. IV, §3). Hence

\[
(G_j^{(s)} : G_{j+1}^{(s)}) = (G_{\varphi_{s-1}(j)}^{(s-1)} : G_{\varphi_{s-1}(j+1)}^{(s-1)})(N_j^{(s)} : N_{j+1}^{(s)})
\]

Now, suppose \( N_j^{(s)} = N_{j+1}^{(s)} \). Then \( N^{(s)} \varphi_{s-1}(j) = N^{(s)} \varphi_{s-1}(j+1) \).

If \( \varphi_{s-1}(j+1) \leq \frac{\varepsilon_{s-1}}{p-1} + \varepsilon_{s-1} \), then we have, from Theorem 3,

\[
\{ \varphi_{s-1}(j) \} = \{ \varphi_{s-1}(j+1) \}
\]

or

\[
p^m h_j - 1 < \varphi_{s-1}(j) < \varphi_{s-1}(j+1) \leq p^m h_j + 1,
\]

where \( h_j \) is a suitable integer. If \( \{ \varphi_{s-1}(j) \} = \{ \varphi_{s-1}(j+1) \} \), then \( G_{\varphi_{s-1}(j)}^{(s-1)} = G_{\varphi_{s-1}(j+1)}^{(s-1)} \).

By the induction hypothesis and Theorem 3 we have

\[
v_0^{(s-1)} \equiv \bar{v}_0^{(s-1)} \equiv \psi_{s-1}(l_0^{(s-1)}) - \psi_{s-1}(1) = 1,
\]

where \( l_0^{(s-1)} \) is the upper ramification number of \( K_{s-1}/K_{s-2} \). Hence \( v_j^{(s-1)} \equiv 1 \mod p \) for \( i \geq 1 \). Therefore, if \( p^m h_j - 1 < \varphi_{s-1}(j) < \varphi_{s-1}(j+1) \leq p^m h_j + 1 \), then \( \varphi_{s-1}(j) \) is not
the ramification number of $K_{s-1}/k$. Thus we obtain

$$G^{|s-1|}_{\varphi_{s-1},-1}(j) = G^{|s-1|}_{\varphi_{s-1},-1}(j+1), \text{ so } G^{|s|}_j = G^{|s+1|}_j.$$ 

For all $i \geq 0$, by the theorem of Dedekind-Hensel-Ore, we have

$$v^{|s-1|}_i \leq \frac{p^{e_i-1}}{g_i^{|s-1|}(p-1)} \leq \frac{e_i}{p-1}$$

where $g_i^{(s-1)} = (g_i^{(s-1)}, 1)$ (cf. Maus [5], 1.4). On the other hand, we have from Theorem 3 \( 0 < \varphi_{s-1}(j+1) - \varphi_{s-1}(j) < 2 \). Therefore, if $\varphi_{s-1}(j+1) > \frac{p^{e_i-1}}{p-1}$, then we obtain

$$G^{|s-1|}_{\varphi_{s-1},-1}(j) = 1 = G^{|s-1|}_{\varphi_{s-1},-1}(j+1), \text{ so } G^{|s|}_j = G^{|s+1|}_j.$$ 

This completes the proof of our assertion.

By the corollary to Theorem 3 and Theorem 4 we can compute an explicit value of $v^{|s|}_i$. For $s=1$, we may replace $f$ in the corollary to Theorem 3 with $f_0$. Let $s \geq 2$ and $\mu_s=1$. Put $e_0 = (p-1)q_0 + r_0$, $0 \leq r_0 < p-1$. Then, in Corollary to Theorem 3 we have

$$\left\lfloor \frac{p^{e_0}}{p^{m-1} + p^{m-2} + \cdots + p+1} \right\rfloor = r_0(p^{m-1} + p^{m-2} + \cdots + p+1)$$

for all $l$ and we may replace $f$, $\varepsilon$ with $f_{s-1} = p^{s-1}f_0$, $\varepsilon_{s-1}$, respectively. Similarly, in the case such that $\mu_s \geq p^{m}$, we may replace $f$, $\varepsilon$ in Remark to Theorem 3 with $f_{s-1} = p^{s-1}f_0$, $\varepsilon_{s-1}$, respectively.

**Remark.** In the case where $m=1$ our result coincides with the result of J. Idd [3].

5. The orders of the ramification groups of $K_i/k$.

The next theorem is a generalization of the result of J. Idd [3] and from this theorem we can compute inductively the orders of the ramification groups of $K_i/k$.

We put $g^{|s|}_j = \left( G^{|s|}_j : 1 \right)$.

**Theorem 5.** Assume that $\mu_s=1$, $f_0 \geq m$ and $s \geq 2$.

1. If $j \neq \frac{p^{m-1}}{p^m}(v^{|s-1|}_i - 1)$ for all $i = 0, 1, \ldots, me_{s-2}-1$, then

$$g^{|s|}_j / g^{|s|}_{j+1} = p^{e_{s-1}}.$$ 

2. If $j = \frac{p^{m-1}}{p^m}(v^{|s-1|}_i - 1)$ for some $i$, then

$$g^{|s|}_j / g^{|s|}_{j+1} = (g^{|s-1|}_i / g^{|s-1|}_{i+1})p^{e_{s-1}}.$$

**Proof.** For each $j$, we have

$$\left( G^{|s|}_j : G^{|s+1|}_j \right) = \left( G^{|s-1|}_{\varphi_{s-1},-1}(v_j) : G^{|s-1|}_{\varphi_{s-1},-1}(v_{j+1}) \right) \left( N^{|s|}_j : N^{|s+1|}_j \right).$$
Now, assume $\varphi_{s+1-k}(\psi_j^{(s)}) = \psi_j^{(s-1)}$ for some $i$. Then we have $\psi_j^{(s)} = \psi_{s+1-k}(\psi_j^{(s-1)}) = \psi_{s+1-k}(\psi_j^{(s-1)})$, because $\psi_j^{(s)} = \psi_j^{(s-1)}$ by Theorem 4. On the other hand, we have $\psi_j^{(s)} = \psi_{s+1-k}(\psi_j^{(s-1)})$ where $\psi_j^{(s)}$ is the upper ramification number of $K_{s+1}/K_{s-1}$. Therefore we obtain $\psi_j^{(s-1)} = \psi_j^{(s)}$. Since $\psi_j^{(s-1)} \leq \frac{p-1}{p^{m-1}}$, we have

$$j < v_j and \psi_j^{(s-1)} = j + \left\lfloor \frac{j+1}{p^{m-1}} \right\rfloor.$$

We put $j = (p^{m-1})q_j + r_j$, with $0 \leq r_j < p^{m-1}$. Then we have $\psi_j^{(s-1)} = p^{m}q_j + r_j + 1$. From the theorem of Hasse-Arf we have

$$\psi_j^{(s-1)} = \psi_j^{(s-1)} \bmod \bar{\psi_j^{(s-1)}}/\bar{\psi_j^{(s-1)}},$$

where $\bar{\psi_j^{(s-1)}} = (N_j^{(s-1)}: 1)$, $N_j^{(s-1)} = \text{Gal}(K_{s-1}/K_{s-2})$. Hence, from Theorem 3 and Theorem 4 we see $\psi_j^{(s-1)} \equiv 1 \bmod p^{m-1}$. Therefore we have $p^{m}q_j + r_j \equiv 0 \bmod p^{m-2}$. Because $r_j < p^{m-1} \leq p^{m-2} - 1$, we obtain $r_j = 0$. Therefore we have $j = p^{m-1}(\psi_j^{(s-1)} - 1)$. This completes the proof of our assertion.

**Remark.** In Theorem 5, the restriction such that $\mu_a = 1$ and $\mu_0 \geq m$ is not essential. Similar results hold for more general case.

**References**