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<td>雑誌名</td>
<td>Bulletin of the Faculty of Education, Kagoshima University. Natural science</td>
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<td>卷</td>
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<td>車</td>
<td>Page range 1-10</td>
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<td>URL</td>
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Shortest Spherical Network of Pentahedra

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(Received 19 October, 2004)

Abstract

We study the problem to divide the spherical surface into five parts of equal area by a network of edges of the shortest total length. It is proved that the regular 3-prism gives the shortest network.

1 Problem and result

Fejes Tóth ([1], [2]) posed the following problem: to divide the surface of the unit sphere into \( n(\geq 4) \) parts of equal area, by the shortest possible net of edges. To study it he invented an ingenious method, but even the method could give solutions only for \( n = 4, 6, \) and 12. In this paper we give a solution for \( n = 5 \) as follows.

Theorem Among all networks of pentahedra, the regular 3-prism has the shortest total length of edges.

(Proof) Note that spherical networks made of pentahedra can have only two topological types, i.e., prism and pyramid. By Proposition 1 and Proposition 2, it is sufficient to compare the total length of edges of the regular 3-prism \( L(3\text{-prism}) \) and that of the regular 4-pyramid \( L(4\text{-pyramid}) \). As is shown in the proofs of these propositions,

\[
L(3\text{-prism}) = 3 \tilde{f}(e_3, e_3), \quad L(4\text{-pyramid}) = 4 \tilde{g}(e_4),
\]

where functions \( \tilde{f}, \tilde{g} \) are defined by (6) and (10), and \( e_n \) is the common length of sides of the regular \( n \)-gon. To find \( e_3 \) and \( e_4 \) we use Lemma 1 below. Then we can evaluate the two total lengths as \( L(3\text{-prism}) \approx 4.28186\pi \) and \( L(4\text{-pyramid}) \approx 4.34633\pi \). Thus the theorem is established. (Q.E.D.)

Proposition 1 Assume \( n \leq 4 \). Then, among all networks of \( n \)-prism type, the regular \( n \)-prism has the shortest total length of edges.

Proposition 2 Assume \( n \leq 5 \). Then, among all networks of \( n \)-pyramid type, the regular \( n \)-pyramid has the shortest total length of edges.

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Lemma 1  Let a be the common length of sides of the regular n-gon of area S. Then it is given by

\[ \cos \frac{a}{2} = \frac{\cos \frac{\pi}{n}}{\cos \frac{2\pi - S}{2n}}. \]

(Proof) Divide the regular n-gon into 2n congruent rectangular triangles, and consider one of them. Let z be a side opposite to the rectangle, y be an other side than a/2, \( \theta \) be an angle opposite to y. Then

\[
\begin{align*}
\theta + \frac{\pi}{n} + \frac{\pi}{2} - \pi &= \frac{S}{2n}, \\
\cos z &= \cos y \cos \frac{a}{2}, \\
\sin z &= \sin y = \sin \frac{\pi}{n}, \\
\sin \frac{\pi}{2} &= \sin \theta = \sin \frac{\pi}{n}.
\end{align*}
\]

Eliminating \( y, z, \theta \) in the above, we obtain the desired formula. (Q.E.D.)

2 Proof of Proposition 1

Lemma 2.1  Consider all convex quadrangles where a pair of opposite sides a, b and an area S are fixed. Then the quadrangle that minimizes the sum of other two sides \( x + y \), is an isosceles trapezoid, ie \( x = y \), that has a circumcircle.

(Proof)
Step 1  Regard the minimum of \( x + y \) as a function of S, and denote it by \( h(S) \). We will show that \( h \) is a strictly increasing function of S. For any S, consider the minimal quadrangle and denote its four angles by \( \phi_i \) \( (i = 1, 2, 3, 4) \). If \( \phi_i \leq \pi/2 \) for all \( i \), then

\[ S = \phi_1 + \phi_2 + \phi_3 + \phi_4 - 2\pi \leq 0, \]

which is a contradiction. Hence \( \phi_i > \pi/2 \) for some \( i \), and thus, without loss of generality, suppose that \( \phi_1 > \pi/2 \). Then consider a triangle that consists of an angle \( \phi_1 \) and two sides of the quadrangle that emanate from the triangle. Without loss of generality we may suppose that these two sides are b and y. Let z be the other side than b, y of the triangle. Then, preserving lengths \( b \) and \( z \), and diminishing \( y \) continuously by \( \delta \), we can diminish area of the triangle and thus area of the quadrangle by \( \epsilon \). Consequently, by the definition of \( h \), we can see \( h(S - \epsilon) \leq h(S) - \delta < h(S) \). Here note that \( \epsilon \) can take an arbitrary positive number as long as it is sufficiently small. Therefore \( h \) is strictly increasing.

Step 2  Consider the minimal quadrangle \( Q \) of area S. Assume that it does not have a circumcircle. It is well-known that the convex quadrangle of given four sides and of the maximal area has a circumcircle. Hence there exists a quadrangle \( Q' \) which has the same area as \( Q \) as has, but has a larger area \( S' \) than S. Repeating the argument in Step 1, we can deduce that there exists a quadrangle \( Q'' \) which has smaller \( x + y \) than \( Q \) has, but has an area \( S'' \) such that \( S < S'' < S' \). But this implies \( h(S) < h(S'') < h(S') = h(S) \), which is a contradiction. Accordingly the minimal quadrangle has a circumcircle.
Step 3  Let $Q$ be the minimal quadrangle of area $S$, and $R$ be the radius of its circumcircle. Divide the quadrangle into four isosceles triangles with bases $a, b, x, y$, and denote them by $T_a, T_b, T_x, T_y$ respectively. Let $\alpha, \beta, \phi, \psi$ be angles of these isosceles at the center of circumcircle.

Consider an isosceles $T_a$ and denote its two angles other than $\alpha$ by $\theta$. Then

$$
\cos \alpha = \frac{\cos a - \cos^2 R}{\sin^2 R} \quad \text{and} \quad \cos \theta = \frac{(1 - \cos a) \cos R}{\sin a \sin R}.
$$

Thus, if we define two functions

$$
f(a, R) = \arccos \left( \frac{(1 - \cos a) \cos R}{\sin a \sin R} \right) \quad \text{and} \quad g(a, R) = \arccos \left( \frac{\cos a - \cos^2 R}{\sin^2 R} \right),
$$

we have $\alpha = g(a, R)$ and area of the isosceles $= g(a, R) + 2f(a, R) - \pi$.

Now note that $\alpha + \beta + \phi + \psi = 2\pi$ and the sum of areas of isosceles $T_a, T_b, T_x, T_y$ equals $S$. Accordingly we have

$$
F(x, y, R) := f(a, R) + f(b, R) + f(x, R) + f(y, R) = \pi + \frac{S}{2} \quad (1)
$$

and

$$
G(x, y, R) := g(a, R) + g(b, R) + g(x, R) + g(y, R) = 2\pi. \quad (2)
$$

Step 4  If we solve (1) and (2) with respect to $x$ and $y$, while $R$ being regarded as a parameter, we have $x = x(R), y = y(R)$. Since it is required to minimize $x(R) + y(R)$ with respect to $R$, it must hold

$$
\frac{dx}{dR} + \frac{dy}{dR} = 0. \quad (3)
$$

Differentiation of (1) and (2) give

$$
\begin{align*}
\frac{\partial F}{\partial x} \frac{dx}{dR} + \frac{\partial F}{\partial y} \frac{dy}{dR} + \frac{\partial F}{\partial R} &= 0, \\
\frac{\partial G}{\partial x} \frac{dx}{dR} + \frac{\partial G}{\partial y} \frac{dy}{dR} + \frac{\partial G}{\partial R} &= 0.
\end{align*}
$$

Then substitution of them into (3) results in

$$
\left( \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} \right) \cdot \left( \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) = \frac{\partial F}{\partial R} : \frac{\partial G}{\partial R}. \quad (4)
$$

Now an elementary computation gives

$$
\begin{align*}
\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} &= -\cos R \left( \frac{1}{\sqrt{1 + \cos x} \cdot h_x} - \frac{1}{\sqrt{1 + \cos y} \cdot h_y} \right), \\
\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y} &= \frac{1 + \cos x}{h_x} - \frac{1 + \cos y}{h_y}, \\
\frac{\partial F}{\partial R} &= \frac{1}{\sin R} \left( \frac{\sqrt{1 - \cos a} + \sqrt{1 - \cos b}}{h_a} + \frac{\sqrt{1 - \cos x} + \sqrt{1 - \cos y}}{h_x} \right), \\
\frac{\partial G}{\partial R} &= -2 \cot R \left( \frac{\sqrt{1 - \cos a} + \sqrt{1 - \cos b}}{h_a} + \frac{\sqrt{1 - \cos x} + \sqrt{1 - \cos y}}{h_x} \right).
\end{align*}
$$
where $h_x = \sqrt{1 + \cos x - 2 \cos^2 R}$ and $h_y, h_a, h_b$ are defined similarly. By substitution of them into the condition (4) it can be rewritten as

$$2 \cos^2 R \cdot h_y \sqrt{1 + \cos y + h_x \sqrt{1 + \cos x}} \cdot (1 + \cos y) = 2 \cos^3 R \cdot h_x \sqrt{1 + \cos x} + h_y \sqrt{1 + \cos y} \cdot (1 + \cos x) \quad (5)$$

Step 5 Squaring both the left- and the right-hand side of (5) and subtracting them, we have

$$2 \cos^2 R (\cos x - \cos y) (w_1 - w_2) = 0,$$

where

$$w_1 = 1 - 4 \cos^2 R + 4 \cos^4 R + \cos x (1 - 2 \cos^2 R) + \cos y (1 - 2 \cos^2 R) + \cos x \cos y,$$

$$w_2 = 2 \sqrt{1 + \cos x} \sqrt{1 + \cos y} h_x \cdot h_y.$$

Suppose that $w_1 = w_2$. Then we have $w_1^2 - w_2^2 = -h_x^2 h_y^2 w_3$, where

$$w_3 = 3(1 + \cos x)(1 + \cos y) + 2 \cos^2 R (2 + \cos x + \cos y) - 4 \cos^4 R.$$

However, as seen in the definition of $h_x, h_y$, we have $1 + \cos x - 2 \cos^2 R > 0$ and $1 + \cos y - 2 \cos^2 R > 0$. Consequently

$$w_3 > 3 \cdot 2 \cos^2 R \cdot 2 \cos^2 R + 2 \cos^2 R \cdot (2 \cos^2 R + 2 \cos^2 R) - 4 \cos^4 R = 16 \cos^2 R > 0.$$

Thus the hypothesis $w_1 = w_2$ can not be maintained. Therefore we obtain $\cos x - \cos y = 0$, i.e. $x = y$. Since the quadrangle is circumscribed by a circle, it must be an isosceles trapezoid. (Q.E.D.)

Consider the minimal isosceles trapezoid in Lemma 2.1. Writing $x, y$ instead of $a, b$, and regarding half of the minimum, i.e. the length of one of its two equal sides as a function of $x, y$, we denote it by $f(x, y)$.

**Lemma 2.2** The function $f$ is given by

$$f(x, y) = \arccos \left( \frac{-1 + s_x^2 + s_x s_y + s_y^2 + k c_x c_y}{1 + s_x s_y - k c_x c_y} \right),$$

where

$$c_x = \cos \frac{x}{2}, c_y = \cos \frac{y}{2}, s_x = \sin \frac{x}{2}, s_y = \sin \frac{y}{2}, \text{ and } k = \cos \frac{S}{2}.$$ 

(Proof) We write simply by $z$ instead of $f(x, y)$. Divide the isosceles trapezoid by its symmetry axis, and consider one of the two quadrangles made by the division. Let $\phi$ be the angle between $x/2$ and $z$, and $\psi$ be the angle between $y/2$ and $z$. If we prolong both sides $x/2$ and $y/2$, then a triangle will be made, of which three sides are $\pi/2 - x/2, \pi/2 - y/2, z$, and two of its three angles are $\pi - \phi, \pi - \psi$. Then

$$\left\{ \begin{array}{ll}
\cos \left( \frac{\pi}{2} - \frac{y}{2} \right) = \cos \left( \frac{\pi}{2} - \frac{x}{2} \right) \cos z + \sin \left( \frac{\pi}{2} - \frac{x}{2} \right) \sin z \cos (\pi - \phi), \\
\cos \left( \frac{\pi}{2} - \frac{x}{2} \right) = \cos \left( \frac{\pi}{2} - \frac{y}{2} \right) \cos z + \sin \left( \frac{\pi}{2} - \frac{y}{2} \right) \sin z \cos (\pi - \psi). \end{array} \right.$$
Hence
\[
\begin{align*}
\cos \phi &= \frac{s_x \cos z - s_y}{c_z \sin z}, & \sin \phi &= \frac{\sqrt{1 - s_x^2 - s_y^2 + 2s_x s_y \cos z - \cos^2 z}}{c_z \sin z}, \\
\cos \psi &= \frac{s_y \cos z - s_x}{c_y \sin z}, & \sin \psi &= \frac{\sqrt{1 - s_y^2 - s_x^2 + 2s_x s_y \cos z - \cos^2 z}}{c_y \sin z}.
\end{align*}
\]

On the other hand, \( \phi + \psi + \pi/2 + \pi/2 - 2\pi = S/2 \), i.e. \( \phi + \psi = \pi + S/2 \). Then, eliminating \( \phi, \psi \) in \( \cos(\phi + \psi) = \cos(\pi + S/2) \), we obtain a quadratic equation for \( w = \cos z \),
\[
(1 + s_x s_y - k c_x c_y)w^2 - (s_x + s_y)^2w + (-1 + s_x^2 + s_x s_y + s_y^2 + k c_x c_y) = 0.
\]
Note that the coefficient of \( w^2 \) in the above equation does not vanish, because, if it vanishes, then we have \( w = 1 \), which is a contradiction. Furthermore note that the quadratic equation always has a root \( w = 1 \). Hence another root is given by \( f(x, y) \).

(Q.E.D.)

**Lemma 2.3** The function \( f(x, y) \) is strictly convex.

(Proof)

**Step 1** We can see
\[
\frac{\partial f}{\partial x} = \frac{n_x}{d_1}, \quad \frac{\partial f}{\partial y} = \frac{n_y}{d_1},
\]
where
\[
\begin{align*}
d_1 &= 2(1 - k c_x c_y + s_x s_y)\sqrt{c_x^2 + c_y^2 - k c_x c_y}, \\
n_x &= k(1 + c_x^2) c_y - c_x (1 + c_y^2) - c_x s_x s_y + k s_x c_y s_y, \\
n_y &= -(1 + c_x^2) c_y + k c_x (1 + c_y^2) + k c_x s_x s_y - s_x c_y s_y.
\end{align*}
\]

**Step 2** Furthermore
\[
\frac{\partial^2 f}{\partial x^2} = \frac{n_{xx}}{d_2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{n_{xy}}{d_2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{n_{yy}}{d_2},
\]
where
\[
\begin{align*}
d_2 &= 4(c_x^2 + c_y^2 - 2k c_x c_y)^3 (1 - k c_x c_y + s_x s_y)^2, \\
n_{xx} &= (1 - k^2)(s_x + s_y)(1 + c_y^2 + s_x^3 s_y - 3k c_x c_y + k c_x^3 c_y), \\
n_{xy} &= -(1 - k^2)(s_x + s_y)(1 - 2c_x^2 - 2c_y^2 + c_x^2 c_y + s_x s_y + 2kc_x c_y - k c_x s_x c_y s_y), \\
n_{yy} &= (1 - k^2)(s_x + s_y)(1 + c_x^2 + s_x s_y^3 - 3k c_x c_y + k c_x c_y^3).
\end{align*}
\]

Hence the Jacobian \( J \) becomes
\[
J := \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = \frac{(1 - k^2)^2(s_x + s_y)^4}{16(c_x^2 + c_y^2 - 2k c_x c_y)^2 (1 - k c_x c_y + s_x s_y)^3}.
\]
Note that if $J > 0$, then the function $f$ is strictly convex. Thus it remains to prove $1 + s_x s_y - k c_x c_y > 0$.

Step 3 Consider again the triangle with three sides $\pi/2 - x/2, \pi/2 - y/2, z$ that appeared in the proof of Lemma 2.2. It must hold $(\pi/2 - x/2) + (\pi/2 - y/2) > z$ i.e. $\pi - z > x/2 + y/2$. Hence, for $w = \cos z$,

$$w > -c_x c_y + s_x s_y.$$  

Then the expression for $z = f(x, y)$ given in Lemma 2 becomes

$$\frac{-1 + s_x^2 + s_x s_y + s_y^2 + k c_x c_y}{1 + s_x s_y - k c_x c_y} > -c_x c_y + s_x s_y.$$  

Suppose that $1 + s_x s_y - k c_x c_y < 0$. Then, after some computation, we can derive $k < 0$. However, since $S = 4\pi/(n + 2)$ with $n \geq 3$, this leads to a contradiction. Thus we have $1 + s_x s_y - k c_x c_y > 0$. (Q.E.D.)

Let us define a function

$$\tilde{f}(x, y) = x + y + f(x, y).$$ (6)

**Lemma 2.4** For $n \leq 4$, the function $\tilde{f}(x, y)$ is strictly increasing.

(Proof) Ask when the following condition holds

$$\frac{\partial}{\partial x} \tilde{f}(x, y) = \frac{\partial}{\partial y} \tilde{f}(x, y) = 0.$$ (7)

Then, by the expressions given in Step 1 of the proof of Lemma 2.3, we have

$$d_1 + n_x = d_1 + n_y = 0.$$  

Since

$$n_x - n_y = -(1 + k)(c_x - c_y)(1 - c_x c_y + s_x s_y).$$  

Hence we can deduce $x = y$. Then we have

$$c_z = \sqrt{\frac{1}{1 + k} \left(2 - \sqrt{\frac{1 - k}{2}}\right)}.$$  

For $n \leq 4$ we see

$$\frac{1}{1 + k} \left(2 - \sqrt{\frac{1 - k}{2}}\right) \geq 1.$$  

Accordingly the condition (7) does not hold. Therefore the proof is completed. (Q.E.D.)
Proof of Proposition 1

A network of \( n \)-prism type consists of \( n \) quadrangles \( Q_i (i = 1, 2, \cdots, n) \), and two \( n \)-gons \( A \) and \( B \). Let \( a_i \) be the common side of \( A \) and \( Q_i \), and \( b_i \) be the common side of \( B \) and \( Q_i \). Denote the total length of \( Q_i \) by \( L_i \). Then it can be seen that the total length of the network \( L \) is given by

\[
2L = \sum_{i=1}^{n} L_i + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i .
\]

Since, by Lemma 2.1,

\[
L_i \geq a_i + b_i + 2f(a_i, b_i),
\]

we have

\[
L \geq \sum_{i=1}^{n} f(a_i, b_i) + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \bar{f}(a_i, b_i) .
\]

Then, Lemma 2.3, with aid of Jensen’s inequality, shows that

\[
\frac{1}{n} \sum_{i=1}^{n} \bar{f}(a_i, b_i) \geq \bar{f}(\bar{a}, \bar{b}) ,
\]

where

\[
\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i \quad \text{and} \quad \bar{b} = \frac{1}{n} \sum_{i=1}^{n} b_i .
\]

Consequently

\[
\frac{L}{n} \geq \bar{f}(\bar{a}, \bar{b}) .
\]

Now recall the isoperimetric property of spherical polygons: among all spherical polygons of area \( S \), the regular polygon has the shortest perimeter length. Thus, if \( e_n \) stands for the length of one side of the regular \( n \)-gon of area \( S \), we have \( \bar{a} \geq e_n, \bar{b} \geq e_n \). Then Lemma 2.4 implies \( \bar{f}(\bar{a}, \bar{b}) \geq \bar{f}(e_n, e_n) \). Therefore we obtain

\[
L \geq n \bar{f}(e_n, e_n),
\]

which proves the theorem. \( \text{(Q.E.D.)} \)

3 Proof of Proposition 2

Lemma 3.1 Consider all triangles where a side \( a \) and an area \( S \) are fixed. Then the triangle that minimizes the sum of other two sides \( x+y \) is an isosceles, ie \( x = y \).

(Proof) Consider a triangle satisfying the given conditions, and let \( R \) be the radius of its circumcircle. (Note that any triangle has a circumcircle.) Divide the triangle into three isosceles triangles with bases \( a, x, y \), and denote them by \( T_a, T_x, T_y \) respectively. By a similar reasoning to that in Step 3 of the proof of Lemma 1.1, we see that it is sufficient to minimize \( x + y \) when \( x, y \) satisfies both conditions

\[
F(x, y, R) := f(a, R) + f(x, R) + f(y, R) = \pi + \frac{S}{2} \quad \text{(8)}
\]

and

\[
G(x, y, R) := g(a, R) + g(x, R) + g(y, R) = 2\pi . \quad \text{(9)}
\]
Once $F, G$ were defined, we can repeat the reasoning in Step 4 and Step 5 of the proof of Lemma 1.1 without any change. Thus we come to the conclusion $x = y$, which completes the proof. (Q.E.D.)

Consider the minimal isosceles triangle in Lemma 3.1. Writing $x$ instead of $a$, and regarding half of the minimum, i.e., the length of one of its two equal sides as a function of $x$, we denote it by $g(x)$.

**Lemma 3.2** The function $g$ is given by

$$g(x) = \arccos \left( \frac{c_x \left( k - c_x \right)}{1 - k c_x} \right),$$

where

$$c_x = \cos \frac{x}{2} \quad \text{and} \quad k = \cos \frac{S}{2}.$$  

(Proof) We write simply by $z$ instead of $g(x)$. Divide the isosceles triangle by its symmetry axis, and consider one of the two triangles made by the division. Let $\phi$ be the angle opposite to the side $x/2$, and $\psi$ be the angle between $x/2$ and $z$. Prolong the side $x/2$ and draw a line which makes an angle $\pi/2 - \phi$ with the side $z$. Then a triangle will be made, of which three sides are $\pi/2, \pi/2 - y/2, z$, and two of its three angles are $\pi/2 - \phi, \pi - \psi$. Then

$$\begin{align*}
\cos \left( \frac{\pi}{2} - \frac{x}{2} \right) &= \cos \frac{\pi}{2} \cos z + \sin \frac{\pi}{2} \sin z \cos \left( \frac{\pi}{2} - \phi \right), \\
\cos \frac{\pi}{2} &= \cos \left( \frac{\pi}{2} - \frac{x}{2} \right) \cos z + \sin \left( \frac{\pi}{2} - \frac{x}{2} \right) \sin z \cos(\pi - \psi). 
\end{align*}$$

Hence

$$\begin{align*}
\sin \phi &= \frac{s_x}{\sin z}, \\
\cos \phi &= \frac{\sqrt{c_x^2 - \cos^2 z}}{\sin z}, \\
\cos \psi &= \frac{s_x \cos z}{c_x \sin z}, \\
\sin \psi &= \frac{\sqrt{c_x^2 - \cos^2 z}}{c_x \sin z},
\end{align*}$$

where $s_x = \sin \frac{x}{2}$. Now, from the assumption on area, we have $\phi + \psi = \frac{\pi}{2} + \frac{S}{2}$. Hence follows a quadratic equation for $w = \cos z$,

$$(1 - k c_x) w^2 - (1 - c_x^2) w + c_x (k - c_x) = 0.$$

It can be factored as

$$(w - 1) \left( (1 - k c_x) w - c_x (k - c_x) \right) = 0,$$

which gives the desired result. (Q.E.D.)

**Lemma 3.3** The function $g$ is strictly convex.
(Proof) By differentiation we have
\[
\frac{\partial g}{\partial x} = \frac{-2c_x + k(1 + c_x^2)}{2(1 - k c_x)\sqrt{1 + c_x^2 - 2k c_x}},
\]
\[
\frac{\partial^2 g}{\partial x^2} = \frac{(1 - k^2)(2 + k(C_x^2 - 3c_x))\sqrt{1 - c_x^2}}{4(1 - k c_x)^2(1 + c_x^2 - 2k c_x)^{3/2}}.
\]
Since \(0 > c_x^3 - 3c_x > -2\) for \(0 < c_x < 1\), we see \(2 + k(C_x^2 - 3c_x) > 2 - 1 \cdot 2 = 0\), and thus the second derivative is positive. Thus the proof is completed. \(\text{(Q.E.D.)}\)

Let us define a function
\[
\tilde{g}(x) = x + g(x).
\]  \hspace{1cm} (10)

**Lemma 3.4** Assume \(n \leq 5\). Then the function \(\tilde{g}\) is strictly increasing.

(Proof) Using the expression for the derivative of \(g\) given in the proof of Lemma 3.3, from the condition that the derivative of \(\tilde{g}\) vanishes, it follows \(h(c_x) = 0\), where
\[
h(\xi) = 3k^2\xi^4 - 4k(1 + 2k^2)\xi^3 + 18k^2\xi^2 - 12k\xi + (4 - k^2).
\]
Since \(h(1) = 4(1 - k)^2(1 - 2k)\), we have \(h(1)\) is non-negative when \(n \leq 5\). Furthermore,
\[
h'(\xi) = -12k(1 + \xi^2 - 2k\xi)(1 - k\xi) < 0
\]
for \(0 < \xi < 1\). Accordingly \(h(\xi) > 0\) for \(0 < \xi < 1\). Hence we obtain the conclusion. \(\text{(Q.E.D.)}\)

**Proof of Proposition 2**

A network of \(n\)-pyramid type consists of \(n\) triangles \(T_i\) \((i = 1, 2, \cdots, n)\) and an \(n\)-gon \(A\). Let \(a_i\) be the common side of \(A\) and \(T_i\). Denote by \(L_i\) the total length of perimeter of \(T_i\). Then the total length of the network \(L\) is given by
\[
2L = \sum_{i=1}^{n} L_i + \sum_{i=1}^{n} a_i.
\]
Since Lemma 3.1 shows that \(L_i \geq a_i + 2g(a_i)\), we have
\[
L \geq \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \tilde{g}(a_i).
\]
Accordingly, by convexity of \(f\) proved in Lemma 3.3, Jensen's inequality implies
\[
\frac{L}{n} \geq \tilde{g}(\bar{a}). \hspace{1cm} (11)
\]
Now the isoperimetric inequality shows that \(\bar{a} \geq e_n\). Then Lemma 3.4 implies that \(\tilde{g}(\bar{a}) \geq \tilde{g}(e_n)\). Therefore we obtain
\[
L \geq n \tilde{g}(e_n),
\]
which completes the proof.
References
