<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>著者</td>
<td>ATSUMI Tsuyoshi</td>
</tr>
<tr>
<td>原出所の学</td>
<td>鹿児島大学理学部紀要・数学・物理学・化学</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>別言語のタイトル</td>
<td>重可移群について</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
著者 | 田村 誠
---|---
誌名 | 鹿児島大学理学部紀要・数学・物理学・化学
巻 | 8
号 | 1
ページ | 23-27
URL | http://hdl.handle.net/10232/00003959
DOUBLY TRANSITIVE PERMUTATION GROUPS
OF DEGREE 5q+1, WHERE q IS A PRIME

By

Tsuyoshi Atsumi
(Received Sept. 30, 1975)

1. Introduction and Summary


THEOREM. Let G be an insoluble transitive permutation group of degree p=4q+1 where p and q are primes, which is not doubly primitive. Then G=\text{PSL}(3,3) and p=13.

Furthermore Atkinson [4] proved the following theorems.

THEOREM. Let G be a doubly transitive group of degree 2q+1, where q is a prime, which is not doubly primitive. Then G is either sharply doubly transitive or a group of automorphisms of a block design with \lambda=1 and k=3.

THEOREM. Let G be a doubly transitive permutation group on \Omega of degree 3q+1, where q is a prime. Then one of the following statements is true.

(1) G is doubly primitive.
(2) G is sharply doubly transitive.
(3) G is a group of automorphisms of a block design on \Omega with \lambda=1 and k=4.
(4) G=\text{PSL}(3,2) and q=2.

In this paper we shall prove the following theorem.

THEOREM. Let G be a doubly transitive permutation group on \Omega of degree 5q+1, where q is a prime and greater than 11, Then one of the following statements is true.

(1) G is doubly primitive.
(2) G is a group of automorphisms of a block design on \Omega with \lambda=1 and k=6.
(3) |G_{\text{af}}|\leq 24.
(4) G has a regular normal subgroup.

Our notation for the parameters of a block design, v, k, r, \lambda, is standard; see [8]. Throughout this paper the term “block” is used only in the block design sense; however, a term such as “K-block” refers to a set of imprimitivity for a group K. In order to prove Theorem we need the several lemmas.
Lemma 1 (E. Witt [12]). Let $X$ be a doubly transitive group on a set $\Omega$, let $a, \beta \in \Omega$ with $a+\beta$ and let $K$ be a weakly closed subgroup of $X_{a\beta}$. Then, if $\Delta = \text{fix}(K)$, in the block design whose blocks are the images under $X$ of $\Delta$ we have $\lambda = 1$.

Proof. We omit the proof of the lemma. (See [4]).

Lemma 2. (Atkinson [4]). Let $X$ be a doubly transitive group on a set $\Omega$, let $a \in \Omega$ and let $\Delta$ be a set of imprimitivity for the action of $X_a$ on $\Omega - \{a\}$. Let $\beta \in \Delta$ and suppose that $\Delta - (\beta)$ is invariant under $X_{(a, \beta)}$. Then, in the block design whose blocks are the images under $X$ of $\Gamma = \Delta \cup \{a\}$ we have $\lambda = 1$.

Proof. (See [4]).

Lemma 3 (Atkinson [4]). Let $X$ be a doubly transitive group on a set $\Omega$. Let $a \in \Omega$ and let $\Delta$ be a set of imprimitivity of size $m$ for the action of $X_a$ on $\Omega - \{a\}$. Then, if $\beta \in \Delta$, $X_{(a, \beta)}$ has an invariant set $\Gamma$ of size $m-1$ on $\Omega - \{a, \beta\}$. Furthermore, if $X_{a\beta}$ is transitive on $\Delta - (\beta)$, $X_{a\beta}$ and $X_{(a, \beta)}$ are transitive on $\Gamma$.

Proof. (See [4]).

Lemma 4. Let $\Omega$ be a set on which there is a non-trivial block design with $\lambda = 1$. Then if $|\Omega| = 5q+1$, where $q$ is a prime, then $q=3$ or $19$ or $k=6$.

Proof. We prove this lemma by considering the incidence equations of a block design.

Lemma 5 (E. Bannai [5]). Let $G$ be a transitive permutation group on $\Omega$ and $a \in \Omega$. Let $H = G_a$ and $x \in G$. Then we have the following equation,

$$\frac{|\Omega|}{|I(x)|} = |\{h \in H | h \text{ is } H\text{-conjugate to } x\}|$$

$$= |\{g \in G | g \text{ is } G\text{-conjugate to } x\}|.$$

Proof. We count the pairs $\{(\delta, g) | \delta \in \Omega, g \in G, \delta x = \delta, g \text{ is } G\text{-conjugate to } x\}$ in two ways. We get the above equation.

We shall frequently use the well-known theorem of Burnside that a transitive group of prime degree is either doubly transitive or is a metacyclic Frobenius group.

2. Proof of the theorem

Let $G$ be a doubly transitive group on a set $\Omega$ of size $5q+1$, where $q$ is a prime. If $G$ is a counterexample to theorem. By a theorem of [1] we have that $q$ divides $|G|$ to the first power only. Let $Q$ be a Sylow $q$-subgroup of $G_a$ where $a \in \Omega$. Let $\Delta_1, \Delta_2, \Delta_3, \ldots$ be a non-trivial system of imprimitivity for the action of $G_a$ on $\Omega - \{a\}$. Let $H = \{x | x \in G_a, \Delta_i x = \Delta_i\}$, $K = \{x \in G_a | \Delta_i x = \Delta_i, i = 1, 2, 3, \ldots\}$ and $\beta \in \Delta_1$. Then $G_{a\beta} \subseteq H_
and $K \triangleleft G_a$. Furthermore we can consider $G_a$ to act on $\Delta$, where $\Delta = \{\Delta_1, \Delta_2, \ldots\}$. There are two cases to consider depending on the size of the $G_a$-blocks.

Case 1. $q$ $G_a$-blocks of size 5

Clearly $H$ acts transitively on $\Delta_1$. At first we assume that $G_a$ acts on $\Delta$ as an insoluble group and $H$ acts on $\Delta_1$ as a soluble group. If $H$ acts on $\Delta_1$ as a regular group of order 5, then $G_{a5}=1$ on $\Delta_1$. Consequently $G_{a5}$ fixes the points of $\Delta_1$. So we get a contradiction by using Lemma 1. If $H$ acts on $\Delta_1$ as a Frobenius group of order 10, then we can assume that $H = \langle (\beta \gamma \delta \varepsilon \eta), (\beta \gamma) (\delta \varepsilon) \rangle$, where $(\beta, \gamma, \delta, \varepsilon, \eta) = \Delta_1$. $G_{a5}$ acts on $\Delta_1 - \{\beta\}$ semi-regularly and $(\delta, \varepsilon, \gamma, \eta)$ are $G_{a5}$-orbits. So $(\delta, \varepsilon)$ and $(\gamma, \eta)$ are $G_{a5}$-invariant. By Lemma 1 we can assume that $G_{a5}$ fixes no points of $\Omega$ except $\alpha$ and $\beta$. So $N(G_{a5}) = G_{\{\alpha, \beta\}}$. $|G_{\{\alpha, \beta\}} : G_{\alpha}| = |G_{\{\gamma, \eta\}} : G_{\beta}| = 2$. Consequently $N(G_{a5}) = G_{\{\alpha, \beta\}}$ is $G_{\{\alpha, \beta\}}$-invariant. Therefore $\Delta_1 - \{\beta\}$ is $G_{\{\alpha, \beta\}}$-invariant. This fact contradicts Lemma 2.

Secondly if $G_a$ acts on $\Delta$ as a doubly transitive group, then $\Delta_1 - \{\beta\}$ is an orbit of $G_{a5}(=H_5)$. Since $G_a$ acts on $\Delta$ as a doubly transitive group, $H$ acts transitively on $\{\Delta_2, \ldots, \Delta_q\}$. As $|H : G_{a5}| = |H : H_5| = 5$, all the orbits of $G_{a5}$ on $\{\Delta_2, \ldots, \Delta_q\}$ have size at least $(q-1)/5 > 4$ when $q>19$, and if $q \leq 19$, then all the orbits of $G_{a5}$ on $\{\Delta_2, \ldots, \Delta_q\}$ have size at least $(q-1)/4$ by Lemma 17. 1 [10]. It follows that $\Delta_1 - \{\beta\}$ is the unique orbit of $G_{a5}$ of size 4 and therefore $\Delta_1 - \{\beta\}$ is $G_{\{\alpha, \beta\}}$-invariant. We may now obtain a contradiction from Lemma 2.

Finally if $G_a$ acts on $\Delta$ as a soluble group and $K$ is insoluble and $K \neq K_1$, then $K$ has an abelian characteristic subgroup $M$ of order $n$ by the result of O'Nan [7]. Consequently $H$ acts on $\Delta_i$ as $A_5$ or $S_5$ for any $i$. Now let be a element $x$ of $N$ of order 3. Then $x$ fixes $1+2q$ points on $\Omega$ because $N$ acts faithfully on $\Delta_i$ for any $i$. The number of the conjugate elements of $x$ in $G_a$ is 20. For $G_a$ acts on $\Delta$ as a Frobenius group and so any element of $G_a-K$ does not fix $1+2q$ points. Therefore the number of the conjugate elements of $x$ is $(5q+1)20/(2q+1)$ by Lemma 5 and this number must be integer. $(5q+1)20/(2q+1)=50-30/(2q+1)$ is an integer $(q>11)$. This is a contradiction.

Finally if $G_a$ acts on $\Delta$ as a soluble group and $K$ is soluble and $K \neq 1$, then $K$ has an abelian characteristic subgroup $M$. Clearly $\pi(M) \subseteq \{2, 3, 5\}$. Let $S$ be a $S_5$-subgroup of $M$. If $S \neq 1$, then $S$ is weakly closed in $G_a$. For $G_a$ acts on $\Delta$ as a Frobenius group and so any element of order 2 in $G_a-K$ fixes at most one $\Delta_i$ as a set, but
every element of order \(2^i\) in \(S\) fixes at least \(q\) points on \(Q - \{a\}\). So any element of \(S\) is not conjugate to any element of \(G_s - K\) in \(G_s\). If \(S^g \subseteq G_s\) for any \(g \in G_s\), then by the above argument \(S^g \subseteq K\) and so \(S^g = S^i\) for some \(k \in K\) because \(S\) is a \(S_2\)-subgroup of \(K\) and \(S\) is normal in \(K\). Thus \(S^g = S\). Clearly \(S \subseteq G_s\) for some \(\gamma \in \Omega\) and \(S\) is weakly closed in \(G_s\) and \(S\) fixes at least \(q\) points on \(\Omega - \{a\}\). This result contradicts our assumption by Lemma 1. If \(S=1\), then we consider \(S_3\)-subgroup of \(M\). Similarly we get a contradiction. So we assume that \(M\) is a \(5\)-group. If \(M\) does not act on \(\Omega - \{a\}\) as semi-regular group, then we can construct a block design with \(\lambda = 1\) by Corollary B1 [6]. This is not our case. So \(M\) acts on \(\Omega - \{a\}\) semi-regularly. Thus \(|M|=5\), \(M\) is cyclic. In this case we have a contradiction by Aschbacher's result [2]. If \(K=1\), then it follows that the \(S_3\)-subgroup of \(G_s\) is cyclic. Consequently \(G\) is known by a result of Aschbacher [3]. We have a contradiction by considering the degree of \(G\).

Case 2. \(5\) \(G_s\)-blocks of size \(q\)

Since \(G_s/K \subseteq S_5\), \(g \in \{G_s: K\}\). Therefore \(Q \subseteq K\) and \(K\) is transitive on each \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \text{ and } \Delta_5\). If \(N\) is the kernel of the action of \(K\) on \(\Delta_1\) and \(N \neq 1\), then \(N\) acts transitively on some \(\Delta_i\) which contradicts the fact that \(q^2 \nmid |G|\). Thus \(K\) acts faithfully on each of \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \text{ and } \Delta_5\). If \(K\) is soluble we shall show that \(K_\beta = 1\). If \(K_\beta = 1\), then \(K_\beta\) fixes precisely one point from each of \(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \text{ and } \Delta_5\) because \(K\) has a unique conjugacy class of subgroups of index \(q\). Thus \(K_\beta\) and any conjugate of \(K_\beta\) fix exactly \(6\) points. Consider some conjugate \(K_\beta^g\) of \(K\) contained in \(G_s\). If \(K_\beta^g \subseteq K\) then some \(\Delta_i\) contains none of the fixed points of \(K_\beta^g\) and hence there is some \(\Delta_j\) which contains at least two of these fixed points; but then \(K_\beta^g\) must fix pointwise the whole of \(\Delta_j\) and so has more than \(5\) fixed points. Thus \(K_\beta^g \subseteq K\) and, as \(K\) has a unique subgroup of index \(q\) which fixes \(\beta\), we have \(K_\beta^g = K_\beta\). Therefore \(K_\beta\) is weakly closed in \(G_s\) and Lemma 1 gives us a contradiction. This means that \(K_\beta = 1\) as we asserted. \(|G_s| = 5q\cdot|G_s| = eq(v|120)\). Consequently \(|G_s| = 24\). This is a contradiction. If \(K\) is insoluble and \(G_s\) is local, then there is a normal \(q\)-subgroup \(Q'\) of \(G_s\). \(|Q'| = q\) by Theorem [1]. So \(G\) is known by Aschbacher's Theorem [2]. Again we have a contradiction by considering the degree of \(G\).

From now on we can assume that \(G_s\) has a unique minimal normal subgroup \(N\) which is simple. If \(C_{G_s}(N) \neq 1\), then \(C_{G_s}(N) \supseteq N\) because \(C_{G_s}(N) < G\) and \(N\) is a unique minimal subgroup. Therefore \(Z(N) = C_N(N) \neq 1\). So \(N\) is a cyclic group of order \(q\). \(G\) is local. This is not our case. So \(C_{G_s}(N) = 1\). \(G_s = N_{G_s}(N)/C_{G_s}(N)\) is considered to be included in \(\text{Aut } N\), where \(\text{Aut } N\) is the group of the automorphisms of \(N\). Since \(N \cong \text{Inn } N\), where \(\text{Inn } N\) is the group of the inner automorphisms of \(N\), we can consider \(G_s/N\) to be included in \(\text{Aut } N/\text{Inn } N\). By a theorem of Wielandt [11] \(\text{Aut } N/\text{Inn } N\) is cyclic. So it follows that \(G_s/N\) is cyclic. Since \(G_s/K\) is a homomorphic image of \(G_s/N\) and \(G_s/K \subseteq S_5\). Thus \(G_s/K\) is cyclic and \(G_s\) acts on \(\Delta\) regularly.\ldots(1)

As \(K \subseteq G_s\), \(\Gamma_1 = \Delta_1 - \{\beta\}\) is a \(G_s\)-orbit of size \(q-1\). If \(\Gamma_1\) is \(G_s(\gamma)\)-invariant, then
we have a contradiction from Lemma 2. If $\Gamma_1$ is not $G_{a,b}$-invariant, then there exists another $G_{a,b}$-orbit $\Gamma_2$ of size $q-1$ such that $\Gamma_1 \cup \Gamma_2$ is an orbit of $G_{a,b}$. By Lemma 3 there is yet another $G_{a,b}$-orbit $\Gamma_3$ of size $q-1$ and since it is a $G_{a,b}$-orbit it is distinct from $\Gamma_1$ and $\Gamma_2$. If either of $\Gamma_2$ or $\Gamma_3$ is contained in any $\Delta_i$ then $G_{a,b}$ leaves $\Delta_i$ invariant and fixes the remaining point of $\Delta_i$; using Lemma 1, this leads to a contradiction. There are two cases to remain. In first case there is $\Delta_k$ ($2 \leq k \leq 5$) such that $\Delta_k \cap \Gamma_2 \neq \emptyset$ and $\Delta_k \cap \Gamma_3 \neq \emptyset$. But $\Gamma_2 \cap \Delta_k$ and $\Gamma_3 \cap \Delta_k$ are invariant under $K_{a,b}$ and, as they are set of imprimitivity for the action of $G_{a,b}$ on $\Gamma_2$ and $\Gamma_3$, we have $|\Delta_k \cap \Gamma_2| \leq (q-1)/2$ and $|\Delta_k \cap \Gamma_3| \leq (q-1)/2$. Consequently, $K$ has at least 3 orbits on $\Delta_k$. Now $K$ acts doubly transitively on each of $\Delta_i$ and $\Delta_k$ with characters $1+x_i$ and $1+x_2$, say, and the number of orbits of $K_{a,b}$ on $\Delta_k$ is $(1+x_i, 1+x_2) \leq 2$ and this is a contradiction. In final case we have $\Gamma_2 \subseteq \Delta_i \cup \Delta_j$, $\Gamma_3 \subseteq \Delta_k \cup \Delta_l$, where $(i, j) \cap (k, l) = \emptyset$. So $G_{a,b}$ has an element of order 2 on $(\Delta_2, \Delta_3, \Delta_4, \Delta_5)$. This fact contradicts (1). Thus we complete the proof of the Theorem.

Acknowledgment

The author is grateful to Prof. H. Nagao and Dr. E. Bannai for their kind suggestions.

References