

A class of random fractal tessellations in hyperbolic planes

Yukinao ISOKAWA, Kagoshima University

1 Introduction

In his famous Essay Mandelbrot (1982) has presented various fractal models for the Universe. He and his predecessors have demanded that these models satisfy the two conditions which on the surface are contradictory each other. The one of these is that the mass $M(\rho)$ in a sphere with radius ρ and center at the Earth grows as ρ^D when ρ tends to infinity. Here D is a fraction such that $0 \leq D \leq 3$, which Mandelbrot call the fractal dimension of the Universe. The other condition is that the mass distribution in the Universe satisfies some cosmographic principle, which roughly states that to every observer at any position, the mass distribution has the same appearance. Mandelbrot has found that in order to satisfy both conditions, it is necessary to introduce randomness into fractal models.

Although these models have great values both theoretically and practically, it seems to the present author that they have an unnecessary restriction. Mandelbrot's study and later studies (for these see Falconer (1993)) have confined themselves to fractal models in Euclidean spaces. In Euclidean spaces, among various types of fractals, the most simple are self-similar ones. On the contrary, in hyperbolic spaces, it is impossible to consider similarity. As is well-known, the existence of similar sets is equivalent to the axiom of parallelism (As for the hyperbolic geometry, consult, for example, Fenchel (1989)). How we define fractals in hyperbolic spaces ?

In this paper we present a class of random tessellations in hyperbolic planes, and show that they have a fractal property. To put it more explicitly, we construct random tessellations with unbounded domains which are determined by ultraparallel straight lines. In a special case these tessellations reduce to non-random ones which are composed of mutually con-

gruent domains. Imagine that the mass lies uniformly on lines which are boundaries of constituent domains of a tessellation, and interiors of these domains are void of the mass. Let $M(\rho)$ be the total mass in a disk with radius ρ and center at some point. Then our main theorem roughly states that the expectation of $M(\rho)$ behaves as $e^{D\rho}$ as ρ tends to infinity, where D is a fraction such that $0 \leq D \leq 1$. Thus we observe a somewhat peculiar phenomenon that tessellation which is composed of strictly or statistically congruent domains exhibit a fractal behaviour.

In Section 2 we first present the definition of random tessellations with which we concern ourselves throughout the paper. And after preparing several lemmas, we offer a heuristic argument which derives an infinite series that approximates the expectation of $M(\rho)$. In Section 3 we study asymptotic behaviour of this series in a special case. In Section 4, based on the result established in the previous section, we prove our main theorem. Before Section 5, we do not pay any attention to any cosmographic principle. In Section 5 we construct tessellations with a cosmographic principle whose composing domains are statistically congruent. Especially we offer non-random tessellations whose domains are strictly congruent.

2 Definitions and preliminaries

Random fractal tessellations which we consider in this paper will be constructed by generating ultraparallel lines according to a branching stochastic process. Thus we introduce a branching stochastic process on $\{0, 1, 2, \dots\}$. We represent a realization of this process by a tree, whose nodes are finite sequences of positive integers $\{1, 2, 3, \dots\}$. We denote this random tree by \mathbf{T} . Now, let \mathbf{i} be a node of \mathbf{T} and let $N_{\mathbf{i}}$ be the number of outgoing edges from the node \mathbf{i} . Particularly when \mathbf{i} is the root node of \mathbf{T} , we denote this number by N_0 . We assume that

(A1) all $N_{\mathbf{i}}$ are mutually independent and identically distributed.

We denote this common probability distribution by $Q = \{q_n : n = 0, 1, 2, \dots\}$. We allow the possibility that $N_{\mathbf{i}} = 0$, that is, $q_0 > 0$.

Now we go into the realm of the hyperbolic geometry a little while. Let \mathbf{D} be the Poincaré disk and $\partial\mathbf{D}$ be the boundary of \mathbf{D} . Furthermore, let

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

H be the half-plane $\{x + iy : y > 0\}$ and l_\emptyset be the line $\{x + iy : y = 0\}$. In \mathbf{D} , a line represented by a circle which is orthogonal to $\partial\mathbf{D}$. Denote by $l(\alpha, \theta)$ the line whose two points of infinity are $e^{i(\theta+\alpha)}$ and $e^{i(\theta-\alpha)}$. Thus α is the parallel angle at the origin (the center of \mathbf{D}). Consider the translation which moves the line l_\emptyset to the line $l(\alpha, \theta)$. There are infinitely many such translations. Out of these we adopt the translation $\phi = \phi(\cdot; \alpha, \theta)$ whose inverse is expressed as

$$\phi^{-1}(z) = ie^{-i\theta} \frac{z - z_0}{1 - \overline{z_0}z}, \quad z \in \mathbf{D},$$

where

$$z_0 = \frac{1 - \sin \alpha}{\cos \alpha} e^{i\theta}.$$

In order to state the manner of generating lines explicitly, we introduce a family of probability distributions $\{Q_n : n = 1, 2, \dots\}$ where each Q_n is a distribution on $\{(\alpha_1, \theta_1, \dots, \alpha_n, \theta_n) : 0 < \alpha_j < \frac{\pi}{2}, 0 < \theta_j < \pi \text{ for every } j\}$. Lines generated according to Q_n lie in the half-plane H . In the following we only consider the case that these generated lines are mutually ultraparallel. Thus we assume that for each n

(A2) the support of Q_n is contained in

$$\{(\alpha_1, \theta_1, \dots, \alpha_n, \theta_n) : 0 < \theta_1 - \alpha_1 < \theta_1 + \alpha_1 < \dots < \theta_n - \alpha_n < \theta_n + \alpha_n < \pi\}.$$

We turn to define tessellations which are determined by ultraparallel lines. We generate these lines in the following manner :

1. First we generate N_\emptyset lines according to the probability distribution Q and the family of probability distributions $\{Q_n : n = 1, 2, \dots\}$. We denote one of the resulting lines by $l(\alpha_{i_1}, \theta_{i_1})$.
2. Suppose that a line $l(\alpha_{i_1 i_2 \dots i_{k-1}}, \theta_{i_1 i_2 \dots i_{k-1}})$ has already been generated. Then we generate $N_{i_1 i_2 \dots i_{k-1}}$ lines. Then we translate these lines by the translation $\phi(\cdot; \alpha_{i_1 i_2 \dots i_{k-1}}, \theta_{i_1 i_2 \dots i_{k-1}})$. We denote one of these lines by $l(\alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k})$.
3. We repeat the procedure stated in step 2 indefinitely.

As soon as we have generated infinitely many ultraparallel lines

$$\{l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) : \mathbf{i} \in \mathbf{T}\},$$

we obtain a tessellation with unbounded domains.

Now we prepare several lemmas concerning lines in the hyperbolic plane.

Lemma 1. If the line $l(\alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k})$ is a translate of a line $l(\alpha_{i_k}, \theta_{i_k})$ by the translation $\phi(\cdot; \alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k})$, then

$$\tan \alpha_{i_1 i_2 \dots i_{k-1} i_k} = \frac{\sin \alpha_{i_1 i_2 \dots i_{k-1}} \sin \alpha_{i_k}}{\cos \alpha_{i_k} + \cos \alpha_{i_1 i_2 \dots i_{k-1}} \sin \theta_{i_k}}.$$

Proof Lemma 1. Denote $\alpha_{i_1 i_2 \dots i_{k-1}}, \theta_{i_1 i_2 \dots i_{k-1}}, \alpha_{i_1 i_2 \dots i_{k-1} i_k}, \theta_{i_1 i_2 \dots i_{k-1} i_k}, \alpha_{i_k}$ and θ_{i_k} by $\alpha, \theta, \alpha', \theta', \alpha_0$ and θ_0 respectively. In \mathbf{D} lines $l(\alpha_0, \theta_0)$ and $l(\alpha', \theta')$ are represented by the equations

$$|z|^2 - (\bar{c}_0 z + c_0 \bar{z}) + 1 = 0 \quad \text{and} \quad |z|^2 - (\bar{c}' z + c' \bar{z}) + 1 = 0$$

respectively, where

$$c = \frac{1}{\cos \alpha_0} e^{i\theta_0} \quad \text{and} \quad c' = \frac{1}{\cos \alpha'} e^{i\theta'}.$$

Then, because $l(\alpha', \theta')$ is a translate of $l(\alpha_0, \theta_0)$ by the translation $\phi(\cdot; \alpha, \theta)$, we can derive

$$(1) \quad c' = -ie^{i\theta} \cdot \frac{2ir + c_0 + \bar{c}_0(ir)^2}{1 + \bar{c}_0 \cdot ir + c_0 \cdot \bar{i}r + |ir|^2},$$

where

$$(2) \quad r = \frac{1 - \sin \alpha}{\cos \alpha}.$$

Using (1) and (2), after an elementary calculation, we obtain

$$\frac{1}{\alpha'} = |c'| = \frac{\cos \alpha_0 + \cos \alpha \sin \theta_0}{\sqrt{\cos^2 \alpha \cos^2 \alpha_0 + \sin^2 \alpha + 2 \cos \alpha \cos \alpha_0 \sin \theta_0 + \cos^2 \alpha \sin^2 \theta_0}}.$$

From this it follows that

$$\tan \alpha' = \frac{\sin \alpha_0 \sin \alpha}{\cos \alpha + \cos \alpha_0 \sin \alpha},$$

which is the result we have to prove.

Lemma 2. Denote the hyperbolic distance between l_\emptyset and $l(\alpha, \theta)$ by $d(l_\emptyset, l(\alpha, \theta))$. Then

$$\cosh d(l_\emptyset, l(\alpha, \theta)) = \frac{\sin \theta}{\sin \alpha}.$$

Proof of Lemma 2. Let u_1, v_1 be points of infinity of l_1 , and u_2, v_2 be those of l_2 . Denote the cross ratio of four points u_1, v_1, u_2, v_2 by r . In the

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

hyperbolic geometry it is known that if two lines l_1 and l_2 are ultraparallel, then $\cosh d(l_1, l_2) = \frac{1+r}{|1-r|}$. In order to prove the lemma, it is sufficient to put $u_1 = 1, v_1 = -1, u_2 = e^{i(\theta+\alpha)}$, and $v_2 = e^{i(\theta-\alpha)}$.

Let D_ρ be the disk with radius ρ and with center at the origin, where ρ denotes the hyperbolic distance. Denote the length of a line segment by $m(\cdot)$.

Lemma 3.

$$m(l(\alpha, \theta) \cap D_\rho) = 2 \log \left(\cosh \rho \sin \alpha + \sqrt{\cosh^2 \rho \sin^2 \alpha - 1} \right).$$

Proof of Lemma 3. Without loss of generality we suppose that $\theta = 0$. In \mathbf{D} the line $l(\alpha, \theta)$ can be represented by the equation $|z|^2 - (\bar{c}z + c\bar{z}) + 1 = 0$, where $c = 1/\cos \alpha$. Moreover, the circle $C_\rho = \partial D_\rho$ can be represented by an Euclidean circle with center at the origin and radius $r = \tanh \frac{\rho}{2}$. Then, letting two points where $l(\alpha, \theta)$ and C_ρ intersect be $re^{\pm i\omega}$, we have

$$(3) \quad \cos \omega = \frac{1+r^2}{2r} \cos \alpha = \coth \rho \cos \alpha.$$

Now, from the hyperbolic geometry, we borrow the knowledge that for two points z_1 and z_2 in \mathbf{D} , the hyperbolic distance between these points is given by

$$\log \frac{1 + \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}{1 + \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}.$$

Then, putting $z_1 = re^{i\omega}$ and $z_2 = re^{-i\omega}$, and substituting (3), we can complete the proof.

In this paper we concern ourself with the total length of the portions of lines $\{l(\alpha_i, \theta_i) : i \in \mathbf{T}\}$ inside the disk D_ρ , that is,

$$M(\rho) = \sum_{i \in \mathbf{T}} m(l(\alpha_i, \theta_i) \cap D_\rho).$$

We are interested in asymptotic behaviour of $E(M(\rho))$ as ρ tends to infinity, where $E(\cdot)$ denotes the expectation, and particularly in comparison with the area of D_ρ . Now it is known that the area of D_ρ is given by $2\pi(\cosh \rho - 1)$, which grows approximately as $\frac{1}{2}e^\rho$ as ρ tends to infinity. Thus it seems reasonable to investigate asymptotic behaviour of $\log E(M(\rho))$ instead of

$E(M(\rho))$. Define the functions $f(t)$ and $f_0(t)$ as

$$f(t) = \begin{cases} 2 \log(t + \sqrt{t^2 - 1}) & \text{for } t \geq 1, \\ 0 & \text{for } t < 1 \end{cases}$$

and

$$f_0(t) = \begin{cases} \log t & \text{for } t \geq 1, \\ 0 & \text{for } t < 1 \end{cases}.$$

Then, by the usual argument in the calculus, we can show that there is a constant K such that

$$2 f_0(t) \leq f(t) \leq K f_0(t).$$

Thus, if we put

$$M_0(\rho) = \sum_{\mathbf{i} \in \mathbf{T}} f_0(\cosh \rho \sin \alpha_{\mathbf{i}}),$$

we have

$$2 M_0(\rho) \leq M(\rho) \leq K M_0(\rho).$$

Accordingly, it is sufficient to study asymptotic behaviour of $\log E(M_0(\rho))$.

Now we give a following heuristic argument which will be rigorously proved later under appropriate assumptions :

1. From *Lemma 1* it follows that

$$\tan \alpha_{i_1 i_2 \dots i_k} \leq \tan \alpha_{i_1 i_2 \dots i_{k-1}} \cdot \frac{\sin \alpha_{i_k}}{\sin \theta_{i_k}}.$$

2. Accordingly, since $\sin \alpha_{i_k} / \sin \theta_{i_k} < 1$, we can expect $\alpha_{i_1 i_2 \dots i_k} \rightarrow 0$ as $k \rightarrow \infty$.

3. Thus, when $k \rightarrow \infty$,

$$\sin \alpha_{i_1 i_2 \dots i_k} \sim \tan \alpha_{i_1 i_2 \dots i_k} \sim \sin \alpha_{i_1 i_2 \dots i_{k-1}} \cdot \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \sin \theta_{i_k}},$$

where the notation " \sim " means "both sides are asymptotically equal".

Based on these observations, in the remainder of this section, we offer a rough estimate for $E(M_0(\rho))$.

Before we set about this task, we prepare some notations. Let \mathbf{T}_k be the set of nodes of \mathbf{T} with length k . Let \mathcal{F}_0 be the trivial σ -fields, and given \mathcal{F}_{k-1} , define

$$\mathcal{F}_k = \sigma(\mathcal{F}_{k-1} \cup \{N_{\mathbf{i}}, l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) : \mathbf{i} \in \mathbf{T}_{k-1}\}).$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Then we may expect

$$\begin{aligned} & \mathbb{E} \left(\sum_{i_1 i_2 \dots i_k} f_0 (\cosh \rho \sin \alpha_{i_1 i_2 \dots i_k}) \mid \mathcal{F}_{k-1} \right) \\ & \sim \mathbb{E} \left(\sum_{i_1 i_2 \dots i_k} f_0 \left(\cosh \rho \sin \alpha_{i_1 i_2 \dots i_{k-1}} \cdot \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \sin \theta_{i_k}} \right) \mid \mathcal{F}_{k-1} \right) \\ & = \sum_{i_1 i_2 \dots i_{k-1}} \mathbb{E} \left(\sum_{j=1}^N f_0 \left(\cosh \rho \sin \alpha_{i_1 i_2 \dots i_{k-1}} \cdot \frac{\sin A_j^{(N)}}{\cos A_j^{(N)} + \sin \Theta_j^{(N)}} \right) \right), \end{aligned}$$

where N is a random variable with probability distribution Q , and when $N = n$, $(A_1^{(n)}, \Theta_1^{(n)}, \dots, A_n^{(n)}, \Theta_n^{(n)})$ is a random vector with probability distribution Q_n .

Now we introduce a random vector $(\Lambda_1^{(n)}, \dots, \Lambda_n^{(n)})$ by setting

$$\Lambda_j^{(n)} = \frac{\sin A_j^{(n)}}{\cos A_j^{(n)} + \sin \Theta_j^{(n)}}$$

for $j = 1, \dots, n$, and denote its probability distribution by P_n . Moreover, we define an operator Λ by

$$(\Lambda f_0)(t) = \mathbb{E} \left(\sum_{j=1}^N f_0(t \Lambda_j^{(N)}) \right).$$

Then we obtain the following

$$\begin{aligned} & \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_k} f_0 (\cosh \rho \sin \alpha_{\mathbf{i}}) \mid \mathcal{F}_{k-1} \right) \\ (4) \quad & \sim \sum_{\mathbf{i} \in \mathbf{T}_{k-1}} (\Lambda f_0)(\cosh \rho \sin \alpha_{\mathbf{i}}). \end{aligned}$$

Applying (4) k times, we can get

$$(5) \quad \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_k} f_0 (\cosh \rho \sin \alpha_{\mathbf{i}}) \right) \sim (\Lambda^k f_0)(\cosh \rho).$$

Accordingly, by a heuristic argument, we have derived

$$\mathbb{E}(M_0(\rho)) \sim \sum_{k=0}^{\infty} (\Lambda^k f_0)(\cosh \rho).$$

In the next section we will investigate asymptotic behaviour of this infinite series.

3 Asymptotic behaviour of an approximated expectation of the mass distribution

Let $\{p_j : j = 1, 2, \dots, m\}$ be positive numbers, $\{\lambda_j : j = 1, 2, \dots, m\}$ be positive numbers such that $\lambda_j < 1$ ($j = 1, 2, \dots, m$), and define an operator Λ by

$$(6) \quad (\Lambda f)(t) = \sum_{j=1}^m p_j f(\lambda_j t),$$

where

$$(7) \quad f(t) = \begin{cases} \log t & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases}.$$

In this section we study asymptotic behaviour of an infinite series

$$(8) \quad F(t) = \sum_{k=0}^{\infty} (\Lambda^k f)(t).$$

as t tends to infinity. In turn, as will be seen later in this section, in order to study asymptotic behaviour of the infinite series (8), we have to know asymptotic behaviour of the following integral

$$(9) \quad I(t) = I(t; c) = (2\pi)^{-\frac{m-1}{2}} \int_0^{\infty} \cdots \int_0^{\infty} z^c \frac{z^{z+\frac{1}{2}}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} \\ \cdot f\left(t \prod_j \lambda_j^{x_j}\right) \prod_j dx_j,$$

where $z = \sum_{j=1}^m x_j$, c is a constant and the index j of every product in (9) runs over $\{1, 2, \dots, m\}$.

In the integral (9) we change variables as

$$\begin{cases} x_j = z u_j & (j = 1, 2, \dots, m-1) \\ x_m = z (1 - \sum' u_j) \end{cases},$$

where the sum \sum' is taken over $\{1, 2, \dots, m-1\}$. Then, since the Jacobian

$$\frac{\partial(x_1, x_2, \dots, x_m)}{\partial(z, u_1, \dots, u_{m-1})} = z^{m-1},$$

we have

$$I(t) = (2\pi)^{-\frac{m-1}{2}} \int_0^{\infty} dz \int_D \cdots \int \left(\prod_j u_j \right)^{-\frac{1}{2}} \left(\frac{\prod_j p_j^{u_j}}{\prod_j u_j^{u_j}} \right)^z \\ \cdot f\left(t \left(\prod_j \lambda_j^{u_j} \right)^z\right) z^{\frac{m-1}{2}+c} \prod_j' du_j,$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

where $u_m = 1 - \sum' u_j$ and

$$D = \{(u_1, u_2, \dots, u_{m-1}) : \sum' u_j \leq 1\}.$$

Now, using the vector notation $\mathbf{u} = (u_1, u_2, \dots, u_{m-1})$, we introduce the following functions

$$\begin{aligned} h(\mathbf{u}) &= -\sum_{j=1}^m u_j \log u_j, \\ a(\mathbf{u}) &= \sum_{j=1}^m a_j u_j, \quad a_j = \log \frac{1}{\lambda_j}, \end{aligned}$$

and

$$b(\mathbf{u}) = \sum_{j=1}^m b_j u_j, \quad b_j = \log \frac{1}{p_j}.$$

Then $I(t)$ can be expressed as

$$\begin{aligned} I(t) &= (2\pi)^{-\frac{m-1}{2}} \int_0^\infty dz \int \cdots \int_D \left(\prod_j u_j \right)^{-\frac{1}{2}} d'\mathbf{u} \\ &\quad \int_0^\infty e^{z(h(\mathbf{u})-b(\mathbf{u}))} f\left(te^{-za(\mathbf{u})}\right) z^{\frac{m-1}{2}+c} dz, \end{aligned}$$

where $d'\mathbf{u} = \prod_j' du_j$.

Moreover we introduce the functions

$$\mu(\mathbf{u}) = \frac{h(\mathbf{u}) - b(\mathbf{u})}{a(\mathbf{u})}$$

and

$$k(\mathbf{u}) = (2\pi)^{-\frac{m-1}{2}} \left(\prod_j u_j \right)^{-\frac{1}{2}} a(\mathbf{u})^{-\frac{m+1}{2}-c}.$$

Then, after the change of variable as $z = \frac{1}{a(\mathbf{u})} \log \frac{1}{y}$, we have

$$\begin{aligned} (10) \quad I(t) &= \int \cdots \int_D k(\mathbf{u}) d'\mathbf{u} \\ &\quad \cdot \int_0^1 f(ty) \left(\log \frac{1}{y} \right)^{\frac{m-1}{2}+c} \frac{dy}{y^{1+\mu(\mathbf{u})}}. \end{aligned}$$

At this point we prepare several lemmas.

Lemma 1. Let μ and δ be positive real constants, and let

$$g(t) = g(t; \mu, \delta) = \int_0^1 f(tx) \left(\log \frac{1}{x} \right)^\delta \frac{dx}{x^{1+\mu}}.$$

Then, as $t \rightarrow \infty$,

$$g(t) = t^\mu (\log t)^{2+\delta} \cdot \frac{1}{\mu^2 (\log t)^2} (1 - (\mu \log t + 1)e^{-\mu \log t}) + \epsilon(t; \mu, \delta),$$

where

$$|\epsilon(t; \mu, \delta)| \leq t^{\frac{\mu}{2}} (\log t)^{2+\delta}.$$

Proof of Lemma 1. Changing variable as $tx = y$, we have

$$\begin{aligned} g(t) &= t^\mu \int_0^t f(y) (\log t - \log y)^\delta \frac{dy}{y^{1+\mu}} \\ &= t^\mu \int_1^t \log y (\log t - \log y)^\delta \frac{dy}{y^{1+\mu}}. \end{aligned}$$

Again changing variable as $\log y = z \log t$, we have

$$g(t) = t^\mu (\log t)^{2+\delta} \int_0^1 z(1-z)^\delta e^{-\mu z \log t} dz.$$

Then, noting that

$$\int_{\frac{1}{2}}^1 z(1-z)^\delta e^{-\mu z \log t} dz \leq \int_{\frac{1}{2}}^1 z e^{-\mu z \log t} dz \leq \frac{1}{2} t^{-\frac{\mu}{2}},$$

we get

$$g(t) = t^\mu (\log t)^{2+\delta} \int_0^1 z e^{-\mu z \log t} dz + \epsilon(t; \mu, \delta).$$

Since

$$\int_0^1 x e^{-\nu x} dx = \frac{1}{\nu^2} (1 - (\nu + 1)e^{-\nu}),$$

where ν is any positive constant, the proof of lemma is completed.

Lemma 2. In the domain D , the function $\mu(\mathbf{u})$ has the unique maximum μ_{max} at a point \mathbf{u}_0 . This maximum μ_{max} is the unique root of the equation

$$\sum_{j=1}^m p_j \lambda_j^\mu = 1,$$

and the point \mathbf{u}_0 can be determined by

$$u_j = p_j \lambda_j^{\mu_{max}}.$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Proof of Lemma 2. Regard the function μ as a function of variables $\tilde{\mathbf{u}} = (u_1, u_2, \dots, u_m)$ with the constraint $\sum_{j=1}^m u_j = 1$, and consider the function

$$\tilde{\mu}(\tilde{\mathbf{u}}) = \mu(\tilde{\mathbf{u}}) - \gamma \cdot \left(\sum_{j=1}^m u_j - 1 \right),$$

where γ is a positive constant. Letting $\frac{\partial \tilde{\mu}}{\partial u_j} = 0$ for all $j = 1, 2, \dots, m$, we have

$$(11) \quad (1 + \log u_j + b_j)a(\tilde{\mathbf{u}}) + a_j(h(\tilde{\mathbf{u}}) - b(\tilde{\mathbf{u}})) + \gamma a(\tilde{\mathbf{u}})^2 = 0$$

Multiplying (11) by u_j and summing over $j = 1, 2, \dots, m$, we get

$$\gamma = -\frac{1}{a(\tilde{\mathbf{u}})}.$$

Putting this into (11), we can deduce that in the interior of the domain D there exists only one extrem point $\tilde{\mathbf{u}}$ which satisfies a system of equations

$$(12) \quad u_j = p_j \lambda_j^{\mu(\tilde{\mathbf{u}})}.$$

Since this extrem point lies on the hyperplane $\sum_{j=1}^m u_j = 1$, the extrem value μ has to satisfy the equation

$$(13) \quad \sum_{j=1}^m p_j \lambda_j^{\mu} = 1.$$

It remains to show that this extrem value is really the maximum. For this purpose, it is sufficient to prove that at this extrem point which satisfies (12), the matrix

$$\left(-\frac{\partial^2 \mu}{\partial u_i \partial u_j} \right)_{1 \leq i, j \leq m-1}$$

is positive definite.

Derivating the function $\mu(\mathbf{u})$ two times and substituting (12), we have

$$(14) \quad \frac{\partial^2 \mu}{\partial u_i \partial u_j} = -\frac{1}{a(\mathbf{u})} \left(\frac{1}{u_m} + \delta_{ij} \frac{1}{u_i} \right).$$

where δ_{ij} denotes the Kronecker delta. Then we can easily show that the matrix

$$\left(-\frac{\partial^2 \mu}{\partial u_i \partial u_j} \right)_{1 \leq i, j \leq m-1}$$

is positive definite. Thus the proof is completed.

Returning to the integral (10), we can rewrite it as

$$I(t) = \int \cdots \int_D k(\mathbf{u}) g(t; \mu(\mathbf{u}), \frac{m-1}{2} + c) d'\mathbf{u},$$

where $g(t; \cdot, \cdot)$ is the function introduced in Lemma 1.

Decompose the domain D into a domain

$$D_1 = \left\{ \mathbf{u} \in D : \mu(\mathbf{u}) \geq \frac{\mu_{max}}{2} \right\}$$

and its complement $D \setminus D_1$. Then we easily have the following estimates.

Lemma 3. For any sufficiently large t ,

$$I(t) \leq \frac{4}{\mu_{max}^2} (\log t)^{\frac{m-1}{2}} \int \cdots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \\ + O\left(t^{\frac{\mu_{max}}{2}} (\log t)^{\frac{m+3}{2}}\right)$$

and

$$I(t) \geq \frac{1}{2\mu_{max}^2} (\log t)^{\frac{m-1}{2}} \int \cdots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \\ - O\left(t^{\frac{\mu_{max}}{2}} (\log t)^{\frac{m+3}{2}}\right)$$

Proof of Lemma 3. Put $\delta = \frac{m-1}{2} + c$. Since

$$\frac{1 - (x+1)e^{-x}}{x^2} \leq \min\left\{\frac{1}{x^2}, \frac{1}{2}\right\} \text{ for } x > 0,$$

from Lemma 1 it follows that

$$I(t) \leq (\log t)^{2+\delta} \int \cdots \int_D k(\mathbf{u}) t^{\mu(\mathbf{u})} \cdot \frac{1}{\mu(\mathbf{u})^2 (\log t)^2} \left(1 - (\mu(\mathbf{u}) \log t + 1)e^{-\mu(\mathbf{u}) \log t}\right) \\ + (\log t)^{2+\delta} \int \cdots \int_D k(\mathbf{u}) t^{\frac{\mu(\mathbf{u})}{2}} d'\mathbf{u} \\ \leq (\log t)^{2+\delta} \int \cdots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} \cdot \frac{1}{\mu(\mathbf{u})^2 (\log t)^2} d'\mathbf{u} \\ + (\log t)^{2+\delta} \int \cdots \int_{D \setminus D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} \cdot \frac{1}{2} d'\mathbf{u} \\ + t^{\frac{\mu_{max}}{2}} (\log t)^{2+\delta} \int \cdots \int_D k(\mathbf{u}) d'\mathbf{u} \\ \leq \frac{4}{\mu_{max}^2} (\log t)^\delta \int \cdots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u}$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

$$+\frac{3}{2}t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta} \int \dots \int_D k(\mathbf{u}) d'\mathbf{u}.$$

On the other hand, since

$$\frac{1 - (x+1)e^{-x}}{x^2} \geq \frac{1}{2x^2} \text{ for any sufficiently large } x > 0,$$

we have

$$\begin{aligned} I(t) &\geq (\log t)^{2+\delta} \int \dots \int_D k(\mathbf{u}) t^{\mu(\mathbf{u})} \cdot \frac{1}{\mu(\mathbf{u})^2(\log t)^2} \left(1 - (\mu(\mathbf{u}) \log t + 1)e^{-\mu(\mathbf{u}) \log t}\right) \\ &\quad - (\log t)^{2+\delta} \int \dots \int_D k(\mathbf{u}) t^{\frac{\mu(\mathbf{u})}{2}} d'\mathbf{u} \\ &\geq (\log t)^{2+\delta} \int \dots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} \cdot \frac{1}{2\mu(\mathbf{u})^2(\log t)^2} d'\mathbf{u} \\ &\quad - t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta} \int \dots \int_D k(\mathbf{u}) d'\mathbf{u} \\ &\geq \frac{1}{2\mu_{max}^2}(\log t)^\delta \int \dots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \\ &\quad - t^{\frac{\mu_{max}}{2}}(\log t)^{2+\delta} \int \dots \int_D k(\mathbf{u}) d'\mathbf{u}. \end{aligned}$$

Thus the proof is completed.

Lemma 4. As $t \rightarrow \infty$,

$$\int \dots \int_{D_1} k(\mathbf{u}) t^{\mu(\mathbf{u})} d'\mathbf{u} \sim a(\mathbf{u}_0)^{-1-c} t^{\mu_{max}} (\log t)^{-\frac{m-1}{2}}$$

Proof of Lemma 4. Denote by $J(t)$ the integral with which we have to concern ourself. Let $\mathbf{u}_0 = (u_1^0, u_2^0, \dots, u_{m-1}^0)$ be the point at which the function μ attains its maximum. Obviously $\mathbf{u}_0 \in D_1$. Since the function μ is twice continuously differentiable, in a neighbourhood of \mathbf{u}_0 it can be expanded as

$$\mu(\mathbf{u}) = \mu_{max} - \frac{1}{2} \sum_{i,j=1}^{m-1} t_{ij}(u_i - u_i^0)(u_j - u_j^0) + \dots,$$

where

$$t_{ij} = - \left(\frac{\partial^2 \mu}{\partial u_i \partial u_j} \right)_{\mathbf{u}_0}.$$

By (14) we have

$$(15) \quad t_{ij} = \frac{1}{a(\mathbf{u})} \left(\frac{1}{u_m} + \delta_{ij} \frac{1}{u_i} \right).$$

Now we apply Laplace's method (Erdélyi(1956) p.36) to $J(t)$. Then we have

$$J(t) \sim k(\mathbf{u}_0) t^{\mu_{max}} \int \cdots \int_{\mathbf{R}^{m-1}} \exp \left(-\frac{\log t}{2} \sum_{i,j=1}^{m-1} t_{ij}(u_i - u_i^0)(u_j - u_j^0) \right) d'\mathbf{u}$$

Since the matrix $T = (t_{ij})_{1 \leq i,j \leq m-1}$ is positive definite, there exists the square root of T , which we denote by $S = (s_{ij})_{1 \leq i,j \leq m-1}$. Changing variables as $v_i = \sum_{j=1}^{m-1} s_{ij}(u_j - u_j^0)$, we get

$$(16) \quad J(t) \sim k(\mathbf{u}_0) t^{\mu_{max}} |S|^{-1} \left(\frac{2\pi}{\log t} \right)^{\frac{m-1}{2}}.$$

Now, using (15), we can easily show that

$$(17) \quad |T| = |S|^2 = \frac{1}{a^{m-1} \prod_{j=1}^m u_j}.$$

Using (16) and (17), we can complete the proof.

Combining Lemma 2, Lemma 3 and Lemma 4, we obtain the following result.

Lemma 5.

$$\lim_{t \rightarrow \infty} \frac{\log I(t)}{\log t} = \mu,$$

where μ is the unique root of the equation

$$(18) \quad \sum_{j=1}^m p_j \lambda_j^\mu = 1.$$

The above lemma yields the following result.

Lemma 6. Let n be a positive integer, c be a constant and set

$$I_n(t; c) = (2\pi)^{-\frac{m-1}{2}} \int_n^\infty \cdots \int_n^\infty z^c \frac{z^{z+\frac{1}{2}}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f \left(t \prod_j \lambda_j^{x_j} \right) \prod_j dx_j,$$

where $z = \sum_{j=1}^m x_j$.

Then

$$\lim_{t \rightarrow \infty} \frac{\log I_n(t; c)}{\log t} = \mu,$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

where μ is the same number as that in *Lemma 5*.

Proof of Lemma 6. Setting

$$I_n^{(i)}(t; c) = (2\pi)^{-\frac{m-1}{2}} \int \cdots \int_{\{x_i \leq n; x_j > n \text{ for all } j \neq i\}} z^c \frac{z^{z+\frac{1}{2}}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right) \prod_j dx_j,$$

for $i = 1, \dots, m$, we have

$$0 \leq I(t; c) - I_n(t; c) \leq \sum_{i=1}^m I_n^{(i)}(t; c).$$

Without loss of generality we argue about $I_n^{(m)}(t; c)$. Then it is easily seen that

$$I_n^{(m)}(t; c) \leq (2\pi)^{-\frac{m-1}{2}} \int_0^m \frac{p_m^{x_m}}{x_m^{x_m+\frac{1}{2}}} dx_m \cdot \int_n^\infty \cdots \int_n^\infty \frac{(z+n)^{z+n+\frac{1}{2}+c}}{\prod_{j=i}^{m-1} x_j^{x_j+\frac{1}{2}}} \prod_{j=1}^{m-1} p_j^{x_j} f\left(t \lambda_m^{x_m} \prod_{j=1}^{m-1} \lambda_j^{x_j}\right) \prod_{j=1}^{m-1} dx_j$$

Since there is a constant K such that $(z+n)^{z+n+\frac{1}{2}+c} < K z^{z+n+\frac{1}{2}+c}$, we have

$$I_n^{(m)}(t; c) \leq K' \int_n^\infty \cdots \int_n^\infty z^{n+c} \frac{z^{z+\frac{1}{2}}}{\prod_{j=i}^{m-1} x_j^{x_j+\frac{1}{2}}} \prod_{j=1}^{m-1} p_j^{x_j} f\left(t \prod_{j=1}^{m-1} \lambda_j^{x_j}\right) \prod_{j=1}^{m-1} dx_j,$$

where K' is a constant. Applying *Lemma 5* to the right hand side of the above inequality, we deduce that $I_n^{(m)}(t; c)$ is of the same order as $t^{\mu'}$, where μ' is the root of the equation $\sum_{j=1}^{m-1} p_j \lambda_j^{\mu'} = 1$. Now it is obvious that μ' is smaller than the root μ of the equation (18). Hence we get

$$I_n^{(m)}(t; c) = o(t^\mu).$$

This implies that

$$I(t; c) - I_n(t; c) = o(t^\mu).$$

Thus by *Lemma 5* we complete the proof.

Now we turn to the infinite series $F(t)$. Introduce the following series which plays a role as a bridge combining $F(t)$ and $I(t)$:

$$\tilde{F}_n(t; c) = \sum_{k=0}^{\infty} \sum_{\{\sum x_j = k, x_j > n \text{ for every } j\}} \cdots \sum (2\pi)^{-\frac{m-1}{2}} \frac{k^{k+\frac{1}{2}+c}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right),$$

where n is a positive integer and c is a constant.

Lemma 7.

$$\lim_{t \rightarrow \infty} \frac{\log \tilde{F}_n(t; c)}{\log t} = \mu,$$

where μ is the same number as that in *Lemma 5*.

Proof of Lemma 7. When $x_j < y_j \leq x_j + 1$, putting $z = \sum_{j=1}^m y_j$, we have

$$\begin{aligned} & \frac{(z-m)^{z-m+\frac{1}{2}+c}}{\prod_j y_j^{y_j+\frac{1}{2}}} \prod_j p_j^{y_j} f\left(t \prod_j \lambda_j^{y_j}\right) \\ & \leq \frac{k^{k+\frac{1}{2}+c}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right) \\ & \leq \frac{z^{z+\frac{1}{2}+c}}{\prod_j (y_j-1)^{y_j-\frac{1}{2}}} \prod_j p_j^{y_j-1} f\left(t \prod_j \lambda_j^{y_j-1}\right). \end{aligned}$$

Summing over $x_j > n$ for every x_j , we get

$$\begin{aligned} & \int \cdots \int_{(n, \infty)^m} (2\pi)^{-\frac{m-1}{2}} \frac{(z-m)^{z-m+\frac{1}{2}+c}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right) \prod_j dx_j \\ & \leq \tilde{F}_n(t; c) \\ & \leq \int \cdots \int_{(n, \infty)^m} (2\pi)^{-\frac{m-1}{2}} \frac{z^{z+\frac{1}{2}+c}}{\prod_j (x_j-1)^{x_j-\frac{1}{2}}} \prod_j p_j^{x_j-1} f\left(t \prod_j \lambda_j^{x_j-1}\right) \prod_j dx_j \\ & = \int \cdots \int_{(n-1, \infty)^m} (2\pi)^{-\frac{m-1}{2}} \frac{(z+m)^{z+m+\frac{1}{2}+c}}{\prod_j x_j^{x_j+\frac{1}{2}}} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right) \prod_j dx_j. \end{aligned}$$

It is easily seen that there are positive constants K_1 and K_2 such that

$$(z-m)^{z-m+\frac{1}{2}+c} \geq K_1 z^{-m+c} z^{z+\frac{1}{2}}$$

and

$$(z+m)^{z+m+\frac{1}{2}+c} \leq K_2 z^{m+c} z^{z+\frac{1}{2}}.$$

Accordingly we get

$$K_1 \cdot I_n(t; -m+c) \leq \tilde{F}_n(t) \leq K_2 \cdot I_{n-1}(t; m+c).$$

Hence, with the help of *Lemma 6*, we can complete the proof.

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Now the time is ripe to state an asymptotic behaviour of $F(t)$ explicitly.

Theorem 1.

$$\lim_{t \rightarrow \infty} \frac{\log F(t)}{\log t} = \mu,$$

where μ is the unique root of the equation

$$\sum_{j=1}^m p_j \lambda_j^\mu = 1.$$

In order to prove *Theorem 1*, it is sufficient to establish the following more general *Lemma 8*. Let c be a constant, and define

$$F(t; c) = \sum_{k=0}^{\infty} \sum_{\{\sum_{j=1}^m x_j = k\}} \sum_{x_j = k} k^c \binom{k}{x_1 \cdots x_m} \prod_j p_j^{x_j} f \left(t \prod_j \lambda_j^{x_j} \right),$$

where

$$\binom{k}{x_1 \cdots x_m} = \frac{k!}{x_1! \cdots x_m!}.$$

Lemma 8.

$$\lim_{t \rightarrow \infty} \frac{\log F(t; c)}{\log t} = \mu,$$

where μ is the same number as that in *Theorem 1*.

Proof of Lemma 8. We prove this lemma by induction on m . It is easy to show that the assertion holds when $m = 1$. Assume that the assertion holds for $m - 1$.

Using Stirling's formula, we have

$$\binom{k}{x_1 \cdots x_m} = \xi \cdot (2\pi)^{-\frac{m-1}{2}} \frac{k^{k+\frac{1}{2}}}{\prod_j x_j^{x_j+\frac{1}{2}}},$$

where

$$\exp \left(\frac{1}{12k+1} - \sum_j \frac{1}{12x_j} \right) < \xi < \exp \left(\frac{1}{12k} - \sum_j \frac{1}{12x_j+1} \right).$$

Since $x_j > n$ for every j , we have $\exp(-m/(12n)) < \xi < 1$. Thus, for arbitrarily small ϵ , we have $1 - \epsilon < \xi < 1$ for all sufficiently large n .

Now we put

$$\tilde{F}_n^{(i)}(t; c) = \sum_{k=0}^{\infty} \sum_{\{\sum_{j=1}^m x_j = k, x_i \leq n\}} \sum_{x_i} k^c \binom{k}{x_1 \cdots x_m} \prod_j p_j^{x_j} f\left(t \prod_j \lambda_j^{x_j}\right),$$

for every $i = 1, 2, \dots, m$. Then we have

$$F(t; c) \leq \tilde{F}_n(t; c) + \sum_{i=1}^m \tilde{F}_n^{(i)}(t; c)$$

and

$$F(t; c) \geq (1 - \epsilon) \tilde{F}_n(t; c).$$

Accordingly, we obtain

$$\begin{aligned} (19) \quad & \left| \log F(t; c) - \log \tilde{F}_n(t; c) \right| \\ & \leq \frac{1}{(1 - \epsilon) \tilde{F}_n(t; c)} \cdot |F(t; c) - \tilde{F}_n(t; c)| \\ & \leq \frac{\epsilon}{1 - \epsilon} + \frac{1}{(1 - \epsilon) \tilde{F}_n(t; c)} \cdot \sum_{i=1}^m \tilde{F}_n^{(i)}(t; c). \end{aligned}$$

Without loss of generality, we argue about $\tilde{F}_n^{(m)}(t; c)$. We can see easily that

$$\begin{aligned} \tilde{F}_n^{(m)}(t; c) & \leq \sum_{x_m=0}^n \frac{p_m^{x_m}}{x_m!} \sum_{k \geq x_m} \sum_{\{\sum_{j=1}^{m-1} x_j = k - x_m\}} \sum_{x_m} k^{x_m+c} \binom{k - x_m}{x_1 \cdots x_{m-1}} \\ & \quad \cdot \prod_{j=1}^{m-1} p_j^{x_j} f\left(t \lambda_m^{x_m} \prod_{j=1}^{m-1} \lambda_j^{x_j}\right), \end{aligned}$$

the right hand side of which we will write as

$$\sum_{x_m=0}^n \frac{p_m^{x_m}}{x_m!} G(t \lambda_m^{x_m}; x_m).$$

Then, because of the assumption of induction, for each x_m ,

$$\lim_{t \rightarrow \infty} \frac{\log G(t; x_m)}{\log t} = \mu'$$

where μ' is the root of the equation

$$(20) \quad \sum_{j=1}^{m-1} p_j \lambda_j^{\mu'} = 1.$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Now it is obvious that the root of the equation (18) is larger than the root of the equation (20). Thus, for each x_m , we can see

$$(21) \quad \tilde{F}_n^{(m)}(t; c) = 0 \left(t^{\mu'} \right)$$

as $t \rightarrow \infty$. Therefore, combining (19) and (21) and using *Lemma 7*, we obtain the conclusion.

Proof of Theorem 1. From the definition (6) we have

$$\begin{aligned} (\Lambda^k f)(t) &= \sum_{j_1=1}^m \cdots \sum_{j_k=1}^m p_{j_1} \cdots p_{j_k} f(t\lambda_{j_1} \cdots \lambda_{j_k}) \\ &= \sum_{\sum_j x_j=k} \cdots \sum \binom{k}{x_1 \cdots x_m} \prod_j p_j^{x_j} f \left(t \prod_j \lambda_j^{x_j} \right). \end{aligned}$$

Hence $F(t)$ coincides $F(t; 0)$. Thus *Theorem 1* is a special case of *Lemma 8*.

4 Main theorem

Let N be a random variable with probability distribution $Q = \{q_n : n = 0, 1, 2, \dots\}$, and for each n , let $(\Lambda_1^{(n)}, \dots, \Lambda_n^{(n)})$ be a random vector with probability distribution P_n whose support contained in $(0, 1)^n$. Define a function f as

$$f(t) = \begin{cases} \log t & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases},$$

and define an operator Λ as

$$(\Lambda f)(t) = \mathbf{E} \left(\sum_{j=1}^N f(t\Lambda_j^{(N)}) \right).$$

We set

$$F(t) = \sum_{k=0}^{\infty} (\Lambda^k f)(t).$$

If all Q and P_n ($n = 1, 2, \dots$) are finite discrete distribution, then from *Theorem 1* of Section 3 it follows that

$$\lim_{t \rightarrow \infty} \frac{\log F(t)}{\log t} = \mu,$$

where μ is the unique root of the equation defined by Q and P_n ($n = 1, 2, \dots$). In this section we first generalize *Theorem 1* without any assumption on P_n ($n = 1, 2, \dots$), while we maintain the assumption on Q .

Theorem 2. Assume that

(A3) Q is a finite distribution, that is, there is an integer n_{max} such that $q_n = 0$ for all $n > n_{max}$.

Then

$$\lim_{t \rightarrow \infty} \frac{\log F(t)}{\log t} = \mu,$$

where μ is the unique root of the equation

$$E \left(\sum_{j=1}^N \left(\Lambda_j^{(N)} \right)^\mu \right) = 1.$$

Proof of Theorem 2. Take an arbitrarily positive integer r , and put $\epsilon_r = 1/2^r$. We divide the interval $(0, 1)^n$ into a collection of subintervals

$$I_{i_1 \dots i_n} = \prod_{j=1}^n (i_j \epsilon_r, (i_j + 1) \epsilon_r],$$

where $i_j = 0, 1, \dots, 2^r$ for every j . We put

$$p_{i_1 \dots i_n} = \int \cdots \int_{I_{i_1 \dots i_n}} dP_n(\lambda_1, \dots, \lambda_n).$$

Since f is non-decreasing,

$$\begin{aligned} & p_{i_1 \dots i_n} \sum_{j=1}^n f(t i_j \epsilon) \\ & \leq \int \cdots \int_{I_{i_1 \dots i_n}} \sum_{j=1}^n f(t \lambda_j) dP_n(\lambda_1, \dots, \lambda_n) \\ & \leq p_{i_1 \dots i_n} \sum_{j=1}^n f(t (i_j + 1) \epsilon). \end{aligned}$$

Summing up with i_1, \dots, i_n , we have

$$\begin{aligned} & \sum_{i_1, \dots, i_n} \cdots \sum_{i_1, \dots, i_n} p_{i_1 \dots i_n} \sum_{j=1}^n f(t i_j \epsilon) \\ & \leq \int \cdots \int_{(0,1)^n} \sum_{j=1}^n f(t \lambda_j) dP_n(\lambda_1, \dots, \lambda_n) \end{aligned}$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

$$\begin{aligned}
 &= \mathbb{E} \left(\sum_{j=1}^n f(t\Lambda_j^{(n)}) \right) \\
 &\leq \sum_{i_1, \dots, i_n} \cdots \sum p_{i_1 \dots i_n} \sum_{j=1}^n f(t(i_j + 1)\epsilon) .
 \end{aligned}$$

Furthermore, multiplying q_n and summing up with n , we get

$$\begin{aligned}
 (22) \quad &\sum_{n=0}^{n_{max}} q_n \sum_{i_1, \dots, i_n} \cdots \sum p_{i_1 \dots i_n} \sum_{j=1}^n f(t i_j \epsilon) \\
 &= (\Lambda f)(t)
 \end{aligned}$$

$$(23) \quad \leq \sum_{n=0}^{n_{max}} q_n \sum_{i_1, \dots, i_n} \cdots \sum p_{i_1 \dots i_n} \sum_{j=1}^n f(t(i_j + 1)\epsilon) .$$

Now we enumerate the set of pairs of numbers $(q_n p_{i_1 \dots i_n}, i_j \epsilon)$ and denote them by $\{(p_j, \lambda_j) : j = 1, \dots, m\}$. Then, defining

$$(\underline{\Lambda}_r f)(t) = \sum_{j=1}^m p_j f(t \lambda_j) ,$$

we can write (22) simply as $(\underline{\Lambda}_r f)(t)$. Similarly, defining

$$(\overline{\Lambda}_r f)(t) = \sum_{j=1}^m p_j f(t \bar{\lambda}_j) ,$$

where $\{(p_j, \bar{\lambda}_j) : j = 1, \dots, m\}$ is made by enumerating the set of pairs of numbers $(q_n p_{i_1 \dots i_n}, (i_j + 1)\epsilon)$, we can write (23) as $(\overline{\Lambda}_r f)(t)$ concisely.

Consider the following infinite series

$$(24) \quad \underline{F}_r(t) = \sum_{k=0}^{\infty} \left(\underline{\Lambda}_r^k f \right) (t)$$

and

$$(25) \quad \overline{F}_r(t) = \sum_{k=0}^{\infty} \left(\overline{\Lambda}_r^k f \right) (t) .$$

Note that

$$(26) \quad \underline{F}_r(t) \leq F(t) \leq \overline{F}_r(t) .$$

To series (24) and (25) applying *Theorem 1*, we see that

$$(27) \quad \lim_{t \rightarrow \infty} \frac{\log \underline{F}_r(t)}{\log t} = \underline{\mu}_r$$

and

$$(28) \quad \lim_{t \rightarrow \infty} \frac{\log \bar{F}_r(t)}{\log t} = \bar{\mu}_r,$$

where $\underline{\mu}_r$ is the unique root of the equation

$$(29) \quad \sum_{j=1}^m p_j \lambda_j^{\underline{\mu}_r} = 1$$

and $\bar{\mu}_r$ is the unique root of the equation

$$(30) \quad \sum_{j=1}^m p_j \lambda_j^{\bar{\mu}_r} = 1.$$

Setting

$$\underline{\chi}_r(\lambda) = i\epsilon_r \quad \text{and} \quad \bar{\chi}_r(\lambda) = (i+1)\epsilon_r$$

for $i\epsilon_r < \lambda \leq (i+1)\epsilon_r$, we define functions \underline{g}_r and \bar{g}_r by

$$\underline{g}_r(\mu) = \sum_{n=0}^{n_{max}} q_n \int \cdots \int_{(0,1)^n} \sum_{j=1}^n \underline{\chi}_r(\lambda_j)^\mu dP_n(\lambda_1, \dots, \lambda_n)$$

and

$$\bar{g}_r(\mu) = \sum_{n=0}^{n_{max}} q_n \int \cdots \int_{(0,1)^n} \sum_{j=1}^n \bar{\chi}_r(\lambda_j)^\mu dP_n(\lambda_1, \dots, \lambda_n)$$

respectively. We can write (29) and (30) concisely by $\underline{g}_r(\mu)$ and $\bar{g}_r(\mu)$ respectively. Furthermore, we define a function g by

$$\begin{aligned} g(\mu) &= \sum_{n=0}^{n_{max}} q_n \int \cdots \int_{(0,1)^n} \sum_{j=1}^n \lambda_j^\mu dP_n(\lambda_1, \dots, \lambda_n) \\ &= E \left(\sum_{j=1}^N \left(\Lambda_j^{(N)} \right)^\mu \right). \end{aligned}$$

Then it is easily seen that

1. For each r , \underline{g}_r and \bar{g}_r are non-increasing continuous functions.
2. Since both $\underline{\chi}_r(\lambda)$ and $\bar{\chi}_r(\lambda)$ converge to λ as $r \rightarrow \infty$, the bounded convergence theorem implies that both $\underline{g}_r(\mu)$ and $\bar{g}_r(\mu)$ converge to $g(\mu)$ for each μ .

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

3. $\{\underline{g}_r : r = 1, 2, \dots\}$ is a non-decreasing sequence of functions, and $\{\bar{g}_r : r = 1, 2, \dots\}$ is a non-increasing sequence of functions. That is, for every r ,

$$\underline{g}_r(\mu) \leq \underline{g}_{r+1}(\mu) \quad \text{and} \quad \bar{g}_r(\mu) \geq \bar{g}_{r+1}(\mu)$$

for every μ .

Accordingly, Dini's theorem implies that in any compact interval of μ , \underline{g}_r and \bar{g}_r converge to g uniformly.

Let μ be the root of the equation $g(\cdot) = 1$. Then from uniform convergence just proved and the fact that \underline{g}_r and \bar{g}_r are non-increasing, it follows immediately that

$$\underline{\mu}_r \rightarrow \mu \quad \text{and} \quad \bar{\mu}_r \rightarrow \mu$$

as $r \rightarrow \infty$. Thus, by letting r large, we can make the difference of two limits in (27) and (28) arbitrarily small. Therefore by (26) the proof of *Theorem 1* is completed.

For each n , let $(A_1^{(n)}, \Theta_1^{(n)}, \dots, A_n^{(n)}, \Theta_n^{(n)})$ be a random vector with probability distribution P_n . Concerning the distribution P_n we temporarily use the assumption.

(\tilde{A}) there is a constant $\delta_{min} (> 1)$ such that

$$\min_{0 \leq n \leq n_{max}} \min_{1 \leq j \leq n} \frac{\sin \Theta_j^{(n)}}{\sin A_j^{(n)}} \geq \delta_{min}.$$

By *Lemma 2* in Section 2, this assumption means that every line $l(A_j^{(n)}, \Theta_j^{(n)})$ is at least $\cosh^{-1}(\delta_{min}) (> 0)$ distant from the line l_\emptyset .

We put

$$\Lambda_j^{(n)} = \frac{\sin A_j^{(n)}}{\cos A_j^{(n)} + \sin \Theta_j^{(n)}}$$

for $j = 1, \dots, n$, and define an operator Λ by

$$(\Lambda f)(t) = E \left(\sum_{j=1}^N f(t \Lambda_j^{(N)}) \right) = \sum_{n=0}^{n_{max}} q_n E \left(\sum_{j=1}^n f(t \Lambda_j^{(n)}) \right).$$

By the assumption (\tilde{A}), there is a positive number ϵ_0 such that the support of P_n is contained in the interval $(0, 1 - \epsilon_0)^n$. Let ϵ be an arbitrary positive number smaller than ϵ_0 , and put

$$(\underline{\Lambda}_{\epsilon,1}^{(n)}, \dots, \underline{\Lambda}_{\epsilon,n}^{(n)}) = ((1 - \epsilon)\Lambda_1^{(n)}, \dots, (1 - \epsilon)\Lambda_n^{(n)})$$

and

$$(\overline{\Lambda}_{\epsilon,1}^{(n)}, \dots, \overline{\Lambda}_{\epsilon,n}^{(n)}) = \left(\frac{1}{1 - \epsilon} \Lambda_1^{(n)}, \dots, \frac{1}{1 - \epsilon} \Lambda_n^{(n)} \right).$$

Denote probability distributions of these random vectors by \underline{P}_ϵ and \overline{P}_ϵ respectively. Because of the assumption (\tilde{A}) the supports of \underline{P}_ϵ and \overline{P}_ϵ are contained in the interval $(0, 1)^n$. Finally we define operators $\underline{\Lambda}_\epsilon$ and $\overline{\Lambda}_\epsilon$ by

$$(\underline{\Lambda}_\epsilon f)(t) = E \left(\sum_{j=1}^N f(t \underline{\Lambda}_{\epsilon,j}^{(N)}) \right)$$

and

$$(\overline{\Lambda}_\epsilon f)(t) = E \left(\sum_{j=1}^N f(t \overline{\Lambda}_{\epsilon,j}^{(N)}) \right).$$

Lemma 1. Let ϵ be an arbitrary positive number smaller than ϵ_0 . Under the assumption (\tilde{A}), there exists an integer k_0 such that

$$(31) \quad \sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \left(\underline{\Lambda}_\epsilon^{k-k_0} f \right) (\cosh \rho \sin \alpha_{\mathbf{i}})$$

$$(32) \quad \leq E \left(\sum_{\mathbf{i} \in \mathbf{T}_k} f(\cosh \rho \sin \alpha_{\mathbf{i}}) \mid \mathcal{F}_{k_0} \right)$$

$$(33) \quad \leq \sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \left(\overline{\Lambda}_\epsilon^{k-k_0} f \right) (\cosh \rho \sin \alpha_{\mathbf{i}})$$

for all $k > k_0$.

Proof of Lemma 1. By Lemma 1 in Section 2 and the assumption (\tilde{A}), we have

$$\tan \alpha_{i_1 \dots i_k} \leq \tan \alpha_{i_1 \dots i_{k-1}} \cdot \frac{\sin \alpha_{i_k}}{\sin \theta_{i_k}} \leq \frac{1}{\delta_{\min}} \cdot \tan \alpha_{i_1 \dots i_{k-1}}.$$

Applying this inequality $(k - 1)$ times, we get

$$\tan \alpha_{i_1 \dots i_k} \leq \left(\frac{1}{\delta_{\min}} \right)^{k-1} \cdot \tan \alpha_{i_1} \leq \frac{1}{\sqrt{\delta_{\min}^2 - 1}} \cdot \left(\frac{1}{\delta_{\min}} \right)^{k-1},$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA
because $\tan \alpha_{i_k} \leq \frac{1}{\sqrt{\delta_{\min}^2 - 1}}$ by (\tilde{A}). Hence, for arbitrarily small ϵ , there is an integer k_0 such that $\cos \alpha_{i_1 \dots i_k} > 1 - \epsilon$ for all $k \geq k_0$. Then we have

$$\sin \alpha_{i_1 \dots i_k} \leq \tan \alpha_{i_1 \dots i_k} \leq \sin \alpha_{i_1 \dots i_{k-1}} \cdot \frac{\lambda_{i_k}}{1 - \epsilon}$$

and

$$\sin \alpha_{i_1 \dots i_k} = \tan \alpha_{i_1 \dots i_k} \cos \alpha_{i_1 \dots i_k} \geq \sin \alpha_{i_1 \dots i_{k-1}} \cdot (1 - \epsilon) \lambda_{i_k} ,$$

where

$$\lambda_{i_k} = \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \sin \theta_{i_k}} .$$

Accordingly, since f is non-decreasing, we see that

$$(34) \quad \begin{aligned} f(\cosh \rho \sin \alpha_{i_1 \dots i_{k-1}} \cdot (1 - \epsilon) \lambda_{i_k}) \\ \leq f(\cosh \rho \sin \alpha_{i_1 \dots i_k}) \end{aligned}$$

$$(35) \quad \leq f\left(\cosh \rho \sin \alpha_{i_1 \dots i_{k-1}} \cdot \frac{\lambda_{i_k}}{1 - \epsilon}\right) .$$

From (35), it follows that

$$\begin{aligned} & \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_k} f(\cosh \rho \sin \alpha_{\mathbf{i}}) \mid \mathcal{F}_{k-1} \right) \\ &= \sum_{i_1 \dots i_{k-1} \in \mathbf{T}_{k-1}} \mathbb{E} \left(\sum_{j=1}^{N_{i_1 \dots i_{k-1}}} f(\cosh \rho \sin \alpha_{i_1 \dots i_k}) \mid \mathcal{F}_{k-1} \right) \\ &\leq \sum_{\mathbf{i} \in \mathbf{T}_{k-1}} \mathbb{E} \left(\sum_{j=1}^N f(\cosh \rho \sin \alpha_{\mathbf{i}} \bar{\Lambda}_{\epsilon, j}^{(N)}) \mid \mathcal{F}_{k-1} \right) \\ &= \sum_{\mathbf{i} \in \mathbf{T}_{k-1}} (\bar{\Lambda}_{\epsilon} f)(\cosh \rho \sin \alpha_{\mathbf{i}}) . \end{aligned}$$

Repeating this procedure $(k - k_0 + 1)$ times, we get an upper estimate (33) for (32). Similarly, from (34), we can derive a lower estimate (31). Thus the proof of *Lemma 1* is completed.

Recall that $M(\rho)$ denotes the total length of portions of lines $\{l(\alpha_{\mathbf{i}}, \theta_{\mathbf{i}}) : \mathbf{i} \in \mathbf{T}\}$ inside the disk D_{ρ} . Let $A(\rho)$ be the area of D_{ρ} .

Lemma 2. Under the assumption (\tilde{A}),

$$\lim_{\rho \rightarrow \infty} \frac{\log \mathbb{E}(M(\rho))}{A(\rho)} = \mu ,$$

where μ is the unique root of the equation

$$E \left(\sum_{j=1}^N \left(\Lambda_j^{(N)} \right)^\mu \right) = 1 .$$

Proof of Lemma 2. Put

$$F_\epsilon(t) = \sum_{k \geq 0} \left(\Lambda_\epsilon^k f \right) (t)$$

and

$$\bar{F}_\epsilon(t) = \sum_{k \geq 0} \left(\bar{\Lambda}_\epsilon^k f \right) (t) .$$

Using *Theorem 2*, we have

$$(36) \quad \lim_{t \rightarrow \infty} \frac{\log F_\epsilon(t)}{\log t} = \underline{\mu}_\epsilon$$

and

$$(37) \quad \lim_{t \rightarrow \infty} \frac{\log \bar{F}_\epsilon(t)}{\log t} = \bar{\mu}_\epsilon ,$$

where $\underline{\mu}_\epsilon$ is the unique root of the equation

$$\underline{g}_\epsilon(\mu) = E \left(\sum_{j=1}^N \left(\Lambda_{\epsilon,j}^{(N)} \right)^\mu \right) = 1$$

and $\bar{\mu}_\epsilon$ is the unique root of the equation

$$\bar{g}_\epsilon(\mu) = E \left(\sum_{j=1}^N \left(\bar{\Lambda}_{\epsilon,j}^{(N)} \right)^\mu \right) = 1 .$$

Now we will show that

$$(38) \quad \lim_{t \rightarrow \infty} \frac{\log E \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} F_\epsilon(t \sin \alpha_{\mathbf{i}}) \right)}{\log t} = \underline{\mu}_\epsilon$$

and

$$(39) \quad \lim_{t \rightarrow \infty} \frac{\log E \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \bar{F}_\epsilon(t \sin \alpha_{\mathbf{i}}) \right)}{\log t} = \bar{\mu}_\epsilon .$$

Because of (36), for any small positive number ξ , there is a sufficiently large t_0 such that

$$C_1 t^{\underline{\mu}_\epsilon - \xi} \leq F_\epsilon(t) \leq C_2 t^{\underline{\mu}_\epsilon + \xi}$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA
 for all $t > t_0$, where C_1 and C_2 are constants. Take a positive number η
 sufficiently small so as the probability of the event $\left\{ \max_{\mathbf{i} \in \mathbf{T}_{k_0}} \sin \alpha_{\mathbf{i}} > \eta \right\}$ be
 positive. Then we have

$$C_1 (t \sin \alpha_{\mathbf{i}})^{\underline{\mu}_\epsilon - \xi} I(\sin \alpha_{\mathbf{i}} > \eta) \leq \underline{F}_\epsilon(t \sin \alpha_{\mathbf{i}}) < C_2 (t \sin \alpha_{\mathbf{i}})^{\underline{\mu}_\epsilon + \xi}$$

for all $t > t_0/\eta$ and for every $\mathbf{i} \in \mathbf{T}_{k_0}$, where $I(\cdot)$ denotes the indicator
 function of events. Summing up with \mathbf{i} and taking expectations, we get

$$\begin{aligned} & C_1 \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} (\sin \alpha_{\mathbf{i}})^{\underline{\mu}_\epsilon - \xi} I(\sin \alpha_{\mathbf{i}} > \eta) \right) t^{\underline{\mu}_\epsilon - \xi} \\ & \leq \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \underline{F}_\epsilon(t \sin \alpha_{\mathbf{i}}) \right) \\ & \leq C_2 \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} (\sin \alpha_{\mathbf{i}})^{\underline{\mu}_\epsilon + \xi} \right) t^{\underline{\mu}_\epsilon + \xi}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} & \underline{\mu}_\epsilon - \xi \\ & \leq \liminf_{t \rightarrow \infty} \frac{\log \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \underline{F}_\epsilon(t \sin \alpha_{\mathbf{i}}) \right)}{\log t} \\ & \leq \limsup_{t \rightarrow \infty} \frac{\log \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \underline{F}_\epsilon(t \sin \alpha_{\mathbf{i}}) \right)}{\log t} \\ & \leq \underline{\mu}_\epsilon + \xi. \end{aligned}$$

Since ξ can be made arbitrarily small, we obtain (38). Similarly we can show
 (39).

Put

$$M_{k_0}(\rho) = \sum_{k \geq k_0} \sum_{\mathbf{i} \in \mathbf{T}_k} f(\cosh \rho \sin \alpha_{\mathbf{i}}).$$

After summing up (31), (32), and (33) over $k \geq k_0$, we take their expecta-
 tions. Then we have

$$\begin{aligned} & \mathbb{E} \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \underline{F}_\epsilon(\cosh \rho \sin \alpha_{\mathbf{i}}) \right) \\ & \leq \mathbb{E}(M_{k_0}(\rho)) \end{aligned}$$

$$\leq E \left(\sum_{\mathbf{i} \in \mathbf{T}_{k_0}} \bar{F}_\epsilon(\cosh \rho \sin \alpha_{\mathbf{i}}) \right),$$

Using (36) and (37), we get

$$(40) \quad \underline{\mu}_\epsilon \leq \liminf_{\rho \rightarrow \infty} \frac{\log E(M_{k_0}(\rho))}{\cosh \rho} \leq \limsup_{\rho \rightarrow \infty} \frac{\log E(M_{k_0}(\rho))}{\cosh \rho} \leq \bar{\mu}_\epsilon.$$

Finally we put

$$g(\mu) = E \left(\sum_{j=1}^N (\Lambda_j^{(N)})^\mu \right).$$

It is easily seen that

1. \underline{g}_ϵ and \bar{g}_ϵ are continuous non-increasing functions.
2. As ϵ decreases, $\underline{g}_\epsilon(\mu)$ decreases (to state exactly, do not increase) and $\bar{g}_\epsilon(\mu)$ increases (do not decrease) for every μ .
3. By the bounded convergence theorem, as ϵ tends to 0, both $\underline{g}_\epsilon(\mu)$ and $\bar{g}_\epsilon(\mu)$ converge to $g(\mu)$ for each μ .

Accordingly, Dini's theorem implies that in any compact set of μ , both \underline{g}_ϵ and \bar{g}_ϵ converge to g uniformly. Hence it follows that both $\underline{\mu}_\epsilon$ and $\bar{\mu}_\epsilon$ converge to a common limit μ which is the root of the equation $g(\cdot) = 1$. Therefore, from (40), we deduce

$$\lim_{\rho \rightarrow \infty} \frac{\log E(M_{k_0}(\rho))}{\cosh \rho} = \mu.$$

Since the number of terms of $M(\rho) - M_{k_0}(\rho)$ does not depend on ρ , we can complete the proof of *Lemma 2*.

We arrive at an appropriate place to state our main theorem. Throwing out the assumption (\tilde{A}), we introduce the the assumption that

(A4) there is a positive constant $\omega_0 < \frac{\pi}{2}$ such that $\max_{1 \leq j \leq n} A_j^{(n)} < \omega_0$ for every n .

Let ϵ be a positive number, and put

$$\chi_\epsilon(t) = \begin{cases} \frac{(1+\epsilon)^2}{t} & \text{for } 1 \leq t \leq (1+\epsilon)^2 \\ 1 & \text{for } t > (1+\epsilon)^2 \end{cases}.$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

For each n , we define random vectors

$$\left(\underline{A}_{\epsilon,1}^{(n)}, \underline{\Theta}_{\epsilon,1}^{(n)} \cdots, \underline{A}_{\epsilon,n}^{(n)}, \underline{\Theta}_{\epsilon,n}^{(n)} \right) \quad \text{and} \quad \left(\overline{A}_{\epsilon,1}^{(n)}, \overline{\Theta}_{\epsilon,1}^{(n)} \cdots, \overline{A}_{\epsilon,n}^{(n)}, \overline{\Theta}_{\epsilon,n}^{(n)} \right)$$

by

$$\begin{cases} \sin \underline{A}_{\epsilon,j}^{(n)} = \frac{1}{1+\epsilon} \sin A_j^{(n)} \\ \sin \underline{\Theta}_{\epsilon,j}^{(n)} = \sin \Theta_j^{(n)} \end{cases}$$

and

$$\begin{cases} \sin \overline{A}_{\epsilon,j}^{(n)} = (1+\epsilon) \sin A_j^{(n)} \\ \sin \overline{\Theta}_{\epsilon,j}^{(n)} = \chi_\epsilon \left(\frac{\sin \Theta_j^{(n)}}{\sin A_j^{(n)}} \right) \sin \Theta_j^{(n)} \end{cases}$$

for every $j = 1, \dots, n$. Then a random vector $\left(\underline{A}_{\epsilon,1}^{(n)}, \underline{\Theta}_{\epsilon,1}^{(n)} \cdots, \underline{A}_{\epsilon,n}^{(n)}, \underline{\Theta}_{\epsilon,n}^{(n)} \right)$ satisfies the assumption (\tilde{A}) , and moreover, if we choose ϵ so that $(1+\epsilon)^2 \sin \omega_0 < 1$, then a random vector $\left(\overline{A}_{\epsilon,1}^{(n)}, \overline{\Theta}_{\epsilon,1}^{(n)} \cdots, \overline{A}_{\epsilon,n}^{(n)}, \overline{\Theta}_{\epsilon,n}^{(n)} \right)$ also satisfies the assumption (\tilde{A}) . Denote a realization of $A_j^{(n)}, \Theta_j^{(n)}, \underline{A}_j^{(n)}, \underline{\Theta}_j^{(n)}, \overline{A}_j^{(n)}$ and $\overline{\Theta}_j^{(n)}$ by $\alpha_j, \theta_j, \underline{\alpha}_j, \underline{\theta}_j, \overline{\alpha}_j$ and $\overline{\theta}_j$ respectively. When α_i and θ_i for $i \in \mathbf{T}_{k-1}$ are given, we define α_i and θ_i for $i \in \mathbf{T}_k$ by the recursive formula stated in Lemma 1 of Section 2.

Lemma 3. For every $i \in \mathbf{T}$,

$$\underline{\alpha}_i \leq \alpha_i \leq \overline{\alpha}_i.$$

Proof of Lemma 3. We prove this lemma by induction on k . Obviously the lemma is true for $k = 1$. Assume that the assertion holds for $k - 1$. Denote $\cos \alpha_{i_1 \dots i_{k-1}}, \cos \underline{\alpha}_{i_1 \dots i_{k-1}}$ and $\cos \overline{\alpha}_{i_1 \dots i_{k-1}}$ by $\xi, \underline{\xi}$ and $\overline{\xi}$ respectively. By the assumption of induction, we have $\underline{\xi} \geq \xi \geq \overline{\xi}$.

We first argue about $\tan \underline{\alpha}_{i_1 \dots i_k}$. We have

$$\begin{aligned} & \frac{\sin \underline{\alpha}_{i_k}}{\cos \underline{\alpha}_{i_k} + \underline{\xi} \sin \underline{\theta}_{i_k}} \\ &= \frac{\frac{1}{1+\epsilon} \sin \alpha_{i_k}}{\sqrt{1 - \left(\frac{1}{1+\epsilon} \right)^2 \sin^2 \alpha_{i_k} + \underline{\xi} \sin \theta_{i_k}}} \\ &\leq \frac{\sin \alpha_{i_k}}{\sqrt{1 - \sin^2 \alpha_{i_k} + \underline{\xi} \sin \theta_{i_k}}} \\ &\leq \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \xi \sin \theta_{i_k}}. \end{aligned}$$

Since $\sin \underline{\alpha}_{i_1 \dots i_{k-1}} \leq \sin \alpha_{i_1 \dots i_{k-1}}$ by the assumption of induction, using Lemma 1 in Section 2, we get

$$(41) \quad \tan \underline{\alpha}_{i_1 \dots i_k} \leq \tan \alpha_{i_1 \dots i_k} .$$

Next we argue about $\tan \bar{\alpha}_{i_1 \dots i_k}$. When $\frac{\sin \theta_{i_k}}{\sin \alpha_{i_k}} > (1 + \epsilon)^2$, we have

$$\begin{aligned} & \frac{\sin \bar{\alpha}_{i_k}}{\cos \bar{\alpha}_{i_k} + \bar{\xi} \sin \bar{\theta}_{i_k}} \\ &= \frac{(1 + \epsilon) \sin \alpha_{i_k}}{\sqrt{1 - (1 + \epsilon)^2 \sin^2 \alpha_{i_k} + \bar{\xi} \sin \theta_{i_k}}} \\ &\geq \frac{\sin \alpha_{i_k}}{\sqrt{1 - \sin^2 \alpha_{i_k} + \bar{\xi} \sin \theta_{i_k}}} \\ &\geq \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \xi \sin \theta_{i_k}} . \end{aligned}$$

Hence follows that

$$(42) \quad \tan \alpha_{i_1 \dots i_k} \leq \tan \bar{\alpha}_{i_1 \dots i_k} .$$

On the other hand, when $\frac{\sin \theta_{i_k}}{\sin \alpha_{i_k}} \leq (1 + \epsilon)^2$, we have

$$\begin{aligned} & \frac{\sin \bar{\alpha}_{i_k}}{\cos \bar{\alpha}_{i_k} + \bar{\xi} \sin \bar{\theta}_{i_k}} \\ &= \frac{(1 + \epsilon) \sin \alpha_{i_k}}{\sqrt{1 - (1 + \epsilon)^2 \sin^2 \alpha_{i_k} + \bar{\xi} \cdot (1 + \epsilon)^2 \sin \alpha_{i_k}}} \\ &= \frac{\sin \alpha_{i_k}}{\sqrt{\left(\frac{1}{1 + \epsilon}\right)^2 - \sin^2 \alpha_{i_k} + \bar{\xi} \cdot (1 + \epsilon) \sin \alpha_{i_k}}} . \end{aligned}$$

It is easily seen that if we set $g(t) = \sqrt{t^2 - a^2} + \frac{ab}{t}$, where both a and b are constants smaller than 1, then $g(t) \leq g(1)$ for all $t \leq 1$ in a neighbourhood of 1. Thus, choosing sufficiently small ϵ , we have

$$\sqrt{\left(\frac{1}{1 + \epsilon}\right)^2 - \sin^2 \alpha_{i_k} + \bar{\xi} \sin \alpha_{i_k} \cdot (1 + \epsilon)} \leq \sqrt{1 - \sin^2 \alpha_{i_k} + \bar{\xi} \sin \alpha_{i_k}} .$$

Accordingly,

$$\frac{\sin \alpha_{i_k}}{\sqrt{\left(\frac{1}{1 + \epsilon}\right)^2 - \sin^2 \alpha_{i_k} + \bar{\xi} \cdot (1 + \epsilon) \sin \alpha_{i_k}}}$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

$$\begin{aligned} &\geq \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \bar{\xi} \sin \theta_{i_k}} \\ &\geq \frac{\sin \alpha_{i_k}}{\cos \alpha_{i_k} + \xi \sin \theta_{i_k}}, \end{aligned}$$

from which follows (42). Thus the proof of *Lemma 3* is completed.

Theorem 3. Assume that (A1), (A2), (A3), and (A4). Then

$$\lim_{\rho \rightarrow \infty} \frac{\log E(M(\rho))}{A(\rho)} = \mu,$$

where μ is the unique root of the equation

$$E \left(\sum_{j=1}^N \left(\Lambda_j^{(N)} \right)^\mu \right) = 1.$$

Proof of Theorem 3. Put

$$\underline{\Lambda}_{\epsilon,j}^{(n)} = \frac{\sin \underline{A}_{\epsilon,j}^{(n)}}{\cos \underline{A}_{\epsilon,j}^{(n)} + \sin \underline{\Theta}_{\epsilon,j}^{(n)}}$$

and

$$\bar{\Lambda}_{\epsilon,j}^{(n)} = \frac{\sin \bar{A}_{\epsilon,j}^{(n)}}{\cos \bar{A}_{\epsilon,j}^{(n)} + \sin \bar{\Theta}_{\epsilon,j}^{(n)}}$$

for $j = 1, \dots, n$, and define

$$\underline{g}_\epsilon(\mu) = E \left(\sum_{j=1}^N \left(\underline{\Lambda}_{\epsilon,j}^{(N)} \right)^\mu \right)$$

and

$$\bar{g}_\epsilon(\mu) = E \left(\sum_{j=1}^N \left(\bar{\Lambda}_{\epsilon,j}^{(N)} \right)^\mu \right).$$

Moreover we put

$$\underline{M}_\epsilon(\rho) = \sum_{\mathbf{i} \in \mathbf{T}} f(\cosh \rho \sin \underline{\alpha}_{\mathbf{i}})$$

and

$$\bar{M}_\epsilon(\rho) = \sum_{\mathbf{i} \in \mathbf{T}} f(\cosh \rho \sin \bar{\alpha}_{\mathbf{i}}).$$

From *Lemma 3* it follows that

$$\underline{M}_\epsilon(\rho) \leq M(\rho) \leq \bar{M}_\epsilon(\rho).$$

Since the assumption (\tilde{A}) holds for both random vectors $(\underline{A}_{\epsilon,1}^{(n)}, \underline{\Theta}_{\epsilon,1}^{(n)} \dots, \underline{A}_{\epsilon,n}^{(n)}, \underline{\Theta}_{\epsilon,n}^{(n)})$ and $(\overline{A}_{\epsilon,1}^{(n)}, \overline{\Theta}_{\epsilon,1}^{(n)} \dots, \overline{A}_{\epsilon,n}^{(n)}, \overline{\Theta}_{\epsilon,n}^{(n)})$, using *Lemma 2*, we get

$$\lim_{\rho \rightarrow \infty} \frac{\log E(\underline{M}_{\epsilon}(\rho))}{A(\rho)} = \underline{\mu}_{\epsilon},$$

where $\underline{\mu}_{\epsilon}$ is the unique root of the equation $\underline{g}_{\epsilon}(\cdot) = 1$, and

$$\lim_{\rho \rightarrow \infty} \frac{\log E(\overline{M}_{\epsilon}(\rho))}{A(\rho)} = \overline{\mu}_{\epsilon},$$

where $\overline{\mu}_{\epsilon}$ is the unique root of the equation $\overline{g}_{\epsilon}(\cdot) = 1$.

Then, using an argument similar to that which we have done in the proof of *Lemma 2* with the help of Dini' theorem, we can complete the proof.

5 Tessellations with strictly or statistically congruent domains

In this section we study tessellations which satisfy a cosmographic principle, that is, tessellations with symmetry. To state it exactly, in case without randomness, we construct tessellations with congruent (unbounded) domains, and in case with randomness, those with "statistically" congruent domains.

Our method of construction is as follows:

1. Consider an experiment that on the circle $\partial\mathbf{D}$, we drop $n + 1$ arcs with a constant length $2\omega_0$ so that they do not mutually overlap. Here the word "arc" denotes the concept in the Euclidean geometry. Parametrize these arcs by position of their center, and denote them by $\{\tilde{t}_j : j = 0, 1, \dots, n\}$, where $0 \leq \tilde{t}_j < 2\pi$ for every j .
2. Out of these arcs we choose an arc, say \tilde{t}_0 , at random, and put $t_j = \tilde{t}_j - \tilde{t}_0 \bmod 2\pi$ for $j = 1, \dots, n$. Note that to each arc corresponds a line in \mathbf{D} . Let ϕ_0 be a translation which moves an arc \tilde{t}_0 to the line l_0 .
3. Moving lines which correspond to arcs $\{t_j : j = 1, \dots, n\}$ by the translation ϕ_0 , we get n lines in the half-plane H , which we denote by $\{l(\alpha_j, \theta_j) : j = 1, \dots, n\}$.

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Suppose that $n + 1$ random arcs $\{\tilde{T}_j : j = 0, 1, \dots, n\}$ are placed according to a symmetric probability distribution \tilde{S}_n , where the word "symmetric" means that

$$\tilde{S}_n(d\tilde{t}_{\sigma(0)}, d\tilde{t}_{\sigma(1)}, \dots, d\tilde{t}_{\sigma(n)}) = \tilde{S}_n(d\tilde{t}_0, d\tilde{t}_1, \dots, d\tilde{t}_n)$$

for any permutation σ on $\{0, 1, \dots, n\}$. Put $T_j = \tilde{T}_j - \tilde{T}_0 \bmod 2\pi$ for $j = 1, \dots, n$. Let Q_n be the probability distribution of a random vector $(A_1, \Theta_1, \dots, A_n, \Theta_n)$ which specify n random lines corresponding to n arcs. We put

$$\Lambda_j = \frac{\sin A_j}{\cos A_j + \sin \Theta_j}$$

for every $j = 1, \dots, n$, and denote by P_n the probability distribution of a random vector $(\Lambda_1, \dots, \Lambda_n)$. Then a random vector $(\Lambda_1, \dots, \Lambda_n)$ can be obtained directly from (T_1, \dots, T_n) .

Lemma 1.

$$\Lambda_j = \frac{1 - \cos \omega_0}{\cos \omega_0 - \cos T_j}$$

for $j = 1, \dots, n$.

Proof Lemma 1. We can see easily that a translation ϕ_0 which moves points $e^{\pm i\omega_0}$ to points ± 1 , is given by

$$\phi_0(z) = \frac{i(r_0 - z)}{1 - r_0 z},$$

where $r_0 = \frac{1 - \sin \omega_0}{\cos \omega_0}$. Let t_j, α_j and θ_j be a realization of T_j, A_j and Θ_j respectively. Since lines $l(\alpha_j, \theta_j)$ are obtained by translating arcs t_j by ϕ_0 , we have

$$e^{i(\theta_j \pm \alpha_j)} = \phi_0 \left(e^{i(t_j \pm \omega_0)} \right)$$

for $j = 1, \dots, n$. After an elementary but tedious calculation, we can obtain

$$(43) \quad \begin{cases} \cot \alpha_j &= \frac{\cos \omega_0}{\sin^2 \omega_0} (1 - \cos t_j) \\ \frac{\cos \theta_j}{\sin \alpha_j} &= \frac{1}{\sin \omega_0} \sin t_j \end{cases}$$

Substituting (43) into

$$\lambda_j = \frac{\sin \alpha_j}{\cos \alpha_j + \sin \theta_j},$$

we get

$$\lambda_j = \frac{1 - \cos \omega_0}{\cos \omega_0 - \cos t_j},$$

which is that we have to prove.

Example 1. Consider a non-random tessellation where

1. at every time of generation, a constant number n new lines are generated, that is, $q_n = 1$.
2. supposing that n lines are arranged so as to be $\theta_1 < \theta_2 < \dots < \theta_n$, all distances between lines

$$d(l(\alpha_j, \theta_j), l(\alpha_{j+1}, \theta_{j+1}))$$

are identical to each other and equal to $\cosh^{-1} \delta$ for $j = 0, \dots, n$, where $l(\alpha_j, \theta_j)$ for $j = 0$ and $j = n + 1$ denote l_0 .

We can explicitly construct this model by dropping $n + 1$ arcs such that $\tilde{t}_j = \frac{2\pi j}{n+1}$ for $j = 0, 1, \dots, n$. Thus, by *Lemma 1*, we have

$$\lambda_j = \frac{1 - \cos \omega_0}{\cos \omega_0 - \cos \frac{2\pi j}{n+1}}.$$

As for an indeterminate value ω_0 , it is determined by solving a system of three equations (43) and $\frac{\sin \theta_j}{\sin \alpha_j} = \delta$ for $j = 1$. By an easy calculation we get

$$\sin^2 \omega_0 = \frac{1 - \cos t_1}{\delta - 1}.$$

The fractal dimension μ of this non-random tessellation are calculated by solving the equation

$$(44) \quad \sum_{j=1}^n \lambda_j^\mu = 1.$$

To our regret, it seems that we can solve this equation only by numerical methods. On the other hand, the simplest case $n = 2$, we can solve (44) and see that

$$\mu = \frac{\log 2}{\log \frac{1}{\lambda_1}} = \frac{\log 2}{\log \left(\sqrt{(\delta + 1)(\delta - \frac{1}{2})} + \delta \right)}.$$

Example 2. Consider a random tessellation where

1. $q_n = 1$.
2. a random vector (T_1, \dots, T_n) has the uniform probability distribution on the set

$$D = \{(t_1, \dots, t_n) : \omega_0 < t_1 - \omega_0 < t_1 + \omega_0 < \dots < t_n - \omega_0 < t_n + \omega_0 < 2\pi - \omega_0\}.$$

A class of random fractal tessellations in hyperbolic planes, Yukinao ISOKAWA

Then the fractal dimension of this random tessellation is equal to the root μ of the equation

$$(45) \quad \sum_{j=1}^n \int \cdots \int_D \left(\frac{1 - \cos \omega_0}{\cos \omega_0 - \cos t_j} \right)^\mu dt_1 \cdots dt_n = 1 .$$

It seems that we can solve this equation only by numerical methods. Even in the simplest case $n = 2$ where the equation (45) reduces to

$$2 \int_{2\omega_0}^{2\pi-4\omega_0} \left(\frac{1 - \cos \omega_0}{\cos \omega_0 - \cos t} \right)^\mu dt = 1 ,$$

we can not solve in the closed form.

Acknowledgements

I am indebted to Professor Hitotsuyanagi for his helpful discussion on the hyperbolic geometry.

References

- Mandelbrot, B. B. (1982) *Fractal Geometry of Nature*. W.H.Freeman, San Francisco.
- Falconer, K. J. (1993) *Fractal Geometry*. Wiley, New York.
- Fenchel, W. (1989) *Elementary Geometry in Hyperbolic Space* W.de Gruyter, Berlin.