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Rep. Fac. Sci. Kagoshima Univ., (Math., Phys. & Chem.),
No. 18, p. 1–18, 1985.

SOME RESULTS ON THE MEYER-KÖNIG AND ZELLER OPERATORS

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(Received September 10, 1985)

Abstract

In the present paper we shall obtain the asymptotic expansions and the explicit expressions for $(M_n e_i)(x)$ when $i=2$ and 3, respectively. We shall also derive some improved estimate for $M_n e_2 - e_2$.

1. Introduction and the results.

Let A_R be the set of all complex-valued functions defined on the half-open interval $[0, 1)$ for which $|f(t)| \leqq P \exp\left(\frac{a}{1-t}\right)$, $t \in [0, 1)$, where P and a are some positive constants depending only upon the function f. Then the Meyer-König and Zeller operators M_n are defined on A_R by

$$(M_n f)(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right) \qquad (x \in [0, 1);\ n \in N). \tag{1.1}$$

It is easily seen that, if $x \in (0, 1)$, then the series (1.1) converges for all $n \geqq 1 + \left[\frac{a}{\log \frac{1}{x}}\right]$, where the square bracket denotes, as usual, the integral part of the argument. If $f(t)$ is continuous to the left at $t=1$ and $f(1)$ exists, then $(M_n f)(1)$ is defined as $(M_n f)(1) := \lim_{x \uparrow 1} (M_n f)(x) = f(1)$. These operators are clearly linear and also positive. It is easily verified that

$$M_n e_i = e_i \qquad (i=0, 1;\ n \in N), \tag{1.2}$$

where the functions e_i are defined by $e_i : x \mapsto x^i$, $(i \in N \cup \{0\})$. It is also well-known that $M_n e_2$ converges uniformly to e_2 in $[0, 1]$.

P. P. Korovkin [2] has proved the following theorem: If the three conditions

$$L_n(1)(x) = 1 + \alpha_n(x), \qquad L_n(t)(x) = x + \beta_n(x),$$
$$L_n(t^2)(x) = x^2 + \gamma_n(x)$$

are satisfied for the sequence of linear positive operators $L_n(f)(x)$, where $\alpha_n(x)$, $\beta_n(x)$, $\gamma_n(x)$ converge uniformly to 0 in $[a, b]$, then the sequence $L_n(f)(x)$ converges uniformly to the function $f(x)$ in $[a, b]$, if $f(x)$ is continuous in $[a, b]$, continuous on the right at $x=b$ and on the left at the point $x=a$. Owing to this theorem, if $f(x)$ is continuous on $[0,1]$, then $(M_n f)(x)$ converges uniformly to $f(x)$ in $[0,1]$. Therefore $M_n e_i$ $(i=0, 1,$

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2) have a conspicuous meaning to study the asymptotic behavior of the operator.

The main purpose of this paper is to study the second moment and the third moment of M_n operator. Especially we shall investigate the asymptotic expansions and explicit expressions for $M_n e_2$ and $M_n e_3$. In §2 we shall follow the line of arguments by Lupas and Müller [3] and improve their results. In §3 we shall make use of the reasoning that is performed in §2 and refine the asymptotic expansion for $M_n e_3$ which was found by Sikkema [5] in 1970. In §4 we shall show some improvements on the estimation relating to $M_n e_2 - e_2$. Finally, in §5 we derive an explicit expression for $(M_n e_3)(x)$ with the aid of a differential equation just as in [1].

The main results obtained in our research work are as follows:

1. $$(M_n e_2)(x)=x^2+\frac{x(1-x)^2}{n}+\frac{x(1-x)^2(2x-1)}{n^2}+\frac{x(1-x)^2(6x^2-6x+1)}{n^3}+O\left(\frac{1}{n^4}\right)$$
$$(n\to\infty) \quad x\in[0,\ 1),$$

2. $$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-9x+11x^2)}{n^2}$$
$$+\frac{x(1-x)^2(-2+27x-72x^2+50x^3)}{n^3}+O\left(\frac{1}{n^4}\right) \qquad (n\to\infty),$$

3. if $(M_n e_2)(x)-x^2$ attains the maximum at the point x_0, then
$$x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}+\frac{4304}{3^9n^4}+O\left(\frac{1}{n^5}\right) \qquad (n\to\infty),$$

4. $$(M_n e_2)(x_0)-x_0^2=\frac{4}{27n}-\frac{4}{3^4n^2}-\frac{20}{3^6n^3}+\frac{964}{3^9n^4}+\frac{356}{3^9n^5}+O\left(\frac{1}{n^6}\right) \qquad (n\to\infty),$$

5. $(M_n e_3)(x)=x+\dfrac{(1-x)^{n+1}}{x^n}u_n(x)$, where
$$u_n(x)=(-1)^n n^2 t_n(x)+(-1)^{n+1}\left\{n^2h(n)+2n\right\}\left\{s_n(x)-\log(1-x)\right\}+(-1)^n n^2\sum_{k=1}^{\infty}\frac{x^k}{k^2},$$
$$s_n(x)=\sum_{k=1}^{n}\frac{(-1)^k x^k}{k(1-x)^k}, \qquad t_n(x)=\sum_{k=1}^{n}\frac{(-1)^k h(k)x^k}{k(1-x)^k}$$
$$h(n)=\sum_{k=1}^{n}\frac{1}{k}.$$

Remarks. We mention the results obtained earlier by various authors to make our results clear.

1. P. C. Sikkema [5] obtained
$$M_n e_2(x)=x^2+\frac{x(1-x)^2}{n}+\frac{x(1-x)^2(2x-1)}{n^2}+o\left(\frac{1}{n^2}\right)$$
as a special case of his Theorem 3.

2. P. C. Sikkema [5] also obtained the following result:
$$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-9x+11x^2)}{n^2}+O\left(\frac{1}{n^3}\right) \qquad (n\to\infty),$$
see expression (20) in [5].

3. J. A. H. Alkemade [1] recently obtained the following asymptotic expansion for x_0:

$$x_0=\frac{1}{3}+\frac{4}{27n}+O\left(\frac{1}{n^2}\right) \qquad (n\to\infty),$$

see p. 270 in [1]

4. J. A. H. Alkemade [1] also established the following asymptotic expansion for $\|F_n\|$:

$$\|F_n\|=\frac{4}{27n}-\frac{4}{81n^2}+O\left(\frac{1}{n^3}\right) \qquad (n\to\infty),$$

where $\|F_n\|=\max\limits_{0\leq x\leq 1}|F_n(x)|=F_n(x_0)=(M_ne_2)(x_0)-x_0^2$.

5. This result seems to be new.

2. An asymptotic expansion for $(M_ne_2)(x)$.

This section is devoted to improve the order of the asymptotic expansion of M_ne_2 due to Lupas and Müller [3]. For the sake of completeness we shall give the detailed proof of Lemma 2.1 below. This proposition is mentioned in [3], but its proof is omitted there. In what follows, for simplicity, we make a convention that

$$m_{n,k}(x)=(1-x)^{n+1}x^k\binom{n+k}{k}.$$

Lemma 2.1. (p. 20 *in* [3])

$$(M_ne_2)(x)=x^2+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x). \tag{2.1}$$

Proof. By definition

$$(M_ne_2)(x)=\sum_{k=0}^{\infty}(1-x)^{n+1}x^k\binom{n+k}{k}\left(\frac{k}{n+k}\right)^2=\sum_{k=1}^{\infty}(1-x)^{n+1}x^k\binom{n+k-1}{k-1}\frac{k}{n+k}.$$

Noting that

$$\frac{k}{n+k}=\frac{kn}{(n+k)(n+k-1)}+\frac{k(k-1)}{(n+k)(n+k-1)},$$

we see that

$$\begin{aligned}
(M_ne_2)(x)&=\sum_{k=1}^{\infty}(1-x)^{n+1}x^k\binom{n+k-1}{k-1}\left\{\frac{kn}{(n+k)(n+k-1)}+\frac{k(k-1)}{(n+k)(n+k-1)}\right\}\\
&=\sum_{k=1}^{\infty}(1-x)^{n+1}x^k\binom{n+k-2}{k-1}\frac{k}{n+k}+\sum_{k=2}^{\infty}(1-x)^{n+1}x^k\binom{n+k-2}{k-2}\frac{k}{n+k}\\
&=\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+1}\binom{n+k-1}{k}\frac{k+1}{n+k+1}+\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k}{k}\frac{k+2}{n+k+2}\\
&=\sum_{k=1}^{\infty}(1-x)^{n+1}x^{k+1}\binom{n+k-1}{k}\frac{k}{n+k+1}+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x)\\
&\qquad+\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k}{k}\frac{k+2}{n+k+2}\\
&=\sum_{k=1}^{\infty}(1-x)^{n+1}x^{k+1}\binom{n+k-1}{k-1}\frac{n}{n+k+1}+\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k}{k}\frac{k+2}{n+k+2}\\
&\qquad+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x)
\end{aligned}$$

$$=\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k}{k}\frac{n}{n+k+2}+\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k}{k}\frac{k+2}{n+k+2}$$
$$+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x)$$
$$=x^2\sum_{k=0}^{\infty}(1-x)^{n+1}x^k\binom{n+k}{k}+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x)$$
$$=x^2+x(1-x)\sum_{k=0}^{\infty}\frac{1}{n+k+1}m_{n-1,k}(x).$$

Theorem 2. 1. (p. 20 *in* [3])

$$(M_ne_2)(x)=e_2(x)+\frac{x(1-x)^2}{n}+o\left(\frac{1}{n}\right)\qquad(n\to\infty). \tag{2.2}$$

Proof. On the basis of Lemma 2. 1, we can prove the theorem, see p. 20 in [3].

Remark. In [3] it has been more generally shown that

$$(M_nf)(x)-f(x)=\frac{x(1-x)^2f''(x)}{2n}+o\left(\frac{1}{n}\right)\qquad(n\to\infty)$$

for the function $f\in C^2[0, 1]$.

Lemma 2. 2.

$$\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{n+k+1}=\frac{(1-x)^n}{x^{n+1}}\int_0^x\left(\frac{t}{1-t}\right)^n dt. \tag{2.3}$$

Proof. It is well know that

$$(1-t)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}t^k,\qquad(|t|<1).$$

Hence

$$t^n(1-t)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}t^{n+k}. \tag{2.4}$$

Integrating the both sides of (2.4) with respect to t from 0 to x, $x\in[0, 1)$,

$$\int_0^x\left(\frac{t}{1-t}\right)^n dt=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^{n+k+1}}{n+k+1}. \tag{2.5}$$

Thus we have

$$\frac{(1-x)^n}{x^{n+1}}\int_0^x\left(\frac{t}{1-t}\right)^n dt=(1-x)^n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{n+k+1},$$

which gives the desired result (2.3).

Lemma 2. 3.

$$\frac{(1-x)^n}{x^{n+1}}\int_0^x\left(\frac{t}{1-t}\right)^n dt=\frac{1-x}{n+1}+\frac{2x(1-x)}{(n+1)(n+2)}+\frac{6x^2(1-x)}{(n+1)(n+2)(n+3)}+A_n(x),$$

where

$$0\leqq A_n(x)\leqq\frac{24}{(n+1)(n+2)(n+3)(n+4)}\frac{x^3}{(1-x)^4}.$$

Proof. Set $I=\int_0^x\left(\frac{t}{1-t}\right)^n dt$. If we make a substitution $u=\frac{t}{1-t}$, then

$$I=\int_0^{\frac{x}{1-x}}\frac{u^n}{(1+u)^2}du.$$

Integrating by parts repeatedly,

$$I=\frac{x^{n+1}}{(n+1)(1-x)^{n-1}}+\frac{2x^{n+2}}{(n+1)(n+2)(1-x)^{n-1}}+\frac{6x^{n+3}}{(n+1)(n+2)(n+3)(1-x)^{n-1}}$$
$$+\frac{24}{(n+1)(n+2)(n+3)}\int_0^{\frac{x}{1-x}}\frac{u^{n+3}}{(1+u)^5}du.$$

Thus

$$\frac{(1-x)^n}{x^{n+1}}I=\frac{1-x}{n+1}+\frac{2x(1-x)}{(n+1)(n+2)}+\frac{6x^2(1-x)}{(n+1)(n+2)(n+3)}$$
$$+\frac{24}{(n+1)(n+2)(n+3)}\frac{(1-x)^n}{x^{n+1}}\int_0^{\frac{x}{1-x}}\frac{u^{n+3}}{(1+u)^5}du.$$

Let

$$A_n(x)=\frac{24}{(n+1)(n+2)(n+3)}\frac{(1-x)^n}{x^{n+1}}\int_0^{\frac{x}{1-x}}\frac{u^{n+3}}{(1+u)^5}du.$$

Then we see that

$$0\leqq A_n(x)\leqq\frac{24(1-x)^n}{(n+1)(n+2)(n+3)x^{n+1}}\int_0^{\frac{x}{1-x}}u^{n+3}du=\frac{24x^3}{(n+1)(n+2)(n+3)(n+4)(1-x)^4}.$$

Theorem 2. 2.

$$(M_ne_2)(x)=x^2+\frac{x(1-x)^2}{n}+\frac{x(1-x)^2(2x-1)}{n^2}+\frac{x(1-x)^2(6x^2-6x+1)}{n^3}$$
$$+O\left(\frac{1}{n^4}\right)\qquad x\in[0,\ 1].$$

Proof. From Lemmas 2. 2 and 2. 3,

$$\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{n+k+1}=\frac{1-x}{n+1}+\frac{2x(1-x)}{(n+1)(n+2)}+\frac{6x^2(1-x)}{(n+1)(n+2)(n+3)}+A_n(x).$$

Noting $A_n(x)=O\left(\frac{1}{n^4}\right)$, we have

$$\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{n+k+1}=\frac{1-x}{n}+\frac{(1-x)(2x-1)}{n^2}+\frac{(1-x)(6x^2-6x+1)}{n^3}+O\left(\frac{1}{n^4}\right).$$

Combining the last expression and Lemma 2. 1, we obtain Theorem 2. 2.

Remark. In 1970 Sikkema [5] proved generally that

$$(M_nf)(x)=f(x)+\frac{x(1-x)^2}{2n}f''(x)+\frac{1}{n^2}\left\{\frac{1}{2}x(1-x)^2(2x-1)f''(x)\right.$$
$$\left.+\frac{1}{6}x(1-x)^3(1-5x)f'''(x)+\frac{1}{8}x^2(1-x)^4f^{\mathrm{iv}}(x)\right\}+o\left(\frac{1}{n^2}\right),$$

as far as the order of the expansion is concerned, Theorem 2. 2 is an improvement on the result mentioned above. However, in 1984 Alkemade found for the first time the explicit expression for M_ne_2. Therefore by using it we can find the asymptotic expansion for M_ne_2 to any higher order without performing the above operation involving the integration. But since the explicit expression for $M_n e_i$ $(i \geqq 3)$ has not yet been found, this procedure seems to be useful slightly.

3. An asymptotic expansion for $(M_n e_3)(x)$.

In this section we derive an asymptotic expansion for $(M_n e_3)(x)$ by appealing to the method dealt with in §2.

Lemma 3. 1.

$$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n-1}-\frac{9x^2(1-x)^2}{n-1}\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{n+k+2} \\ -x^2(1-x)\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+2)^2}+x(1-x)^2\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{(n+k+1)^2}. \tag{3.1}$$

Proof. By definition

$$(M_n e_3)(x)=(1-x)^{n+1}\sum_{k=0}^{\infty}x^k\binom{n+k}{k}\left(\frac{k}{n+k}\right)^3,$$

which implies that

$$(M_n e_3)(x)=(1-x)^{n+1}\sum_{k=1}^{\infty}x^k\binom{n+k-1}{k-1}\left(\frac{k}{n+k}\right)^2.$$

Since

$$\frac{k}{n+k}=\frac{kn}{(n+k)(n+k-1)}+\frac{k(k-1)}{(n+k)(n+k-1)},$$

$$(M_n e_3)(x)=(1-x)^{n+1}\sum_{k=1}^{\infty}x^k\binom{n+k-1}{k-1}\frac{k^2n}{(n+k)^2(n+k-1)} \\ +(1-x)^{n+1}\sum_{k=1}^{\infty}x^k\binom{n+k-1}{k-1}\frac{k^2(k-1)}{(n+k)^2(n+k-1)}$$

$$=(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+1}\binom{n+k-1}{k}\left(\frac{k+1}{n+k+1}\right)^2+(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+2}\binom{n+k}{k}\left(\frac{k+2}{n+k+2}\right)^2$$

$$=(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+1}\binom{n+k-1}{k}\frac{k^2}{(n+k+1)^2}+(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+1}\binom{n+k-1}{k}\frac{2k}{(n+k+1)^2} \\ +(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+1}\binom{n+k-1}{k}\frac{1}{(n+k+1)^2}+(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+2}\binom{n+k}{k}\left(\frac{k+2}{n+k+2}\right)^2$$

$$=(1-x)^{n+1}\sum_{k=1}^{\infty}x^{k+1}\binom{n+k-1}{k-1}\frac{kn}{(n+k+1)^2}+(1-x)^{n+1}\sum_{k=1}^{\infty}x^{k+1}\binom{n+k-1}{k-1}\frac{2n}{(n+k+1)^2} \\ +(1-x)x\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+1)^2}+(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+2}\binom{n+k}{k}\left(\frac{k+2}{n+k+2}\right)^2$$

$$=(1-x)x\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+1)^2} \\ +(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+2}\binom{n+k}{k}\left\{\frac{(k+1)n}{(n+k+2)^2}+\frac{2n}{(n+k+2)^2}+\frac{(k+2)^2}{(n+k+2)^2}\right\}$$

$$=(1-x)x\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+1)^2}+(1-x)^{n+1}\sum_{k=0}^{\infty}x^{k+2}\binom{n+k}{k}\frac{(n+k+2)^2-n(n+k+1)}{(n+k+2)^2}$$

$$=(1-x)x\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+1)^2}+x^2-\sum_{k=0}^{\infty}(1-x)^{n+1}x^{k+2}\binom{n+k+1}{k}\frac{n(n+1)}{(n+k+2)^2}$$

$$=\frac{x(1-x)^{n+1}}{(n+1)^2}+x^2+\frac{x}{1-x}\sum_{k=0}^{\infty}\frac{(1-x)^2m_{n-1,k+1}(x)-n(n+1)xm_{n+1,k}(x)}{(n+k+2)^2}.$$

Inasmuch as

$$\begin{aligned}(1-x)^2m_{n-1,k+1}(x)&-n(n+1)xm_{n+1,k}(x)\\&=(1-x)^{n+2}\binom{n+k}{k+1}x^{k+1}-n(n+1)x^{k+1}(1-x)^{n+2}\binom{n+k+1}{k}\\&=-x^{k+1}(1-x)^{n+2}\binom{n+k}{k+1}(k^2+nk+2k+n),\end{aligned}$$

we obtain that

$$\begin{aligned}(M_ne_3)(x)&=\frac{x(1-x)^{n+1}}{(n+1)^2}+x^2-x(1-x)\sum_{k=0}^{\infty}\frac{\binom{n+k}{k+1}x^k(1-x)^n(k^2+nk+2k+n)}{(n+k+2)^2}\\&=\frac{x(1-x)^{n+1}}{(n+1)^2}+x^2-x^2(1-x)\sum_{k=0}^{\infty}m_{n-1,k}(x)\frac{(n+k)(k^2+nk+2k+n)}{(n+k+1)^2(k+1)}\\&=\frac{x(1-x)^{n+1}}{(n+1)^2}+x^2-x^2(1-x)\Bigg\{\sum_{k=0}^{\infty}m_{n-1,k}(x)\\&\quad+\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)[(n+k)\{k^2+(n+2)k+n\}-(n+k+2)^2(k+1)]}{(n+k+2)^2(k+1)}\Bigg\}\\&=x^3+\frac{x(1-x)^{n+1}}{(n+1)^2}+x^2(1-x)\sum_{k=0}^{\infty}m_{n-1,k}(x)\frac{(3k+3n+6)(k+1)+n-k-2}{(n+k+2)^2(k+1)}.\end{aligned}$$

Thus we are led to

$$\begin{aligned}(M_ne_3)(x)=x^3+\frac{x(1-x)^{n+1}}{(n+1)^2}&+3x^2(1-x)\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{n+k+2}\\&+x^2(1-x)\sum_{k=0}^{\infty}m_{n-1,k}(x)\frac{n-1-(k+1)}{(n+k+2)^2(k+1)}.\end{aligned}$$

Since

$$m_{n-1,k}(x)=\frac{1-x}{n-1}\left\{(1-x)^{n-1}x^k\binom{n+k-2}{k}(n+k+2-3)\right\},$$

we see that

$$\begin{aligned}(M_ne_3)(x)=x^3&+\frac{x(1-x)^{n+1}}{(n+1)^2}+\frac{3x^2(1-x)^2}{n-1}\sum_{k=0}^{\infty}m_{n-2,k}(x)\\&-\frac{9x^2(1-x)^2}{n-1}\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{n+k+2}+x^2(1-x)\sum_{k=0}^{\infty}m_{n-1,k}(x)\frac{n-1}{(n+k+2)^2(k+1)}\\&-x^2(1-x)\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+2)^2}.\end{aligned}\tag{3.2}$$

As

$$x^2(1-x)\sum_{k=0}^{\infty}m_{n-1,k}(x)\frac{n-1}{(n+k+2)^2(k+1)}=x^2(1-x)\sum_{k=0}^{\infty}\binom{n+k-1}{k+1}(1-x)^nx^k\frac{1}{(n+k+2)^2}$$

$$=x^2(1-x)\sum_{k=1}^{\infty}\binom{n+k-2}{k}(1-x)^n x^{k-1}\frac{1}{(n+k+1)^2}=x(1-x)^2\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{(n+k+1)^2}-\frac{x(1-x)^{n+1}}{(n+1)^2},$$

the substitution of this expression into (3.2) yields Lemma 3. 1.

Lemma 3. 2.

$$\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{n+k+2}=\frac{(1-x)^{n-1}}{x^{n+2}}\int_0^x\frac{t^{n+1}}{(1-t)^{n-1}}dt. \tag{3.3}$$

Proof. In exactly the same way as in Lemma 2. 2 of §2, we have

$$t^{n+1}(1-t)^{-n+1}=\sum_{k=0}^{\infty}\binom{n+k-2}{k}t^{n+k+1},$$

hance

$$\int_0^x t^{n+1}(1-t)^{-n+1}dt=\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{x^{n+k+2}}{n+k+2}. \tag{3.4}$$

Multiplying the both sides of (3,4) by $x^{-n-2}(1-x)^{n-1}$, we obtain (3.3).

Lemma 3. 3.

$$\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{n+k+2}=\frac{1-x}{n+2}+\frac{4x(1-x)}{(n+2)(n+3)}+\frac{20x^2(1-x)}{(n+2)(n+3)(n+4)}+A_n(x), \tag{3.5}$$

where $0\leqq A_n(x)\leqq\dfrac{120x^3}{(n+2)(n+3)(n+4)(n+5)(1-x)^6}$.

Proof. Let $I=\int_0^x\frac{t^{n+1}}{(1-t)^{n-1}}dt$. Making a substitution $u=\frac{t}{1-t}$,

$$I=\int_0^{\frac{x}{1-x}}\frac{u^{n+1}}{(1+u)^4}du.$$

Integrating by parts successively, it follows that

$$I=\frac{x^{n+2}}{(n+2)(1-x)^{n-2}}+\frac{4x^{n+3}}{(n+2)(n+3)(1-x)^{n-2}}+\frac{20x^{n+4}}{(n+2)(n+3)(n+4)(1-x)^{n-2}}$$
$$+\frac{120}{(n+2)(n+3)(n+4)}\int_0^{\frac{x}{1-x}}\frac{u^{n+4}}{(1+u)^7}du.$$

Hence

$$\frac{(1-x)^{n-1}}{x^{n+2}}I=\frac{1-x}{n+2}+\frac{4x(1-x)}{(n+2)(n+3)}+\frac{20x^2(1-x)}{(n+2)(n+3)(n+4)} \tag{3.6}$$
$$+\frac{120(1-x)^{n-1}}{(n+2)(n+3)(n+4)x^{n+2}}\int_0^{\frac{x}{1-x}}\frac{u^{n+4}}{(1+u)^7}du.$$

Denoting the last term on the right-hand side of (3.6) by $A_n(x)$, we get

$$0\leqq A_n(x)\leqq\frac{120(1-x)^{n-1}}{(n+2)(n+3)(n+4)x^{n+2}}\int_0^{\frac{x}{1-x}}u^{n+4}du \tag{3.7}$$
$$=\frac{120x^3}{(n+2)(n+3)(n+4)(n+5)(1-x)^6}.$$

Lemma 3. 2, (3.6) and (3.7) lead us to Lemma 3. 3.

Lemma 3. 4.

$$\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+2)^2}=\frac{(1-x)^2}{(n+2)^2}+\frac{4x(1-x)^2}{(n+2)^2(n+3)}+\frac{3x(1-x)^2}{(n+2)(n+3)^2}+O\Big(\frac{1}{n^4}\Big). \tag{3.8}$$

Proof. Obviously

$$t^{n+1}(1-t)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}t^{n+k+1}.$$

Thus

$$\int_0^s t^{n+1}(1-t)^{-n}dt=\sum_{k=0}^{\infty}\frac{\binom{n+k-1}{k}s^{n+k+2}}{n+k+2}$$

follows. Hence we have

$$\frac{1}{s}\int_0^s t^{n+1}(1-t)^{-n}dt=\sum_{k=0}^{\infty}\frac{\binom{n+k-1}{k}s^{n+k+1}}{n+k+2},$$

which implies that

$$\int_0^x\Big(\frac{1}{s}\int_0^s t^{n+1}(1-t)^{-n}dt\Big)ds=\sum_{k=0}^{\infty}\frac{\binom{n+k-1}{k}x^{n+k+2}}{(n+k+2)^2}.$$

Therefore

$$\sum_{k=0}^{\infty}\frac{m_{n-1,k}(x)}{(n+k+2)^2}=\frac{(1-x)^n}{x^{n+2}}\int_0^x\Big(\frac{1}{s}\int_0^s\frac{t^{n+1}}{(1-t)^n}dt\Big)ds. \tag{3.9}$$

Let $I(s)=\int_0^s\frac{t^{n+1}}{(1-t)^n}dt$ and set $u=\frac{t}{1-t}$, then we have

$$I(s)=\int_0^{\frac{s}{1-s}}\frac{u^{n+1}}{(1+u)^3}du.$$

Integrating by parts, we see that

$$I(s)=\frac{s^{n+2}}{(n+2)(1-s)^{n-1}}+\frac{3s^{n+3}}{(n+2)(n+3)(1-s)^{n-1}}+A_n(s),$$

where

$$0\leqq A_n(s)\leqq\frac{12}{(n+2)(n+3)}\int_0^{\frac{s}{1-s}}u^{n+3}du=\frac{12}{(n+2)(n+3)(n+4)}\Big(\frac{s}{1-s}\Big)^{n+4},$$

hence

$$\frac{1}{s}I(s)=\frac{s^{n+1}}{(n+2)(1-s)^{n-1}}+\frac{3s^{n+2}}{(n+2)(n+3)(1-s)^{n-1}}+B_n(s),$$

where

$$0\leqq B_n(s)\leqq\frac{12s^{n+3}}{(n+2)(n+3)(n+4)(1-s)^{n+4}}.$$

Let $J(x)=\int_0^x\frac{1}{s}I(s)ds$ and let $k=\frac{s}{1-s}$, then

$$J(x)=\frac{1}{n+2}\int_0^{\frac{x}{1-x}}\frac{k^{n+1}}{(1+k)^4}dk+\frac{3}{(n+2)(n+3)}\int_0^{\frac{x}{1-x}}\frac{k^{n+2}}{(1+k)^5}dk+\int_0^x B_n(s)ds.$$

Integration by parts yields

$$\frac{(1-x)^n}{x^{n+2}}J(x)=\frac{(1-x)^2}{(n+2)^2}+\frac{4x(1-x)^2}{(n+2)^2(n+3)}+\frac{3x(1-x)^2}{(n+2)(n+3)^2}+O\Big(\frac{1}{n^4}\Big).$$

Now (3.8) follows from the last expression and (3.9).

Lemma 3. 5.

$$\sum_{k=0}^{\infty}\frac{m_{n-2,k}(x)}{(n+k+1)^2}=\frac{(1-x)^2}{(n+1)^2}+\frac{4x(1-x)^2}{(n+1)^2(n+2)}+\frac{3x(1-x)^2}{(n+1)(n+2)^2}+O\left(\frac{1}{n^4}\right).$$

Proof. All we have to do is only replacing n by $n-1$ in Lemma 3. 4.

Theorem 3. 1.

$$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-9x+11x^2)}{n^2}$$
$$+\frac{x(1-x)^2(-2+27x-72x^2+50x^3)}{n^3}+O\left(\frac{1}{n^4}\right).$$

Proof. Taking into account of Lemmas 3. 1, 3. 2, 3. 3, 3. 4, and 3. 5,

$$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n-1}-\frac{9x^2(1-x)^2}{n-1}\left\{\frac{1-x}{n+2}+\frac{4x(1-x)}{(n+2)(n+3)}\right\}$$
$$-x^2(1-x)\left\{\frac{(1-x)^2}{(n+2)^2}+\frac{4x(1-x)^2}{(n+2)^2(n+3)}+\frac{3x(1-x)^2}{(n+2)(n+3)^2}\right\}$$
$$+x(1-x)^2\left\{\frac{(1-x)^2}{(n+1)^2}+\frac{4x(1-x)^2}{(n+1)^2(n+2)}+\frac{3x(1-x)^2}{(n+1)(n+2)^2}\right\}+O\left(\frac{1}{n^4}\right).$$

Therefore from this result Theorem 3. 1 follows at once.

Remark. In 1970 Sikkema [5] proved that

$$(M_n e_3)(x)=x^3+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-9x+11x^2)}{n^2}+O\left(\frac{1}{n^3}\right).$$

Theorem 3. 1 is an improvement of this expansion.

4. Some improvements on the estimation relating to $M_n e_2 - e_2$.

Let

(4.1) $$F_n(x)=(M_n e_2)(x)-x^2 \qquad (x\in[0,\ 1],\ n\in N)$$

and let $\|f\|$ denote the supremum norm of $f\in C[0,\ 1]$. We observe that

(4.2) $$F_n(x)=\frac{1}{n+1}x(1-x)^2{}_2F_1(1,\ 2;\ n+2;\ x),$$

which is established by J. A. H. Alkemade [1]. In this connection we refer to two theorems ([5], [1]).

Theorem A. (Sikkema, [5]) *Let* F_n *be defined by* (4.1). *Then we have*

(*a*) $\|F_1\|\leqq 0.1113$

(*b*) $\|F_n\|\leqq\frac{4}{27n}\left(1-\frac{n^2-5}{4(n^2-1)^2}\right)$ $\qquad (n\geqq 2).$

Theorem B. (Alkemade, [1]) *Let* F_n *be defined by* (4.1). *Then the following statements hold* :

(*a*) $\|F_1\|=0.0999032$ *(exact up to the last digit shown)* (*b*) $\|F_n\| \leqq \dfrac{4}{27n+9}$ $(n \geqq 2)$

(*c*) $\|F_n\| = \dfrac{4}{27n} - \dfrac{4}{81n^2} + O\left(\dfrac{1}{n^3}\right)$ $(n \to \infty)$.

For the part (*c*) of Theorem *B* we deduce the following refinement:

$$\|F_n\| = \frac{4}{27n} - \frac{4}{3^4 n^2} - \frac{20}{3^6 n^3} + \frac{964}{3^9 n^4} + \frac{356}{3^9 n^5} + O\left(\frac{1}{n^6}\right),$$

which is stated as Theorem 4. 1. To prove this we need a lemma.

Lemma 4. 1. *The asymptotic expansion of the value* $x_0 \in (0, 1)$, *which is uniquely determined by* $\|F_n\| = |F_n(x_0)|$, *is given by*

$$x_0 = \frac{1}{3} + \frac{4}{27n} - \frac{16}{3^5 n^2} - \frac{208}{3^7 n^3} + \frac{4304}{3^9 n^4} + O\left(\frac{1}{n^5}\right). \tag{4.3}$$

Proof. First we notice that x_0 satisfies the equation

$${}_2F_1(1, 2\,; n+2\,; x_0) = \frac{n+1}{n+x_0}, \tag{4.4}$$

see, p. 268 in [1]. We begin with showing

$$x_0 = \frac{1}{3} + \frac{4}{27n} - \frac{16}{3^5 n^2} + O\left(\frac{1}{n^3}\right).$$

(4.4) is transformed into

$$1 + \frac{2x_0}{n+2} + \frac{6x_0^2}{(n+2)(n+3)} + \frac{24x_0^3}{(n+2)(n+3)(n+4)} + O\left(\frac{1}{n^4}\right) = \frac{n+1}{n}\left(\frac{1}{1+\dfrac{x_0}{n}}\right).$$

By a simple calculation we obtain

$$1 + \frac{2x_0}{n}\left(1 - \frac{2}{n} + \frac{4}{n^2} - \frac{8}{n^3}\right) + \frac{6x_0^2}{n^2}\left(1 - \frac{5}{n} + \frac{19}{n^2}\right) + \frac{24x_0^3}{n^3}\left(1 - \frac{9}{n}\right)$$
$$= 1 - \frac{x_0}{n} + \frac{x_0^2}{n^2} - \frac{x_0^3}{n^3} + \frac{1}{n} - \frac{x_0}{n^2} + \frac{x_0^2}{n^3} + O\left(\frac{1}{n^4}\right).$$

Finally from this we see that

$$25x_0^3 + (5n-31)x_0^2 + (3n^2-3n+8)x_0 - n^2 + O\left(\frac{1}{n}\right) = 0. \tag{4.5}$$

Motivated by Alkemade's result $x_0 = \dfrac{1}{3} + \dfrac{4}{27n} + O\left(\dfrac{1}{n^2}\right)$, (see, p. 270 in [1]) we may assume that

$$x_0 = \frac{1}{3} + \frac{4}{27n} + \frac{k}{n^2} + O\left(\frac{1}{n^3}\right) \qquad (k \text{ is a constant}),$$

then

$$x_0^2 = \frac{1}{9} + \frac{8}{81n} + O\left(\frac{1}{n^2}\right), \qquad x_0^3 = \frac{1}{27} + O\left(\frac{1}{n}\right). \tag{4.6}$$

Substituting (4.6) into (4.5) yields $\dfrac{16}{81} + 3k + O\left(\dfrac{1}{n}\right) = 0$, and consequently we see that $k = -\dfrac{16}{243}$. Conversely, $x_0 = \dfrac{1}{3} + \dfrac{4}{27n} - \dfrac{16}{3^5 n^2} + O\left(\dfrac{1}{n^3}\right)$ is clearly the solution of (4.4).

This shows that $x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}+O\left(\frac{1}{n^3}\right)$.

Next we put $x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}+\frac{k}{n^3}+O\left(\frac{1}{n^4}\right)$. By substituting this expression into the asymptotic equation

$$119x_0^4+(25n-215)x_0^3+(5n^2-31n+114)x_0^2+(3n^3-3n^2+8n-16)x_0-n^3+O\left(\frac{1}{n}\right)=0,$$

which is obtained from (4.4) by expanding the both sides of it up to the term of $\frac{1}{n^4}$ explicitly, we get

$$x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}+O\left(\frac{1}{n^4}\right).$$

Finally we put $x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}+\frac{k}{n^4}+O\left(\frac{1}{n^5}\right)$. In a similar way, from the asymptotic equation

$$721x_0^5+(119n-1681)x_0^4+(25n^2-215n+1320)x_0^3+(5n^3-31n^2+114n-390)x_0^2$$
$$+(3n^4-3n^3+8n^2-16n+32)x_0-n^4+O\left(\frac{1}{n}\right)=0,$$

we have (4.3).

Remark. By means of the above method we can derive the asymptotic expansion for x_0 giving to as any higher order as we please. But the calculation involved is troublesome.

Theorem 4. 1. *For F_n we have*

$$\|F_n\|=\frac{4}{27n}-\frac{4}{3^4n^2}-\frac{20}{3^6n^3}+\frac{964}{3^9n^4}+\frac{356}{3^9n^5}+O\left(\frac{1}{n^6}\right)\quad (n\to\infty).$$

Proof. In view of (4.2) we obtain

$$\|F_n\|=F_n(x_0)=\frac{1}{n+1}x_0(1-x_0)^2{}_2F_1(1,\ 2\ ;\ n+2\ ;\ x_0),$$

thus by the definition of the hypergeometric series

$$\|F_n\|=\frac{1}{n+1}x_0(1-x_0)^2\left\{1+\frac{2x_0}{n}\left(1-\frac{2}{n}+\frac{4}{n^2}-\frac{8}{n^3}\right)+\frac{6x_0^2}{n^2}\left(1-\frac{5}{n}+\frac{19}{n^2}\right)+\frac{24x_0^3}{n^3}\left(1-\frac{9}{n}\right)+\frac{120x_0^4}{n^4}\right\}+O\left(\frac{1}{n^6}\right). \tag{4.7}$$

From Lemma 4. 1 it follows that

$$\left\{\begin{array}{ll} x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}+\frac{4304}{3^9n^4}+O\left(\frac{1}{n^5}\right), & x_0^2=\frac{1}{9}+\frac{8}{3^4n}-\frac{16}{3^6n^2}-\frac{544}{3^8n^3}+O\left(\frac{1}{n^4}\right), \\ x_0^3=\frac{1}{27}+\frac{4}{3^4n}+O\left(\frac{1}{n^3}\right), & x_0^4=\frac{1}{81}+\frac{16}{3^6n}+O\left(\frac{1}{n^2}\right), \\ (1-x_0)^2=\frac{4}{9}-\frac{16}{3^4n}+\frac{80}{3^6n^2}+\frac{704}{3^8n^3}-\frac{6208}{3^9n^4}+O\left(\frac{1}{n^5}\right). & \end{array}\right. \tag{4.8}$$

Substituting (4.8) into (4.7), we obtain Theorem 4. 1.

In the meantime, Alkemade's theorem and Lemma 4. 1 lead us to the following estimation for $\|F_n\|$.

Theorem 4. 2. *For sufficiently large* $n \in N$,

$$\|F_n\| \leqq \frac{324n^3}{2187n^4+729n^3+324n^2-144n-208}$$

holds.

Proof. Let $f_n(x)=\dfrac{n+1}{n+x}$. we already know from (4.4) that

$${}_2F_1(1,\ 2\ ;\ n+2\ ;\ x_0)=f_n(x_0).$$

The function ${}_2F_1(1,\ 2\ ;\ n+2\ ;\ x)$ is monotonically increasing and $f_n(x)$ is monotonically decreasing in (0, 1). From Lemma 4. 1 we have

$$x_0=\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}+\frac{4304}{3^9n^4}+O\left(\frac{1}{n^5}\right),$$

hence the monotony of ${}_2F_1(1,\ 2\ ;\ n+2\ ;\ x)$ and $f_n(x)$ implies that

$${}_2F_1(1,\ 2\ ;\ n+2\ ;\ x_0) \leqq f_n\left(\frac{1}{3}+\frac{4}{27n}-\frac{16}{3^5n^2}-\frac{208}{3^7n^3}\right)$$

$$=\frac{3^7(n+1)n^3}{3^7n^4+3^6n^3+3^4\times 4n^2-3^2\times 16n-208}$$

for sufficiently large $n \in N$. From (4.2) and the above inequality we conclude that

$$\|F_n\| \leqq \frac{1}{n+1}\|x(1-x)^2\|\,{}_2F_1(1,\ 2\ ;\ n+2\ ;\ x_0)$$

$$\leqq \frac{324n^3}{2187n^4+729n^3+324n^2-144n-208}.$$

This completes the proof of Theorem 4. 2.

We make use of Theorem 4. 2 to improve slightly upon the known theorems on M_n operators. We have the theorem by Shisha and Mond [4].

Theorem 4. 3. [4] *Let* L_n : $C[0,\ 1] \to C[0,\ 1]$ $(n \in N)$ *be a sequence of linear positive operators satisfying* $L_n e_i = e_i$ $(i=0,\ 1)$. *Then for any* $\delta > 0$

(4.9) $$|(L_nf)(x)-f(x)| \leqq \left\{1+\delta^{-2}((L_ne_2)(x)-x^2)\right\}\omega(f\ ;\ \delta)$$

where $\omega(f\ ;\ \delta)$ *denotes the modulus of continuity of* f *on* [0, 1].

If we replace L_n by M_n and set $\delta=n^{-\frac{1}{2}}$ in (4.9) and use Theorem 4. 2, we obtain the following theorem.

Theorem 4. 4. *For sufficiently large* $n \in N$,

$$\|M_nf-f\| \leqq \left\{1+\frac{324n^4}{2187n^4+729n^3+324n^2-144n-208}\right\}\omega\left(f\ ;\ \frac{1}{\sqrt{n}}\right)$$

holds.

Note that $\|M_nf-f\| \leqq \left\{1+\dfrac{4n}{27n+9}\right\}\omega\left(f\ ;\ \dfrac{1}{\sqrt{n}}\right)$, which was previously obtained by

Alkemade [1]. Further we know the following theorem due to Lupas and Müller [3].

Theorem 4. 5. [3] *Let* $L_n: C[0, 1] \to C[0, 1]\ (n \in N)$ *be a sequence of linear positive operators satisfying* $L_n e_i = e_i\ (i=0, 1)$. *If* f' *exists and continuous on* $[0, 1]$, *then for any* $\delta > 0$

$$\|L_n f - f\| \leqq (1+\delta^{-1}) \|L_n e_2 - e_2\| \omega(f'; \delta).$$

If we replace L_n by M_n and set $\delta = n^{-\frac{1}{2}}$ and use Theorem 4. 2, we obtain the following theorem.

Theorem 4. 6. *For sufficiently large* $n \in N$,

$$\|M_n f - f\| \leqq (1+\sqrt{n}) \frac{324 n^3}{2187 n^4 + 729 n^3 + 324 n^2 - 144 n - 208} \omega\left(f'; \frac{1}{\sqrt{n}}\right).$$

Note that $\|M_n f - f\| \leqq \left\{1 + \frac{2\sqrt{n}}{3\sqrt{3n+1}}\right\} \frac{2}{3\sqrt{3n+1}} \omega\left(f'; \frac{1}{\sqrt{n}}\right)$, which was also obtained by Alkemade [1].

5. An explicit expression for $(M_n e_3)(x)$.

In this section we search for the explicit expression for $(M_n e_3)(x)$ by means of the way developed in [1]; namely the way appealing to a differential equation. Let us recall a theorem in [1]:

Theorem C. [1] *Let* $g(t) = \frac{t}{1-t}$, $t \in [0, 1)$. *For each* $n \in N$, $x \in [0, 1)$ *and* $f \in A_R$, $(M_n f)(x)$ *defined by* (1.1) *satisfies the differential equation*

$$x(1-x)\frac{d}{dx}(M_n f)(x) = -(n+1)x(M_n f)(x) + n(1-x)(M_n(gf))(x).$$

Lemma 5. 1. *For each* $n \in N$, $M_n e_3$ *satisfies the differential equation*

$$x^2(1-x)^2 y'' + x(1-x)(1+2n+x)y' + (x^2 + (3n+1)x + n^2)y = n^2 x^3 + (3n+1)x^2 + x, \tag{5.1}$$

with the condition $y(0)=0$ and $y'(0) = \frac{1}{(n+1)^2}$.

Proof. By the definition of $M_n e_3$ it is clear that

$$y(0)=0, \quad y'(0) = \frac{1}{(n+1)^2},$$

where $y(x) = (M_n e_3)(x)$. We set $f = e_2 - e_3$ in Theorem C. Then we have

$$x(1-x)\frac{d}{dx}(M_n(e_2 - e_3))(x) = -(n+1)x(M_n(e_2 - e_3))(x) + n(1-x)(M_n e_3)(x).$$

From the linearity of M_n operators, we have

$$x(1-x)\frac{d}{dx}(M_n e_2)(x) - x(1-x)\frac{d}{dx}(M_n e_3)(x)$$
$$= -(n+1)x(M_n e_2)(x) + (n+x)(M_n e_3)(x),$$

namely,

(5.2) $$x(1-x)\frac{d}{dx}(M_ne_2)(x)+(n+1)x(M_ne_2)(x)$$
$$=x(1-x)\frac{d}{dx}(M_ne_3)(x)+(n+x)(M_ne_3)(x).$$

From Lemma 1 in [1] (see. p. 263)

(5.3) $$x(1-x)\frac{d}{dx}(M_ne_2)(x)=-(n+x)(M_ne_2)(x)+nx^2+x.$$

Substituting (5.3) into (5.2) yields that

$$(nx-n)(M_ne_2)(x)+nx^2+x=x(1-x)\frac{d}{dx}(M_ne_3)(x)+(x+n)(M_ne_3)(x).$$

In other words, $(M_ne_3)(x)$ is a solution of the differential equation

(5.4) $$x(1-x)y'+(x+n)y=n(x-1)(M_ne_2)(x)+nx^2+x.$$

Differentiating (5.4), we have

(5.5) $$x(1-x)y''+(1+n-x)y'+y$$
$$=n(M_ne_2)(x)+n(x-1)((M_ne_2)(x))'+2nx+1.$$

Further from (5.3)

$$n(x-1)((M_ne_2)(x))'=\frac{n(x+n)(M_ne_2)(x)}{x}-n^2x-n.$$

Therefore from the last equation and (5.5) we get

(5.6) $$x(1-x)y''+(1+n-x)y'+y=\frac{n(n+2x)(M_ne_2)(x)}{x}-(n^2-2n)x-n+1.$$

On the other hand, since M_ne_3 is a solution of (5.4),

(5.7) $$n(M_ne_2)(x)=\frac{x(1-x)y'+(x+n)y-nx^2-x}{x-1}.$$

Substituting (5.7) into (5.6), we obtain (5.1), as required.

Lemma 5.2. *Let* $w_n(x)=(M_ne_3)(x)-x$. *Then* $w_n(x)$ *satisfies the following differential equation*

(5.8) $$x^2(1-x)^2w_n''(x)+x(1-x)(1+2n+x)w_n'(x)+\left\{x^2+(3n+1)x+n^2\right\}w_n(x)$$
$$=-nx(1-x)(nx+n+2).$$

Proof. By making use of the relation $w_n(x)=(M_ne_3)(x)-x$ and Lemma 5. 1, we obtain the result.

Lemma 5. 3. $y=x^{-n}(1-x)^{n+1}$ *is a solution of the differential equation*

(5.9) $$x^2(1-x)^2y''+x(1-x)(1+2n+x)y'+\left\{x^2+(3n+1)x+n^2\right\}y=0.$$

Proof. A straightforward calculation gives the result.

The following two lemmas are concerned with some definite integrals. In the sequel, we employ the notation:

$$s_n(x)=\sum_{k=1}^{n}\frac{(-1)^kx^k}{k(1-x)^k},\quad t_n(x)=\sum_{k=1}^{n}\frac{(-1)^kh(k)x^k}{k(1-x)^k},\quad \text{where } h(k)=\sum_{i=1}^{k}\frac{1}{i}.$$

Lemma 5. 4. *For each* $n\in N$ *and for each* $x\in[0,\ 1)$,

$$\int_0^x \frac{t^n}{(1-t)^{n+1}}dt=(-1)^n\left\{s_n(x)-\log(1-x)\right\}.$$

Proof. Denoting $I_n=\int_0^x \frac{t^n}{(1-t)^{n+1}}dt$, we obtain the relation

$$I_n=\frac{x^{n+1}}{(n+1)(1-x)^{n+1}}-I_{n+1},$$

by the integration by parts. Thus $(-1)^{n+1}I_{n+1}-(-1)^nI_n=(-1)^{n+1}\frac{x^{n+1}}{(n+1)(1-x)^{n+1}}$. Adding the equalities obtained by setting $n=0, 1, 2, \cdots, m-1$ in the last equation, we have

$$(-1)^mI_m-I_0=\sum_{n=1}^m\frac{(-1)^nx^n}{n(1-x)^n}=s_m(x).$$

Observing $I_0=-\log(1-x)$, we establish the lemma.

Lemma 5. 5. *For each* $n\in N$ *and for each* $x\in[0, 1)$,

$$\int_0^x \frac{t^n}{(1-t)^{n+2}}dt=\frac{x^{n+1}}{(n+1)(1-x)^{n+1}}.$$

Proof. By putting $u=\frac{t}{1-t}$, we easily obtain the result.

By employing these lemmas we are now in a position to solve the differential equation (5.8). Motivated by Lemma 5. 3 we may set $y=w_n(x)=(M_ne_3)(x)-x=y_1(x)u_n(x)$, where $y_1(x)=x^{-n}(1-x)^{n+1}$. Then obviously we have

(5.10) $$w_n'(x)=y_1'u_n+y_1u_n', \quad w_n''(x)=y_1''u_n+2y_1'u_n'+y_1u_n''.$$

Substituting (5.10) into (5.8), we have

(5.11) $$x^2(1-x)^2y_1u_n''+\left\{2x^2(1-x)^2y_1'+x(1-x)(1+2n+x)y_1\right\}u_n' =-nx(1-x)(nx+n+2).$$

As $y_1'=-x^{-n-1}(1-x)^n(n+x)$, we have from (5.11)

$$xu_n''(x)+u_n'(x)=-\frac{n^2x^{n+1}}{(1-x)^{n+2}}-\frac{n(n+2)x^n}{(1-x)^{n+2}}.$$

Solving this equation, we get

$$xu_n'(x)=-n^2\int_0^x\frac{t^{n+1}}{(1-t)^{n+2}}dt-n(n+2)\int_0^x\frac{t^n}{(1-t)^{n+2}}dt,$$

here we use the condition $[xu_n'(x)]_{x=0}=0$. By appealing to Lemmas 5. 4 and 5. 5 we obtain

$$xu_n'(x)=(-1)^nn^2\left\{s_{n+1}(x)-\log(1-x)\right\}-\frac{n(n+2)x^{n+1}}{(n+1)(1-x)^{n+1}}.$$

Thus we have

$$u_n'(x)=(-1)^nn^2\left\{\frac{s_{n+1}(x)}{x}-\frac{\log(1-x)}{x}\right\}-\frac{n(n+2)x^n}{(n+1)(1-x)^{n+1}}.$$

Finally, from the above equation we get

(5.12) $$u_n(x)=(-1)^nn^2\left\{\int_0^x\frac{s_{n+1}(t)}{t}dt-\int_0^x\frac{\log(1-t)}{t}dt\right\}$$

$$-\frac{n(n+2)}{n+1}\int_0^x \frac{t^n}{(1-t)^{n+1}}dt,$$

here we use $u_n(0)=0$.

Now by definition

$$\int_0^x \frac{s_{n+1}(t)}{t}dt=\sum_{k=1}^{n+1}\frac{(-1)^k}{k}\int_0^x \frac{t^{k-1}}{(1-t)^k}dt$$
$$=\sum_{k=2}^{n+1}\frac{(-1)^k}{k}\int_0^x \frac{t^{k-1}}{(1-t)^k}dt-\int_0^x \frac{1}{1-t}dt,$$

by making use of Lemma 5. 4 again we have

$$\int_0^x \frac{s_{n+1}(t)}{t}dt=-\sum_{k=2}^{n+1}\frac{1}{k}\left\{s_{k-1}(x)-\log(1-x)\right\}+\log(1-x) \tag{5.13}$$
$$=-\sum_{k=2}^{n+1}\frac{1}{k}s_{k-1}(x)+h(n+1)\log(1-x).$$

Next we prove that

$$\sum_{k=2}^{n+1}\frac{1}{k}s_{k-1}(x)=h(n+1)s_n(x)-t_n(x). \tag{5.14}$$

In fact, by the change of order of double summation, we get

$$\sum_{k=2}^{n+1}\frac{1}{k}s_{k-1}(x)=\sum_{k=1}^{n}\frac{1}{k+1}s_k(x)=\sum_{k=1}^{n}\frac{1}{k+1}\sum_{i=1}^{k}\frac{(-1)^i x^i}{i(1-x)^i}$$
$$=\sum_{i=1}^{n}\frac{(-1)^i x^i}{i(1-x)^i}\sum_{k=i}^{n}\frac{1}{k+1}=\sum_{i=1}^{n}\frac{(-1)^i x^i}{i(1-x)^i}\left\{h(n+1)-h(i)\right\}$$
$$=h(n+1)s_n(x)-t_n(x).$$

Combining (5.13) and (5.14) we get

$$\int_0^x \frac{s_{n+1}(t)}{t}dt=t_n(x)-h(n+1)\left\{s_n(x)-\log(1-x)\right\}. \tag{5.15}$$

Again in view of Lemma 5. 4 we conclude from (5.12) that

$$u_n(x)=(-1)^n n^2 t_n(x)+(-1)^{n+1}n^2 h(n+1)\left\{s_n(x)-\log(1-x)\right\}$$
$$+(-1)^n n^2\sum_{k=1}^{\infty}\frac{x^k}{k^2}-\frac{n(n+2)}{n+1}(-1)^n\left\{s_n(x)-\log(1-x)\right\},$$

that is,

$$u_n(x)=(-1)^n n^2 t_n(x)+(-1)^{n+1}\left\{n^2h(n)+2n\right\}\left\{s_n(x)-\log(1-x)\right\} \tag{5.16}$$
$$+(-1)^n n^2\sum_{k=1}^{\infty}\frac{x^k}{k^2}.$$

Recall the relation set at the outset

$$(M_n e_3)(x)=x+\frac{(1-x)^{n+1}}{x^n}u_n(x). \tag{5.17}$$

(5.16) and (5.17) lead us to the following theorem.

Theorem 5. 1. *We have*

$$(M_n e_3)(x)=x+\frac{(1-x)^{n+1}}{x^n}u_n(x),$$

where $u_n(x)=(-1)^n n^2 t_n(x)+(-1)^{n+1}\left\{n^2 h(n)+2n\right\}\left\{s_n(x)-\log(1-x)\right\}+(-1)^n n^2\sum_{k=1}^{\infty}\frac{x^k}{k^2}$,

$$s_n(x)=\sum_{k=1}^{n}\frac{(-1)^k x^k}{k(1-x)^k},\quad t_n(x)=\sum_{k=1}^{n}\frac{(-1)^k h(k)x^k}{k(1-x)^k},\quad h(n)=\sum_{k=1}^{n}\frac{1}{k}.$$

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