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On Minimality of Axiomatic Systems of Remarkable Finsler Connections

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Abstract

In general, an axiomatic system is nowadays considered as a mere assumption of the theory, but the requisites are various. Consistency, completeness and independence are usual required. Consistency and completeness of an axiomatic system are assured by the existence and uniqueness theorems for the system respectively. Independence of an axiomatic system means that each axiom is not the logical consequence of the rest. If an axiomatic system is not independent, some axiom may be removed from the system, so we shall call an independent axiomatic system minimal. The purpose of the present paper is to discuss minimality of the axiomatic systems of remarkable Finsler connections known in a Finsler space systematically.
Key words: Finsler space, Cartan connection, Berwald connection, minimal axiomatic system.

1. Introduction

Let (M, L) be a Finsler space, that is, a differentiable manifold M endowed with a metric function $L(x, y)$ $(y^i=\dot{x}^i)$. The fundamental tensor field g_{ij} is given by $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$, where $\dot{\partial}_i=\partial/\partial y^i$. We put $(g^{ij})=(g_{ij})^{-1}$. We shall express a Finsler connection $F\Gamma$ in terms of its coefficients as $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$.

In a Finsler space there are known four remarkable Finsler connections $C\Gamma=(G^i{}_k, \overset{c}{F}_j{}^i{}_k, \overset{c}{C}_j{}^i{}_k)$, $B\Gamma=(G^i{}_k, G_j{}^i{}_k, 0)$, $R\Gamma=(G^i{}_k, \overset{c}{F}_j{}^i{}_k, 0)$ and $H\Gamma=(G^i{}_k, G_j{}^i{}_k, \overset{c}{C}_j{}^i{}_k)$, named the *Cartan, Berwald, Rund and Hashiguchi connections* respectively. As is well known, $C\Gamma$ and $B\Gamma$ are uniquely determined as Finsler connections $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the following axiomatic systems due to Matsumoto [5] and Okada [9] respectively:

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$$C\Gamma \quad \begin{cases} \text{(C1)}\ D^i{}_k=0, & \text{(C2)}\ T_j{}^i{}_k=0, \quad \text{(C3)}\ g_{ij|k}=0, \\ \text{(C4)}\ S_j{}^i{}_k=0, & \text{(C5)}\ g_{ij}|_k=0; \end{cases}$$

$$B\Gamma \quad \begin{cases} \text{(B1)}\ D^i{}_k=0, & \text{(B2)}\ P^i{}_{jk}=0, \quad \text{(B3)}\ T_j{}^i{}_k=0, \quad \text{(B4)}\ L_{|k}=0, \\ \text{(B5)}\ C_j{}^i{}_k=0, \end{cases}$$

where $_{|k}$ and $|_k$ denote the respective h- and v-covariant differentiations, $D^i{}_k$ $(=y^r F_r{}^i{}_k - N^i{}_k)$ the deflection tensor field, and $T_j{}^i{}_k$ $(=F_j{}^i{}_k - F_k{}^i{}_j)$, $S_j{}^i{}_k$ $(=C_j{}^i{}_k - C_k{}^i{}_j)$ and $P^i{}_{jk}$ $(=\dot{\partial}_k N^i{}_j - F_k{}^i{}_j)$ the respective $(h)h$-, $(v)v$- and $(v)hv$-torsion tensor fields. These axiomatic systems are evidently consistent and complete.

Now, we shall call an axiomatic system *minimal* if the axioms are independent each other, that is, if each axiom is not the logical consequence of the rest. The purpose of the present paper is to discuss minimality of the axiomatic systems of remarkable Finsler connections systematically. Minimality of an axiomatic system is assured by showing that the modified system obtained by omitting any axiom and adding its contradictory is consistent, that is, by the existence theorem for the modified system. From such a modified system we may get an interesting Finsler connection as a generalization of a remarkable Finsler connection. Thus, in order to make such a generalization systematically, the problem of minimality is important.

Let $F\Gamma = (N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ be any Finsler connection. It is shown in Aikou-Hashiguchi [1] that the coefficients $N^i{}_k$ and $F_j{}^i{}_k$ are expressed in terms of its $D^i{}_k$, $T_j{}^i{}_k$ and $g_{ij|k}$, and the coefficients $C_j{}^i{}_k$ in terms of its $S_j{}^i{}_k$ and $g_{ij}|_k$. Conversely, in the expressions the tensor fields $D^i{}_k$, $T_j{}^i{}_k$, $g_{ij|k}$, and $S_j{}^i{}_k$, $g_{ij}|_k$ are arbitrarily given, so Matsumoto's axiomatic system is minimal, which we shall review in §2.

On the other hand, it is also shown in [1] that in any Finsler connection $F\Gamma = (N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ the coefficients $N^i{}_k$ and $F_j{}^i{}_k$ are expressed in terms of its $D^i{}_k$, $P^i{}_{jk}$, $T_j{}^i{}_k$ and $L_{|k}$. The coefficients $C_j{}^i{}_k$ are arbitrarily given. Contrary to the former case, however, in the expressions of $N^i{}_k$ and $F_j{}^i{}_k$ the tensor fields $D^i{}_k$, $P^i{}_{jk}$, $T_j{}^i{}_k$ and $L_{|k}$ are not arbitrarily given, so from the expressions it is not clear if Okada's axiomatic system is minimal. The fact is that there does not exist a positively homogeneous Finsler connection such that $D^i{}_k \neq 0$, $P^i{}_{jk}=0$, since the homogeneity yields $D^i{}_k = -P^i{}_{k0}$, where a subscript 0 means a contraction by y^i. If a Finsler connection is confined to what is positively homogeneous, then (B2) implies (B1), so Okada's axiomatic system is not minimal. We allow a Finsler connection which is not positively homogeneous, and in §3 we shall show the minimality. Incidentally, in §4 we shall remark on the existence of the consistent, complete, minimal axiomatic system for each of the Rund and Hashiguchi connections.

The terminology and notation are referred to Matsumoto [6]. In reference to $T_j{}^i{}_k$, however, we use the notation $S_j{}^i{}_k$ instead of $S^i{}_{jk}$. The tensor fields g_{ij} and g^{ij} will serve for lowering and raising indices, e.g., $K_{jk}{}^i = g_{ks}\, g^{ir}\, K_j{}^s{}_r$, where the position and order of indices are essential.

The present paper was first suggested by R. Miron, the first author, in order to dedicate to M. Hashiguchi, the third author, on the occasion of his sixtieth birthday together with T. Aikou, the second author, and has been completed based on [1].

2. Minimality of Matsumoto's axiomatic system

For any Finsler tensor field $K_j{}^i{}_k$ we put

$$K^*{}_j{}^i{}_k = (K_{jk}{}^i + K_{kj}{}^i - K_k{}^i{}_j)/2. \tag{2.1}$$

Then we have from Theorem 2.2 in [1]

Theorem 2.1. *In a Finsler space, let* $D^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$, $S_j{}^i{}_k$ $(=-S_k{}^i{}_j)$, *and* U_{ikj} $(=U_{jki})$, V_{ikj} $(=V_{jki})$ *be any Finsler tensor fields. Then there exists a unique Finsler connection* $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ *such that the deflection tensor field, the* $(h)h$*- and* $(v)v$*-torsion tensor fields, and the h- and v-covariant derivatives of* g_{ij} *are the given* $D^i{}_k$, $T_j{}^i{}_k$, $S_j{}^i{}_k$, *and* U_{ikj}, V_{ikj} *respectively*:

$$F_0{}^i{}_k - N^i{}_k = D^i{}_k, \tag{2.2}$$

$$F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k, \quad C_j{}^i{}_k - C_k{}^i{}_j = S_j{}^i{}_k, \tag{2.3}$$

$$g_{ij|k} = U_{ikj}, \quad g_{ij}|_k = V_{ikj}. \tag{2.4}$$

We construct the Finsler tensor fields $K^i{}_k$, $X^i{}_k$ *and* $Y_j{}^i{}_k$ *by*

$$K^i{}_k = D^i{}_k - T^*{}_0{}^i{}_k + U^*{}_0{}^i{}_k, \tag{2.5}$$

$$X^i{}_k = K^i{}_k - \overset{c}{C}_k{}^i{}_r K^r{}_0, \tag{2.6}$$

$$Y_j{}^i{}_k = \overset{c}{C}_j{}^i{}_r X^r{}_k + \overset{c}{C}_k{}^i{}_r X^r{}_j - \overset{c}{C}_{jkr} X^{ri} \tag{2.7}$$

successively. Then the coefficients of $F\Gamma$ *is given by*

$$N^i{}_k = G^i{}_k - X^i{}_k, \tag{2.8}$$

$$F_j{}^i{}_k = \overset{c}{F}_j{}^i{}_k + Y_j{}^i{}_k + T^*{}_j{}^i{}_k - U^*{}_j{}^i{}_k, \tag{2.9}$$

$$C_j{}^i{}_k = \overset{c}{C}_j{}^i{}_k + S^*{}_j{}^i{}_k - V^*{}_j{}^i{}_k. \tag{2.10}$$

Since the Finsler tensor fields $D^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$, $S_j{}^i{}_k$ $(=-S_k{}^i{}_j)$, and U_{ikj} $(=U_{jki})$, V_{ikj} $(=V_{jki})$ in Theorem 2.1 are arbitrarily given, the independence of each axiom is clear. For example, putting $D^i{}_k=\delta^i{}_k$, $T_j{}^i{}_k=S_j{}^i{}_k=U_{ikj}=V_{ikj}=0$, we have $X^i{}_k=\delta^i{}_k$, $Y_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k$, so we have a Finsler connection which assures the independence of the axiom (C1) as follows.

Example 2.1. The Finsler connection given by

$$N^i{}_k=G^i{}_k-\delta^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k+\overset{c}{C}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k \tag{2.11}$$

satisfies $D^i{}_k=\delta^i{}_k$, $T_j{}^i{}_k=g_{ij|k}=S_j{}^i{}_k=g_{ij}|_k=0$.

We have generally

Theorem 2.2. *A Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the axioms* (C2), (C3), (C4), (C5) *is obtained from*

$$N^i{}_k=G^i{}_k-X^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k+Y_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k, \tag{2.12}$$

if we put $X^i{}_k=D^i{}_k-\overset{c}{C}_k{}^i{}_r D^r{}_0$, $Y_j{}^i{}_k=\overset{c}{C}_j{}^i{}_r X^r{}_k+\overset{c}{C}_k{}^i{}_r X^r{}_j-\overset{c}{C}_{jkr} X^{ri}$ for any Finsler tensor field $D^i{}_k$. Then the deflection tensor field of $F\Gamma$ is the given $D^i{}_k$.

Such Finsler connections were discussed in [2]. Especially, interesting Finsler connections are given by $D^i{}_k=\lambda\delta^i{}_k+\mu l^i l_k$, e.g., $D^i{}_k=-\delta^i{}_k$, $D^i{}_k=h^i{}_k$, where λ and μ are any Finsler scalar fields.

Next, we shall discuss the independence of the axioms (C2) and (C4). We have generally

Theorem 2.3. *A Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the axioms* (C1), (C3), (C4), (C5) *is obtained from*

$$N^i{}_k=G^i{}_k-X^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k+Y_j{}^i{}_k+T^*{}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k, \tag{2.13}$$

if we put $X^i{}_k=-T^*{}_0{}^i{}_k+\overset{c}{C}_k{}^i{}_r T^*{}_0{}^r{}_0$, $Y_j{}^i{}_k=\overset{c}{C}_j{}^i{}_r X^r{}_k+\overset{c}{C}_k{}^i{}_r X^r{}_j-\overset{c}{C}_{jkr} X^{ri}$ for any Finsler tensor field $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$. Then the $(h)h$-torsion tensor field of $F\Gamma$ is the given $T_j{}^i{}_k$.

A Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the axioms (C1), (C2), (C3), (C5) *is obtained from*

$$N^i{}_k=G^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k+S^*{}_j{}^i{}_k, \tag{2.14}$$

where $S_j{}^i{}_k$ $(=-S_k{}^i{}_j)$ *is any Finsler tensor field. Then the* (v) *v-torsion tensor field of* $F\Gamma$ *is the given* $S_j{}^i{}_k$.

As simple examples we have

Example 2.2. The Finsler connection given by

$$N^i{}_k=G^i{}_k+Lh^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k-L\overset{c}{C}_j{}^i{}_k+l_j\,\delta^i{}_k-l^i\,g_{jk},\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k \tag{2.15}$$

satisfies $T_j{}^i{}_k=l_j\,\delta^i{}_k-l_k\,\delta^i{}_j$, $D^i{}_k=g_{ij|k}=S_j{}^i{}_k=g_{ij}|_k=0$.

The Finsler connection given by

$$N^i{}_k=G^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k+l_j\,\delta^i{}_k-l^i\,g_{jk} \tag{2.16}$$

satisfies $S_j{}^i{}_k=l_j\,\delta^i{}_k-l_k\,\delta^i{}_j$, $D^i{}_k=T_j{}^i{}_k=g_{ij|k}=g_{ij}|_k=0$.

A Finsler connection such that $T_j{}^i{}_k=t_j\,\delta^i{}_k-t_k\,\delta^i{}_j$ and/or $S_j{}^i{}_k=s_j\,\delta^i{}_k-s_k\,\delta^i{}_j$ is generally called *semi-symmetric*, where t_j, s_j are any Finsler covariant vector fields (cf. [7]). Especially, a semi-symmetric Finsler connection satisfying the axioms (C1), (C3), (C4), (C5) is called a *Wagner connection*, and is important as a typical *generalized Cartan connection* (cf. [4]).

Lastly, we shall discuss the independence of the axioms (C3) and (C5). We have generally

Theorem 2.4. *A Finsler connection* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *satisfying the axioms* (C1), (C2), (C4), (C5) *is obtained from*

$$N^i{}_k=G^i{}_k-X^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k+Y_j{}^i{}_k-U^*{}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k, \tag{2.17}$$

if we put $X^i{}_k=U^*{}_0{}^i{}_k-\overset{c}{C}_k{}^i{}_r\,U^*{}_0{}^r{}_0$, $Y_j{}^i{}_k=\overset{c}{C}_j{}^i{}_r\,X^r{}_k+\overset{c}{C}_k{}^i{}_r\,X^r{}_j-\overset{c}{C}_{jkr}\,X^{ri}$ *for any Finsler tensor field* U_{ikj} $(=U_{jki})$. *Then we have* $g_{ij|k}=U_{ikj}$ *with respect to* $F\Gamma$, *where* U_{ikj} *is the given one.*

A Finsler connection $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *satisfying the axioms* (C1), (C2), (C3), (C4) *is obtained from*

$$N^i{}_k=G^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k-V^*{}_j{}^i{}_k, \tag{2.18}$$

where V_{ikj} $(=V_{jki})$ *is any Finsler tensor field. Then we have* $g_{ij}|_k=V_{ikj}$ *with respect to* $F\Gamma$, *where* V_{ikj} *is the given one.*

As typical examples we have

Example 2.3. The Hashiguchi connection $H\Gamma=(G^i{}_k, G_j{}^i{}_k, \overset{c}{C}_j{}^i{}_k)$ satisfies $g_{ij|k}=-2\overset{c}{P}_{ijk}$, $D^i{}_k=T_j{}^i{}_k=S_j{}^i{}_k=g_{ij}|_k=0$.

The Rund connection $R\Gamma=(G^i{}_k, \overset{c}{F}_j{}^i{}_k, 0)$ satisfies $g_{ij}|_k=2\overset{c}{C}_{ijk}$, $D^i{}_k=T_j{}^i{}_k=g_{ij|k}=S_j{}^i{}_k=0$.

In a Landsberg space, where $\overset{c}{P}_{ijk}=0$, the above first example becomes trivial. In a Riemannian space, where $\overset{c}{C}_{ijk}=0$, the above second one is also trivial. As non-trivial examples we have *recurrent Finsler connections* such that $g_{ij|k}=2a_k\, g_{ij}$ or $g_{ij}|_k=2b_k\, g_{ij}$, where a_k and b_k are any Finsler covariant vector fields (cf. [3]). Especially, putting $a_k=2l_k$ and $b_k=2l_k$ we have

Example 2.4. The Finsler connection given by

$$N^i{}_k=G^i{}_k-L\delta^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k+L\overset{c}{C}_j{}^i{}_k-l_j\,\delta^i{}_k-l_k\,\delta^i{}_j+l^i\,g_{jk},\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k \tag{2.19}$$

satisfies $g_{ij|k}=2l_k\, g_{ij}$, $D^i{}_k=T_j{}^i{}_k=S_j{}^i{}_k=g_{ij}|_k=0$.

The Finsler connection given by

$$N^i{}_k=G^i{}_k,\ F_j{}^i{}_k=\overset{c}{F}_j{}^i{}_k,\ C_j{}^i{}_k=\overset{c}{C}_j{}^i{}_k-l_j\,\delta^i{}_k-l_k\,\delta^i{}_j+l^i\,g_{jk} \tag{2.20}$$

satisfies $g_{ij}|_k=2l_k\, g_{ij}$, $D^i{}_k=T_j{}^i{}_k=g_{ij|k}=S_j{}^i{}_k=0$.

3. Minimality of Okada's axiomatic system

We shall discuss minimality of Okada's axiomatic system by the following theorem (cf. Theorem 3.2 and Theorem 3.4 of [1]).

Theorem 3.1. *In a Finsler space, let $D^i{}_k$, $P^i{}_{jk}$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$, L_k, and $C_j{}^i{}_k$ be any Finsler tensor fields. We construct the Finsler tensor fields $Q_j{}^i{}_k$, $H^i{}_k$, $E^i{}_k$, and $Z^i{}_k$ by*

$$Q_j{}^i{}_k=T_j{}^i{}_k-(P^i{}_{jk}-P^i{}_{kj}), \tag{3.1}$$

$$H^i{}_k=D^i{}_k+P^i{}_{k0}+Q_k{}^i{}_0\ (=D^i{}_k+P^i{}_{0k}+T_k{}^i{}_0), \tag{3.2}$$

$$E_k=\dot\partial_k(LL_0)-2LL_k, \tag{3.3}$$

$$Z^i{}_k=(H^i{}_k+\dot\partial_k(H_0{}^i+E^i))/2 \tag{3.4}$$

successively.

If the three conditions

$$Z^i{}_k - y^r \dot{\partial}_r Z^i{}_k = D^i{}_k + P^i{}_{k0},\ \dot{\partial}_k Z^i{}_j - \dot{\partial}_j Z^i{}_k = Q_j{}^i{}_k,\ y_r Z^r{}_k = LL_k \tag{3.5}$$

are satisfied, there exists a unique Finsler connection $F\Gamma = (N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ *such that the deflection tensor field, the* $(v)hv$- *and* $(h)h$-*torsion tensor fields, and the h-covariant derivative of L are the given* $D^i{}_k$, $P^i{}_{jk}$, $T_j{}^i{}_k$, *and* L_k:

$$F_0{}^i{}_k - N^i{}_k = D^i{}_k,\ \dot{\partial}_k N^i{}_j - F_k{}^i{}_j = P^i{}_{jk}, \tag{3.6}$$

$$F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k,\ L_{|k} = L_k, \tag{3.7}$$

and the coefficients $C_j{}^i{}_k$ *are the given one. The coefficients* $N^i{}_k$, $F_j{}^i{}_k$ *of* $F\Gamma$ *are given by*

$$N^i{}_k = G^i{}_k - Z^i{}_k,\ F_j{}^i{}_k = G_j{}^i{}_k - \dot{\partial}_j Z^i{}_k - P^i{}_{kj}. \tag{3.8}$$

Especially, in the case where the given Finsler tensor fields $D^i{}_k$, $P^i{}_{jk}$, $T_j{}^i{}_k$ *and* L_k *are positively homogeneous of respective degrees* 1, 0, 0, 1, *that is, in the case where the obtained Finsler connection is positively homogeneous, the above conditions* (3.5) *are equivalent to the following two conditions*:

$$D^i{}_k + P^i{}_{k0} = 0,\ y^r (\dot{\partial}_k Q_j{}^i{}_r - \dot{\partial}_j Q_k{}^i{}_r) = 0. \tag{3.9}$$

Since a Finsler connection $F\Gamma$ satisfying the axioms (B1), (B2), (B3), (B4) is given by $F\Gamma = (G^i{}_k, G_j{}^i{}_k, C_j{}^i{}_k)$ for any Finsler tensor field $C_j{}^i{}_k$, the independence of (B5) in Okada's axiomatic system is clear. In a non-Riemannian space, the Hashiguchi connection $H\Gamma = (G^i{}_k, G_j{}^i{}_k, \overset{c}{C}_j{}^i{}_k)$ is a typical example of a Finsler conenction satisfing $C_j{}^i{}_k \neq 0$, $P^i{}_{jk} = T_j{}^i{}_k = L_{|k} = 0$.

Since there does not exist a positively homogeneous Finsler connection such that $D^i{}_k \neq 0$, $P^i{}_{jk} = 0$, in order to show the independence of (B1) in Okada's axiomatic system, we should search a Finsler connection with a surviving $D^i{}_k$ which is not positively homogeneous of degree 1. From Theorem 3.1 we have generally

Theorem 3.2. *A Finsler connection* $F\Gamma = (N^i{}_k, F_j{}^i{}_k, 0)$ *satisfying the axioms* (B2), (B3), (B4) *is given by*

$$N^i{}_k = G^i{}_k - Z^i{}_k,\ F_j{}^i{}_k = G_j{}^i{}_k - \dot{\partial}_j Z^i{}_k, \tag{3.10}$$

if the Finsler tensor field $Z^i{}_k=(D^i{}_k+\dot{\partial}_k D_0{}^i)/2$, *constructed from any Finsler tensor field* $D^i{}_k$ *such that* $\dot{\partial}_j D^i{}_k=\dot{\partial}_k D^i{}_j$, *satisfies*

(3.11) $$Z^i{}_k-y^r\dot{\partial}_r Z^i{}_k=D^i{}_k,\ y_r Z^r{}_k=0.$$

Then the deflection tensor field of $F\Gamma$ *is the given* $D^i{}_k$.

A typical example is given by $D^i{}_k=\dot{\partial}_k s^i$, where s^i is any positively homogeneous Finsler contravariant vector field of degree 0 such that $y_r\dot{\partial}_k s^r=0$. Especially, taking $s^i=2l^i$ we have from $\dot{\partial}_k l^i=h^i{}_k/L$

Example 3.1. The Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ given by

(3.12) $$N^i{}_k=G^i{}_k-h^i{}_k/L,\ F_j{}^i{}_k=G_j{}^i{}_k+(l^i h_{jk}+l_j h^i{}_k+l_k h^i{}_j)/L^2$$

satisfies $D^i{}_k=2h^i{}_k/L$, $P^i{}_{jk}=T_j{}^i{}_k=L_{|k}=0$.

In order to discuss the independence of the axioms (B2), (B3), and (B4), it is sufficient to take a positively homogeneous Finsler connection with $C_j{}^i{}_k=0$. First, we shall discuss the independence of (B2). We have generally

Theorem 3.3. *A positively homogeneous Finsler connection* $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ *satisfying the axioms* (B1), (B3), (B4) *is given by* (3.8), *if we put* $Z^i{}_k=(P^i{}_{0k}+\dot{\partial}_k P_{00}{}^i)/2$ *for any positively homogeneous Finsler tensor field* $P^i{}_{jk}$ *of degree* 0 *such that* $P^i{}_{k0}=0$, $y^r(\dot{\partial}_k Q_j{}^i{}_r-\dot{\partial}_j Q_k{}^i{}_r)=0$, *where* $Q_j{}^i{}_k=P^i{}_{kj}-P^i{}_{jk}$. *Then the* (v)*hv-torsion tensor field of* $F\Gamma$ *is the given* $P^i{}_{jk}$.

A simple example of $P^i{}_{jk}$ satisfying the conditions in Theorem 3.3 is given by

Example 3.2. Let $P^i{}_{jk}$ $(=P^i{}_{kj})$ be any positively homogeneous Finsler tensor field of degree 0 such that $P^i{}_{k0}=0$. Then the Finsler connection $F\Gamma=(G^i{}_k, G_j{}^i{}_k-P^i{}_{kj}, 0)$ satisfies $D^i{}_k=T_j{}^i{}_k=L_{|k}=0$. The $(v)hv$-torsion tensor field of $F\Gamma$ is the given $P^i{}_{jk}$.

A typical example is given by $P^i{}_{jk}=\overset{c}{P}{}^i{}_{jk}$, for which $G_j{}^i{}_k-\overset{c}{P}{}^i{}_{jk}=\overset{c}{F}_j{}^i{}_k$. Then we have the Rund connection $R\Gamma$. In a Landsberg space we can take $l^i h_{jk}$ as an example of $P^i{}_{jk}\neq 0$. Thus we have

Example 3.3. The Rund connection $R\Gamma=(G^i{}_k, \overset{c}{F}_j{}^i{}_k, 0)$ satisfies $P^i{}_{jk}=\overset{c}{P}{}^i{}_{jk}$, $D^i{}_k=T_j{}^i{}_k=L_{|k}=0$.

The Finsler connection $F\Gamma=(G^i{}_k, G_j{}^i{}_k-l^i h_{jk}, 0)$ satisfies $P^i{}_{jk}=l^i h_{jk}\neq 0$, $D^i{}_k=T_j{}^i{}_k=L_{|k}=0$.

Next, we shall discuss the independence of (B3). We have generally

Theorem 3.4. *A positively homogeneous Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ satisfying the axioms* (B1), (B2), (B4) *is given by* (3.10), *if we put $Z^i{}_k=(T_k{}^i{}_0+\dot{\partial}_k T^i{}_{00})/2$ for any positively homogeneous Finsler tensor field $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$ of degree* 0 *such that $y^r(\dot{\partial}_k T_j{}^i{}_r-\dot{\partial}_j T_k{}^i{}_r)=0$. Then the $(h)h$-torsion tensor field of $F\Gamma$ is the given $T_j{}^i{}_k$.*

A typical example is given by

Example 3.4. The Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ given by

$$N^i{}_k=G^i{}_k+Lh^i{}_k/2,\ F_j{}^i{}_k=G_j{}^i{}_k+(l_j h^i{}_k-l_k h^i{}_j-l^i h_{jk})/2 \tag{3.13}$$

satisfies $T_j{}^i{}_k=l_j\delta^i{}_k-l_k\delta^i{}_j$. $D^i{}_k=P^i{}_{jk}=L_{|k}=0$.

Lastly, we shall discuss the independence of (B4). We have generally

Theorem 3.5. *A positively homogeneous Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ satisfying the axioms* (B1), (B2), (B3) *is given by* (3.10), *if we put $Z^i{}_k=(\dot{\partial}_k E^i)/2$ for any positively homogeneous Finsler tensor field L_k of degreee* 1. *Then we have $L_{|k}=L_k$ with respect to $F\Gamma$, where L_k is the given one.*

A simple example is given by

Example 3.5. The Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, 0)$ given by

$$N^i{}_k=G^i{}_k-L(\delta^i{}_k+l^i l_k),\ F_j{}^i{}_k=G_j{}^i{}_k-(l_j\delta^i{}_k+l_k\delta^i{}_j+l^i h_{jk}) \tag{3.14}$$

satisfies $L_{|k}=2y_k$, $D^i{}_k=P^i{}_{jk}=T_j{}^i{}_k=0$.

Thus we have shown minimality of Okada's axiomatic system. Contrary to the case of the Cartan expression (2.8), (2.9), (2.10) shown in Theorem 2.1, the tensor fields appeared in the Berwald expression (3.8) shown in Theorem 3.1 are not arbitrarily given, which never negates the excellence of Okada's axiomatic system, but it seems to be an interesting problem to search various characterizations for the Berwald connection (cf. [8]).

4. The Rund and Hashiguchi connections

The Rund connection $R\Gamma=(G^i{}_k, \overset{c}{F}{}_j{}^i{}_k, 0)$ and the Hashiguchi connection $H\Gamma=(G^i{}_k, G_j{}^i{}_k, \overset{c}{C}{}_j{}^i{}_k)$ are obtained by interchanging $\overset{c}{C}{}_j{}^i{}_k$ and 0 in the Cartan connection $C\Gamma=(G^i{}_k, \overset{c}{F}{}_j{}^i{}_k, \overset{c}{C}{}_j{}^i{}_k)$ and in the Berwald connection $B\Gamma=(G^i{}_k, G_j{}^i{}_k, 0)$ respectively.

Since the coefficients $G^i{}_k$, $\overset{c}{F}{}_j{}^i{}_k$ (or $G^i{}_k$, $G_j{}^i{}_k$) and $\overset{c}{C}{}_j{}^i{}_k$ (or 0) are independently determined by the conditions $D^i{}_k=0$, $T_j{}^i{}_k=0$, $g_{ij|k}=0$ (or $D^i{}_k=0$, $P^i{}_{jk}=0$, $T_j{}^i{}_k=0$, $L_{|k}=0$), and $S_j{}^i{}_k=0$, $g_{ij}|_k=0$ (or $C_j{}^i{}_k=0$) respectively, the connections $R\Gamma$ and $H\Gamma$ are uniquely determined as Finsler connections $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the following axiomatic systems respectively:

$$R\Gamma \begin{cases} \text{(R1) } D^i{}_k=0, \quad \text{(R2) } T_j{}^i{}_k=0, \quad \text{(R3) } g_{ij|k}=0, \\ \text{(R4) } C_j{}^i{}_k=0; \end{cases}$$

$$H\Gamma \begin{cases} \text{(H1) } D^i{}_k=0, \quad \text{(H2) } P^i{}_{jk}=0, \quad \text{(H3) } T_j{}^i{}_k=0, \quad \text{(H4) } L_{|k}=0, \\ \text{(H5) } S_j{}^i{}_k=0, \quad \text{(H6) } g_{ij}|_k=0. \end{cases}$$

These axiomatic systems are clearly consistent and complete. These are also minimal. In fact, a Finsler connection which assures the independence of each axiom is obtained from some example given in §2 or §3 by interchanging the coefficients. We also remark that the connections $C\Gamma$, $B\Gamma$, $R\Gamma$ and $H\Gamma$ satisfy the conditions listed below, where a bold-faced 0 indicates that it is assumed as an axiom.

	$D^i{}_k$	$T_j{}^i{}_k$	$g_{ij\|k}$	$P^i{}_{jk}$	$L_{\|k}$	$S_j{}^i{}_k$	$g_{ij}\|_k$	$C_j{}^i{}_k$
$C\Gamma$	**0**	**0**	**0**	$\overset{c}{P}{}^i{}_{jk}$	0	**0**	**0**	$\overset{c}{C}{}_j{}^i{}_k$
$B\Gamma$	**0**	**0**	$-2\overset{c}{P}_{ijk}$	**0**	**0**	0	$2\overset{c}{C}_{ijk}$	**0**
$R\Gamma$	**0**	**0**	**0**	$\overset{c}{P}{}^i{}_{jk}$	0	0	$2\overset{c}{C}_{ijk}$	**0**
$H\Gamma$	**0**	**0**	$-2\overset{c}{P}_{ijk}$	**0**	**0**	**0**	**0**	$\overset{c}{C}{}_j{}^i{}_k$

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