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ON DETERMINATIONS OF FINSLER CONNECTIONS BY DEFLECTION TENSOR FIELDS

By

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The author [2] discussed parallel displacements in Finsler spaces and showed that the connection Γ defined by E. Cartan [1] is the shortest and fittest from a natural standpoint. In that case we imposed as a natural condition the torsion tensor field to vanish, but in its definition the supporting elements are confined to be parallel. And, M. Matsumoto [4] has proposed, from the standpoint of his modern Finsler theory, the following elegant axioms that determine uniquely that connection Γ and the associated non-linear connection $\underline{N}$:

(C1) the connection Γ be metrical,

(C2) the deflection tensor field $D=0$,

(C3) the (h)h-torsion tensor field $T=0$,

(C4) the (v)v-torsion tensor field $S^1=0$,

where the axiom C2 expresses the geometrical meaning as above stated.

So, from the standpoint that the supporting elements may be displaced with respect to any non-linear connection N in the tangent bundle, we shall replace the condition C2 by some weaker conditions and find the conditions to be imposed thereon in order that the connection Γ defined by E. Cartan be obtained (Theorem A).

As a result of this consideration we shall notice that Finsler connections with the deflection tensor field $D=-\delta$ are somewhat canonical. We shall give an example of such a Finsler connection (Theorem B).

Throughout the present paper we shall use the terminology and notations described in M. Matsumoto [5]. In §1, we shall briefly sketch the materials in need of our discussions.

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§1. Preliminaries

1°. Given a differentiable manifold M of dimension n, we denote by $L(M)(M, \pi, GL(n, R))$ the bundle of linear frames and by $T(M)$ $(M, \tau, F, GL(n, R))$ the tangent bundle, where the standard fiber F is a vector space of dimension n with a fixed base $\{e_a\}$.

The induced bundle $\tau^{-1}L(M)=\{(y, z) \in T(M)\times L(M) \mid \tau(y)=\pi(z)\}$ is called the *Finsler bundle* of M and denoted by $F(M)$ $(T(M), \pi_1, GL(n, R))$. The projection π_1 is the mapping

$$\pi_1: F(M) \rightarrow T(M) \mid (y, z) \rightarrow y,$$

and we shall denote by π_2 the mapping

$$\pi_2: F(M) \rightarrow L(M) \mid (y, z) \rightarrow z.$$

The Lie algebra of the structural group $GL(n, R)$ of $L(M)$ or $F(M)$ is denoted by $L(n, R)$ and the canonical base by $\{\boldsymbol{L}_a^b\}$.

2°. A *Finsler connection* (Γ, N) is by definition a pair of a connection Γ in the Finsler bundle $F(M)$ and a non-linear connection N in the tangent bundle $T(M)$.

Given a Finsler connection (Γ, N), let $l_u(u \in F(M))$ and $l_y(y \in T(M))$ be the respective lifts with respect to Γ and N. In terms of a canonical coordinate system (x^i, y^i, z_a^i) of $F(M)$, they are expressed by

$$l_u\left(\frac{\partial}{\partial x^k}\right)_y=\left(\frac{\partial}{\partial x^k}\right)_u-z_b^j\Gamma_{jk}^i\left(\frac{\partial}{\partial z_b^i}\right)_u, \tag{1}$$

$$l_u\left(\frac{\partial}{\partial y^k}\right)_y=\left(\frac{\partial}{\partial y^k}\right)_u-z_b^jC_{jk}^i\left(\frac{\partial}{\partial z_b^i}\right)_u, \tag{2}$$

and

$$l_y\left(\frac{\partial}{\partial x^k}\right)_x=\left(\frac{\partial}{\partial x^k}\right)_y-F_k^i\left(\frac{\partial}{\partial y^i}\right)_y, \tag{3}$$

where Γ_{jk}^i, C_{jk}^i are called the *components of* Γ and the F_k^i the *components of* N. C_{jk}^i are also the components of the (h)hv-torsion tensor field C.

For each $f \in F$ the *h*- and the *v*- *basic vector fields* $B^h(f)$ and $B^v(f)$ are defined by

$$B^h(f)_u=l_ul_y(zf) \tag{4}$$

and

$$B^v(f)_u=l_ul_y^v(zf) \tag{5}$$

at $u=(y, z)$ respectively, where l_y^v is the *vertical lift* expressed by

$$l_y^v\left(\frac{\partial}{\partial x^i}\right)_x=\left(\frac{\partial}{\partial y^i}\right)_y. \tag{6}$$

The *h*- and the *v*- *basic forms* θ^h and θ^v constitute, with the connection form ω of Γ, the dual system of $(B^h(f), B^v(f), Z(A))$, where $Z(A)$ is the fundamental vector field corresponding to $A \in L(n, R)$. They are expressed by

$$\theta^h=z^{-1}{}_i^a dx^i \boldsymbol{e}_a, \tag{7}$$

$$\theta^v = z^{-1}{}^a_i(dy^i + F^i_k dx^k)\boldsymbol{e}_a \tag{8}$$

and

$$\omega = z^{-1}{}^a_i(dz^i_b + z^j_b \Gamma^i_{jk} dx^k + z^j_b C^i_{jk} dy^k)\boldsymbol{L}^b_a. \tag{9}$$

If we denote by θ the basic form in $L(M)$ then

$$\theta^h = \pi_2 \theta. \tag{10}$$

3°. Given a Finsler connection (Γ, N), we get the associated non-linear connection $\underline{N}$ with the subordinate F-connection Γ_F to (Γ, N). The pair $(\Gamma, \underline{N})$ is a Finsler connection and is called the *associated connection* with the given one. We shall denote by putting $\_$ the quantities with respect to $(\Gamma, \underline{N})$.

If we put

$$F^i_{jk} = \Gamma^i_{jk} - C^i_{jm} F^m_k, \tag{11}$$

the components $\underline{F}^i_k$ of $\underline{N}$ are

$$\underline{F}^i_k = y^j F^i_{jk}, \tag{12}$$

and differ by $y^j F^i_{jk} - F^i_k$ from F^i_k. The quantities

$$D^i_k = y^j F^i_{jk} - F^i_k \tag{13}$$

are the components of the *deflection tensor field* D defined by

$$D(f) = B^h(f)\gamma, \tag{14}$$

where γ is the *characteristic field* defined by

$$\gamma\colon F(M) \to F \mid (y, z) \to z^{-1}y = z^{-1}{}^a_i y^i \boldsymbol{e}_a.$$

Between the h-basic vector fields $B^h(f)$ and $\underline{B}^h(f)$ there exists the relation

$$B^h(f) = \underline{B}^h(f) + B^v(D(f)), \tag{15}$$

therefore, as the dual relation, we have

$$\theta^v = \underline{\theta}^v - D(\theta^h). \tag{16}$$

4°. Given a Finsler metric function L, the usual metric tensor field G is defined, its components g_{ij} being given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}. \tag{17}$$

A *Finsler space* means here a differentiable manifold M endowed with such a metric tensor field G.

We put

$$\gamma_{jhk}=\frac{1}{2}\left(\frac{\partial g_{jh}}{\partial x^k}+\frac{\partial g_{hk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^h}\right), \tag{18}$$

$$G^i=\frac{1}{2}\gamma^i_{jk}y^jy^k, \tag{19}$$

and

$$G^i_k=\frac{\partial G^i}{\partial y^k}, \tag{20}$$

where $\gamma^i_{jk}=g^{ih}\gamma_{jhk}$.

And we shall sometimes use the notations

$$l^i=\frac{y^i}{L},\ l_j=g_{ij}l^i. \tag{21}$$

5°. Let a Finsler connection (Γ, N) be given in a Finsler space (M, G). The conditions C1–C4 are expressed as follows:

(C1) (22)
$$\Gamma_{jhk}+\Gamma_{hjk}=\frac{\partial g_{jh}}{\partial x^k},$$

(23)
$$C_{jhk}+C_{hjk}=\frac{\partial g_{jh}}{\partial y^k},$$

(C2) (24)
$$F^i_k=y^jF^i_{jk},$$

(C3) (25)
$$F_{jhk}=F_{khj},$$

(C4) (26)
$$C_{jhk}=C_{khj},$$

where $\Gamma_{jhk}=g_{ih}\Gamma^i_{jk}$, $C_{jhk}=g_{ih}C^i_{jk}$ and $F_{jhk}=g_{ih}F^i_{jk}$. We shall here explain some geometrical meanings of these conditions.

Let C be a differentiable curve in M and $\tilde{C}$ be a differentiable curve in $T(M)$ mapped on the C by the projection τ. Tangent vectors $X(t)$ along C are said to be *parallel along C with respect to $\tilde{C}$*, if the equations

$$\frac{dX^i}{dt}+\Gamma^i_{jk}(x, y)X^j\frac{dx^k}{dt}+C^i_{jk}(x, y)X^j\frac{dy^k}{dt}=0 \tag{27}$$

are satisfied, where C is expressed by $x^i(t)$ and $\tilde{C}$ by $x^i(t)$, $y^i(t)$.

Under the parallel displacement along a curve C, if we take in particular $\tilde{C}$ to be a lift $\tilde{C}_N$ with respect to the non-linear connection N, i.e.

$$\frac{dy^i}{dt}+F^i_k(x, y)\frac{dx^k}{dt}=0, \tag{28}$$

the equations (27) may be written in the form

$$\frac{dX^i}{dt}+F^i_{jk}(x, y)X^j\frac{dx^k}{dt}=0. \tag{29}$$

The supporting elements y^i (the points of the lift $\tilde{C}_N$) are parallel with respect to $\tilde{C}_N$, i.e.

$$\frac{dy^i}{dt}+F^i_{jk}(x, y)y^j\frac{dx^k}{dt}=0, \tag{30}$$

if and only if the equations (24) are satisfied, which is a geometrical meaning of the condition C2.

The connection Γ is called to be *metrical* if the length of a vector remains unchanged under the parallel displacement along any curve C with respect to any $\tilde{C}$, which is a geometrical meaning of the condition C1. On the other hand, the non-linear connection N is called to be *metrical* if the supporting elements as the points of a lift $\tilde{C}_N$ of any curve C have a constant length, that is, the (28) yields

$$\frac{d}{dt}(g_{ij}(x, y)y^iy^j)=0. \tag{31}$$

In the case that the Γ is metrical, the non-linear connection N is metrical if and only if

$$g_{jh}y^jD^h_k=0, \qquad \text{or} \quad l_iD^i_k=0. \tag{32}$$

This is easily verified by (22), (23), (28) and (13). Hence, if the condition C2 is satisfied, the non-linear connection N is metrical.

Let $T(x)$ be the fibre $\tau^{-1}x$ over a point $x \in M$ and $F(x)$ be the Finsler subbundle $\pi_1^{-1}T(x)$. If we denote by Γ^v the restriction of the distribution Γ to $F(x)$, the Γ^v is regarded as a linear connection on the differentiable manifold $T(x)$, whose components are C^i_{jk}. Since the (v)v-torsion tensor field S^1 is expressed by $S^i{}_{jk}=C^i_{jk}-C^i_{kj}$, the condition C4 requires this connection Γ^v to be without-torsion.

If we restrict the metric tensor field G to $T(x)$, then the $T(x)$ becomes a Riemannian space. Thus, the connection satisfying (23) and (26) is the Riemannian connection, which is uniquely determined by the G as follows:

$$C_{jhk}=\frac{1}{2}\frac{\partial g_{jh}}{\partial y^k}. \tag{33}$$

Therefore, C_{jhk} are symmetric and the relations

$$C_{jhk}y^k=0, \qquad \text{or} \quad C_{jhk}l^k=0 \tag{34}$$

hold good.

Now, since $F^i_{jk}=\Gamma^i_{jk}-C^i_{jm}F^m_k$, the (h)h-torsion tensor field T, which is expressed by $T^i_{jk}=F^i_{jk}-F^i_{kj}$, depends not only on the Γ but on the N. However, the conditions C1 and C4 do not depend on the N. So, the condition C2 gives an influence upon the definition of the T only. Hence, to determine the Γ only, it seems that the condition C2 is replaced by some weaker conditions.

§ 2. Determinations of Finsler connections by deflection tensor fields

6°. First, we shall consider the case that any non-linear connection is given in the tangent bundle of a Finsler space.

PROPOSITION 1. *Given a non-linear connection N in the tangent bundle of a Finsler space, there exists a unique Finsler connection (Γ, N) satisfying the following four conditions:*

(C1) *the connection Γ be metrical,*

(C2′) *the non-linear connection be the given N,*

(C3) *the (h)h-torsion tensor field $T=0$,*

(C4) *the (v)v-torsion tensor field $S^1=0$.*

The components Γ_{jhk} and C_{jhk} of the Γ are

$$\Gamma_{jhk}=\gamma_{jhk}+\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial y^m}F^m_h-\frac{\partial g_{hk}}{\partial y^m}F^m_j\right), \tag{35}$$

$$C_{jhk}=\frac{1}{2}\frac{\partial g_{jh}}{\partial y^k}, \tag{33}$$

where F^i_k are the components of the given non-linear connection N.
In this case F_{jhk} are

$$F_{jhk}=\gamma_{jhk}-\frac{1}{2}\left(\frac{\partial g_{jh}}{\partial y^m}F^m_k+\frac{\partial g_{hk}}{\partial y^m}F^m_j-\frac{\partial g_{jk}}{\partial y^m}F^m_h\right), \tag{36}$$

and if we put

$$\frac{\delta}{\partial x^k}=\frac{\partial}{\partial x^k}-F^m_k\frac{\partial}{\partial y^m}, \tag{37}$$

then they are expressed by

$$F_{jhk}=\frac{1}{2}\left(\frac{\delta g_{jh}}{\partial x^k}+\frac{\delta g_{hk}}{\partial x^j}-\frac{\delta g_{jk}}{\partial x^h}\right). \tag{38}$$

PROOF. (33) follows from (23) and (26) as remarked in 5°. If we put

$$\Gamma_{jhk}=\gamma_{jhk}+\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial y^m}F^m_h-\frac{\partial g_{hk}}{\partial y^m}F^m_j\right)+A_{jhk}, \tag{39}$$

then we obtain by (22) and (18)

$$A_{jhk}+A_{hjk}=0, \tag{40}$$

and by (11), (33) and (25)

$$A_{jhk}=A_{khj}. \tag{41}$$

From these equations it follows that $A_{jhk}=0$. Hence, (39) becomes (35), and (36) follows.

And the Γ defined by (35) and (33) satisfies with the N our conditions.

From (36) and (34), we have

$$y^j F^i_{jk}=y^j \gamma^i_{jk}-\frac{1}{2}g^{ih}\frac{\partial g_{hk}}{\partial y^m}F^m_j y^j. \tag{42}$$

We may solve F^i_k from (13) and (42), and obtain

$$F^i_k=G^i_k+C^i_{kl}D^l_s y^s-D^i_k. \tag{43}$$

Substituting (43) into (35), we have

PROPOSITION 2. *Given a Finsler tensor field D of type* (1, 1) *in a Finsler space, there exists a unique Finsler connection* (Γ, N) *satisfying the following four conditions*:

(C1) *the connection Γ be metrical*,

(C2$''$) *the deflection tensor field be the given D*,

(C3) *the (h)h-torsion tensor field $T=0$*,

(C4) *the (v)v-torsion tensor field $S^1=0$*.

The components Γ_{jhk}, C_{jhk} and F^i_k of the (Γ, N) *are*

$$\begin{aligned}\Gamma_{jhk}=\gamma_{jhk}+\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial y^m}G^m_h-\frac{\partial g_{hk}}{\partial y^m}G^m_j\right)\\+C_{jkm}C^m_{hl}D^l_s y^s-C_{hkm}C^m_{jl}D^l_s y^s-C_{jkm}D^m_h+C_{hkm}D^m_j,\end{aligned} \tag{44}$$

$$C_{jhk}=\frac{1}{2}\frac{\partial g_{jh}}{\partial y^k}, \tag{33}$$

and

$$F^i_k=G^i_k+C^i_{kl}D^l_s y^s-D^i_k, \tag{43}$$

where D^i_k are the components of the given Finsler tensor field D.

7°. Proposition 2 shows that the connection Γ determined in Proposition 1 or 2 is the one defined by E. Cartan if and only if

$$C_{jkm}C^m_{hl}D^l_s y^s-C_{hkm}C^m_{jl}D^l_s y^s-C_{jkm}D^m_h+C_{hkm}D^m_j=0. \tag{45}$$

It is easily verified by (34) that (45) is equivalent to

(46) $$C_{jkm}D^m_h = C_{hkm}D^m_j, \quad \text{or} \quad C_{jhm}D^m_k = C_{khm}D^m_j.$$

Thus we have

THEOREM A. *Given a Finsler tensor field D of type* (1, 1) *in the Finsler bundle of a Finsler space, there exists a unique Finsler connection* (Γ, N) *satisfying the following four conditions*:

(C1) *the connection Γ be metrical,*

(C2″) *the deflection tensor field be the given D,*

(C3) *the (h)h-torsion tensor field $T=0$,*

(C4) *the (v)v-torsion tensor field $S^1=0$.*

And, a necessary and sufficient condition that the Γ thus determined be the one defined by E. Cartan is that the deflection tensor field D satisfies the condition

(47) $$C(f_1, D(f_2)) = C(f_2, D(f_1)),$$

where C is the (h)hv-torsion tensor field of the (Γ, N), or equivalently that the components D^i_k of the deflection tensor field D satisfy the conditions

(48) $$\frac{\partial g_{jh}}{\partial y^m} D^m_k = \frac{\partial g_{kh}}{\partial y^m} D^m_j.$$

In this case the conditions

(49) $$\frac{\partial g_{jh}}{\partial y^m} D^m_s y^s = 0$$

hold good, and the components Γ_{jhk}, C_{jhk} and F^i_k of the (Γ, N) are

(50) $$\Gamma_{jhk} = \gamma_{jhk} + \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial y^m} G^m_h - \frac{\partial g_{hk}}{\partial y^m} G^m_j\right),$$

(33) $$C_{jhk} = \frac{1}{2}\frac{\partial g_{jh}}{\partial y^k},$$

and

(51) $$F^i_k = G^i_k - D^i_k.$$

8°. As a special example of the D satisfying the condition (48), we have

PROPOSITION 3. *In a Finsler space there exists a unique Finsler connection (Γ, N) satisfying the following four conditions*:

(C1) *the connection Γ be metrical,*

(C2‴) *the deflection tensor field D be given by*

(52) $$D^i_k = \lambda l^i l_k + \mu \delta^i_k,$$

where λ and μ are scalar functions on the tangent bundle,

(C3) *the (h)h-torsion tensor field* $T=0$,

(C4) *the (v)v-torsion tensor field* $S^1=0$.

The connection Γ is the one defined by E. Cartan. And, the non-linear connection N is metrical if and only if $\lambda+\mu=0$.

This is easily proved by (34) and (32). Thus, we have noticed that, in order to determine the connection Γ defined by E. Cartan, the condition (C2) may be replaced by the weaker condition (C2‴). If we take D in (C2‴) such that

$$D^i_k=\lambda(l^i l_k-\delta^i_k), \tag{53}$$

then the non-linear connection N is metrical, and so we have a generalization of the $(\Gamma, \underline{N})$ defined by E. Cartan.

However, in order to obtain the Γ only, it does not need the non-linear connection to be metrical. In particular, if $\lambda=0$, $\mu=-1$ (i.e. $D=-\delta$) then the components F^i_k of the non-linear connection N become $F^i_k=G^i_k+\delta^i_k$, which are somewhat canonical in features. So, it seems to be interesting that, apart from Finsler metrics, we treat Finsler connections with the deflection tensor field $D=-\delta$. Next, we shall give an example of such a Finsler connection.

§ 3. Finsler connections derived from affine connections

9°. Let $F(M)$ $(M, \tilde{\pi}, \tilde{G})$ be the *affine bundle* over M, where $\tilde{G}=GL(n, R)\times F$ is the affine group with the multiplication

$$(g_1, v_1)(g_2, v_2)=(g_1 g_2, g_1 v_2+v_1). \tag{54}$$

Each $(g, v)\in\tilde{G}$ acts on $F(M)$ by

$$\tilde{T}_{(g,v)}\colon F(M)\to F(M)\,|\,(y, z)\to(y+zv, zg),$$

so we have the restrictions

$$T_g\colon F(M)\to F(M)\,|\,(y, z)\to(y, zg)$$

and

$$S_v\colon F(M)\to F(M)\,|\,(y, z)\to(y+zv, z).$$

Therefore, a connection in the affine bundle is invariant not only by T_g but by S_v.

The Lie algebra of the structural group $\tilde{G}$ is $L(n, R)+F$, if we identify the Lie algebra of the additive group F with F itself. If we denote by $Z(A)$ and $Y(f)$ the respective fundamental vector fields corresponding to $A\in L(n, R)+0$ and $f\in 0+F$, then $Z(A)$ is also the fundamental vector field in the Finsler bundle $F(M)$, and $Y(f)$ is the induced fundamental vector field.

The *induced vertical distribution* F^i defined by

$$F(M) \ni u \to \{X \in F(M)_u \mid \pi_2 X = 0\}$$

is spanned by $Y(f)$, where $F(M)_u$ is the tangent space at $u \in F(M)$.

10°. Let $\tilde{\Gamma}$ be a connection in the affine bundle $F(M)$. Then, a Finsler connection (Γ, N) is obtained by pairing $\tilde{\Gamma}$ with the induced vertical distribution F^i. In this case the v-basic vector field $B^v(f)$ is $Y(f)$.

Since the $\tilde{\Gamma}$ is S_v-invariant, the h-basic vector field $B^h(f)$ is S_v-invariant. Therefore, the subordinate F-connection to (Γ, N) is a linear connection and the deflection tensor field D of (Γ, N) is S_v-invariant.

Now, we shall treat the connection forms.

PROPOSITION 4. *Let $\tilde{\omega}$ and ω be the connection forms of $\tilde{\Gamma}$ and Γ respectively. If we consider the form $\omega + \theta^v$ to take values in the Lie algebra $L(n, R) + F$, then*

$$\tilde{\omega} = \omega + \theta^v. \tag{55}$$

PROOF. Since $(\theta^h, \theta^v, \omega)$ constitutes the dual system of $(B^h(f), Y(f), Z(A))$, we have

$$(\omega + \theta^v)\,(B^h(f)) = 0, \tag{56}$$

$$(\omega + \theta^v)\,(Z(A)) = A, \quad (\omega + \theta^v)(Y(f)) = f. \tag{57}$$

These relations show that $\omega + \theta^v$ is just the connection form $\tilde{\omega}$ of the $\tilde{\Gamma}$. Because, with respect to the connection in the affine bundle $F(M)$ over M, the horizontal subspace is spanned by $B^h(f)$ and the vertical subspace by the fundamental vector fields $Z(A)$ and $Y(f)$.

PROPOSITION 5. *Let $\tilde{\omega}$ be the connection form of $\tilde{\Gamma}$, and $\underline{\omega}$ be the connection form of the subordinate linear connection to (Γ, N). If ι is the injection*

$$\iota : L(M) \to F(M) \mid z \to (0, z),$$

then

$$\iota^* \tilde{\omega} = \underline{\omega} - D(\theta). \tag{58}$$

The proof will be obtained from (55), (16), (12) and (10). A connection Γ in the affine bundle is canonical, if the $\iota^* \tilde{\omega}$ has the form

$$\iota^* \tilde{\omega} = \underline{\omega} + \theta, \tag{59}$$

and is called the *affine connection* [3]. The formula (58) shows that the connection $\tilde{\Gamma}$ is affine if and only if

$$D = -\delta. \tag{60}$$

Thus we have

THEOREM B. *Let $\tilde{\Gamma}$ be a connection in the affine bundle $F(M)$ over M. Then, a Finsler connection (Γ, N) of M may be defined by the Finsler pair $(\tilde{\Gamma}, F^i)$, where F^i is the induced vertical distribution. Its subordinate F-connection becomes a linear connection and its deflection tensor field D is S_v-invariant. In particular, the connection $\tilde{\Gamma}$ is an affine connection of M if and only if $D=-\delta$.*

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