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By

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§ 1. Introduction

V. KRISHNAMURTHY [1] has intended to estimate the number of the possible topologies in a given finite set, in which he has mentioned that a topology of a finite set can be represented by a suitable matrix. It seems that such matrix is a powerful tool to study the structure of finite topological spaces.

After defining a topogenous matrix in section 2, we shall consider some elementary properties of finite spaces in section 3. Especially, it is shown that the topogenous matrix for the product space $X \times Y$ is the direct product of the topogenous matrices of X and Y . In section 4, a concept of a dual space of a finite space is introduced. In section 5, it is shown that the topogenous matrix of a finite T_0 -space is equivalent to a triangular matrix. In section 6, a topological invariant which we call be the degree of connection is defined.

§ 2. Topogenous matrix

Let X be a finite set $\{a_1, a_2, \dots, a_n\}$ and let τ be a topology on X , where τ is the family of all open subsets of X .

Since X is finite, each point a_i of X has a unique minimal neighborhood U_i , which is the intersection of open neighborhoods of a_i . Hence if X is the set $\{a_1, a_2, \dots, a_n\}$, then $\{U_1, U_2, \dots, U_n\}$ is an open neighborhood basis of the space (X, τ) .

Now, we shall introduce the topogenous matrices which play an important role in our investigation of finite topological spaces.

DEFINITION 1. In a finite space (X, τ) , let a (n, n) matrix $A = [a_{ij}]$ be defined as follows:

$$\begin{aligned} a_{ij} &= 1 && \text{if } a_j \in U_i \\ &= 0 && \text{otherwise} \quad (i, j = 1, 2, \dots, n). \end{aligned}$$

Then, the matrix A is said to be a *topogenous matrix* of the space (X, τ) .

A topogenous matrix has the following important properties.

THEOREM 1. (V. KRISHNAMURTHY). *Let A be the topogenous matrix of a finite space X , then the matrix A satisfies the following three conditions*

- (1) $a_{ij}=0$ or 1 ,
- (2) $a_{ii}=1$,
- (3) $a_{ik}=a_{kj}=1 \Rightarrow a_{ij}=1$,

where $i, j, k=1, 2, \dots, n$.

Conversely, if a matrix $A=[a_{ij}]$ satisfies these three conditions, then A induces a topology in X .

DEFINITION 2. We define the *permutation matrix*, which corresponds to the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ p(1) & p(2) & \dots & p(n) \end{pmatrix}$, by $T=[\delta_{ip(j)}]$ where $\delta_{ip(j)}$ is the Kronecker's delta.

DEFINITION 3. Given two topogenous matrices A and B , if there exists a permutation matrix T such that

$$B = T'AT,$$

then A and B are said to be *equivalent* to each other, and are noted as

$$A \sim B.$$

THEOREM 2. Let τ_1 and τ_2 be two topologies in a finite set X , and let A_1 and A_2 be the topogenous matrices of (X, τ_1) and (X, τ_2) respectively. Then (X, τ_1) and (X, τ_2) are homeomorphic if and only if A_1 and A_2 are equivalent.

PROOF. Let $\mathbf{B}_1 = \{U_1, U_2, \dots, U_n\}$ and $\mathbf{B}_2 = \{V_1, V_2, \dots, V_n\}$ be the minimal basic neighborhood systems of (X, τ_1) and (X, τ_2) , respectively, and let $f : (X, \tau_1) \rightarrow (X, \tau_2)$ be a homeomorphism such that

$$f(U_i) = V_{p(i)} \quad (i=1, 2, \dots, n).$$

f induces a mapping

$$f(U_i) = V_{p(i)} \quad (i=1, 2, \dots, n),$$

which preserves the inclusion relation in \mathbf{B}_1 and \mathbf{B}_2 . If A_1 is noted as $[a_{ij}]$, then we have

$$\begin{aligned} a_{ij}=1 &\Leftrightarrow U_j \subset U_i \Leftrightarrow V_{p(j)} \subset V_{p(i)} \\ &\Leftrightarrow a_{p(i)p(j)}=1, \end{aligned}$$

and

$$a_{ij}=a_{p(i)p(j)} \quad (i, j=1, 2, \dots, n).$$

Set

$$T=[\delta_{ip(j)}],$$

where $\delta_{ip(j)}$ is the Kronecker's delta.

Then we have

$$A_2 = T' A_1 T,$$

i. e.

$$A_1 \sim A_2.$$

Conversely assume $A_1 \sim A_2$, then there exists a permutation matrix $T = [\delta_{ip(j)}]$ such that $A_2 = T' A_1 T$. Define $f : (X, \tau_1) \rightarrow (X, \tau_2)$ by

$$f(a_i) = a_{p(i)},$$

then f is a homeomorphism.

From the above proof we also obtain:

COROLLARY. *A matrix which is equivalent to a certain topogenous matrix is a topogenous matrix.*

§ 3. Some elementary properties

We consider a topological space (X, τ) with a finite set $X = \{a_1, a_2, \dots, a_n\}$ and a family $\{U_1, U_2, \dots, U_n\}$ of the corresponding minimal basic neighborhoods. And between elements of (X, τ) , we define the following order:

$$a_i \leq a_j \Leftrightarrow U_i \subset U_j \quad (\text{or } a_i \in U_j),$$

then this relation \leq is transitive and reflexive. This means that (X, \leq) is a quasi ordered set.

The following is the known result.

THEOREM 3. (ALEXANDROFF [2]). *A finite space (X, τ) is a T_0 -space if and only if (X, \leq) is a partially ordered set.*

REMARK 1. Let $A = [a_{ij}]$ be the topogenous matrix of a finite space (X, τ) , then

$$a_{ij} = 1 \Leftrightarrow a_j \leq a_i.$$

REMARK 2. Given two finite spaces (X, τ) and (X, σ) , mapping $f : X \rightarrow Y$ is continuous if and only if the following is satisfied:

$$a \leq b(a, b \in X) \Rightarrow f(a) \leq f(b).$$

Let A be the topogenous matrix of a finite space (X, τ) , where $X = \{a_1, a_2, \dots, a_n\}$. For a subset $Y = \{b_1, b_2, \dots, b_k\}$ of X , we consider the subspace (Y, τ_Y) and its topogenous matrix A_Y .

We have the following theorem.

THEOREM 4. *(Y, τ_Y) is closed in (X, τ) if and only if the following holds:*

$$A \sim \left[\begin{array}{c|c} A_Y & * \\ \hline 0 & A_{Y^c} \end{array} \right] \quad \text{or} \quad A \sim \left[\begin{array}{c|c} A_{Y^c} & 0 \\ \hline * & A_Y \end{array} \right].$$

PROOF. Y is closed in (X, τ) if and only if $U_x \cap Y = \emptyset$ for $x \in Y^c$ and for the minimal neighborhood U_x of x . Hence from the definition of the topogenous matrix, the theorem follows immediately.

COROLLARY 1. Y is open and closed if and only if the following holds:

$$A \sim \left[\begin{array}{c|c} A_Y & 0 \\ \hline 0 & A_{Y^c} \end{array} \right].$$

COROLLARY 2. Let A be the topogenous matrix of a finite space (X, τ) . Then we have

$$A \sim \left[\begin{array}{ccc|c} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \dots & \\ 0 & & & \boxed{A_k} \end{array} \right]$$

where A_i is a topogenous matrix of a component of (X, τ) .

For finite spaces (X, τ) and (Y, σ) , let A_X and A_Y be the topogenous matrices of (X, τ) and (Y, σ) , respectively, and let $A_{X \times Y}$ be the topogenous matrix of the product space $(X \times Y, \tau \times \sigma)$. As for the relations between the matrices A_X , A_Y and the matrix $A_{X \times Y}$. We have the following theorem.

THEOREM 5. Let A_X , A_Y and $A_{X \times Y}$ be the topogenous matrices of finite spaces (X, τ) , (Y, σ) and the product space $(X \times Y, \tau \times \sigma)$, respectively. Then $A_{X \times Y}$ is equivalent to the direct product of A_X and A_Y , i. e.

$$A_{X \times Y} \sim A_X \times A_Y.$$

PROOF. Let $X = \{a_1, a_2, \dots, a_m\}$ and $Y = \{b_1, b_2, \dots, b_n\}$, $\{U_1, U_2, \dots, U_m\}$ and $\{V_1, V_2, \dots, V_n\}$ be families of the minimal basic neighborhoods, respectively. And let $A_X = [a_{ij}]$ and $A_Y = [b_{ij}]$ be the topogenous matrices of these spaces.

Now, consider

$$X \times Y = \{(a_i, b_j) \mid i=1, 2, \dots, m; \quad j=1, 2, \dots, n\}.$$

If U_i and V_j are minimal neighborhoods of a_i in X and b_j in Y , then $U_i \times V_j$ is the minimal neighborhood of (a_i, b_j) in $X \times Y$. Then the topogenous matrix $A_{X \times Y}$ is noted in the form

$$A_{X \times Y} = \begin{matrix} U_1 \times V_1 \\ \vdots \\ U_i \times V_j \\ \vdots \\ U_m \times V_n \end{matrix} \left[\begin{matrix} (a_1, b_1) \cdots (a_h, b_k) \cdots (a_m b_n) \\ \vdots \\ \cdots \cdots c_{(i,j)(h,k)} \cdots \cdots \\ \vdots \end{matrix} \right],$$

where

$$\begin{aligned} c_{(i,j)(h,k)} = 1 &\Leftrightarrow (a_h, b_k) \in U_i \times V_j \\ &\Leftrightarrow a_h \in U_i \quad \text{and} \quad b_k \in V_j \\ &\Leftrightarrow a_{ih} = 1 \quad \text{and} \quad b_{jk} = 1. \end{aligned}$$

Hence, we have $A = A_X \times A_Y$.

§ 4. Dual spaces

A finite space is a quasi ordered set (X, \geq) . Then there exists a quasi ordered set $(X, <)$ in which the ordering $a < b$ means $a \geq b$ in (X, \geq) . The *dual space* of a finite topological space (X, τ) is a topological space corresponding to $(X, <)$. The following is evident.

LEMMA 1. *Let M be the (n, n) topogenous matrix of a finite space (X, τ) . Then a space $(X, \tilde{\tau})$ is a dual space of (X, τ) if and only if the topogenous matrix N of $(X, \tilde{\tau})$ is equivalent to the transposed matrix M' of M .*

Let $M = [a_{ij}]$ be a (n, n) topogenous matrix. Then we define a matrix $M^* = [a_{ij}^*]$ by

$$a_{ij}^* = a_{n-j+1 \ n-i+1}.$$

The matrix M^* is obtained from mapping reflectively all elements of M in the diagonal which is not principal. We have the following.

THEOREM 6. *Let M be the (n, n) topogenous matrix of a finite space (X, τ) . Then*

$$M' \sim M^*$$

is satisfied.

PROOF. If we put $M = [a_{ij}]$, $M' = [a'_{ij}]$, $M^* = [a_{ij}^*]$, then we have

$$a_{ij}^* = a_{n-j+1 \ n-i+1} = a'_{n-i+1 \ n-j+1}.$$

Now let $T(i, j)$ be the permutation matrix which corresponds to a transposition (i, j) , and consider

$$T = T(n, 1)T(n-1, 2) \dots T(n-k, k+1) \dots \quad \left(k \leq \frac{n-1}{2}\right).$$

Then we have

$$T' M^* T = M',$$

i. e.

$$M^* \sim M'.$$

§ 5. Finite T_0 -spaces

In the present section, we shall find a condition for the topogenous matrix that a given finite topological space is a T_0 -space.

THEOREM 7. *A finite space (X, τ) is a T_0 -space if and only if the topogenous matrix A of (X, τ) is equivalent to a certain triangular matrix.*

PROOF. Assume that (X, τ) is a finite T_0 -space with a topogenous matrix A . Let $X = \{a_1, a_2, \dots, a_n\}$, and let U_i be the minimal neighborhood of a_i . We note N_i the number of the elements of U_i , and rearrange X as $X = \{a_{p_1}, a_{p_2}, \dots, a_{p_n}\}$ such that if $i \leq j$, then $N_{p_i} \leq N_{p_j}$. We consider the topogenous matrix $B = [b_{ij}]$ which corresponds to the new basis $(a_{p_1}, a_{p_2}, \dots, a_{p_n})$ of X . If $i < j$, then we have $a_{p_i} \notin U_{p_j}$, and $b_{ij} = 0$. Therefore, B is a triangular matrix which is equivalent to A .

Conversely, assume that for a topogenous matrix A of (X, τ) , there exists a triangular matrix $A_1 = [a_{ij}]$ such that $A \sim A_1$. If $a_i, a_j \in X$ and $a_i < a_j$, then $a_{ij} = 0$ since A_1 is a triangular matrix. Hence $a_j \notin U_i$, that is, (X, τ) is a T_0 -space.

§ 6. Degree of connection

Let (X, τ) be a finite T_0 -space and let $A = [a_{ij}]$ be the topogenous matrix of X , and E be the unit matrix.

Now, set

$$dA = A - E,$$

and define

$$d^p A = (dA)^p \quad (p=1, 2, \dots),$$

$$d^0 A = E.$$

DEFINITION 4. Let (X, τ) be a finite T_0 -space and let A be the topogenous matrix of X . If

$$d^n A \neq 0 \quad \text{and} \quad d^{n+1} A = 0,$$

then we say that the *degree of connection* of X is n , and we denote

$$\text{deg} X = n.$$

Let (X, \geq) be a partially ordered set, and x be an element of X . Consider the family of chains in X such that

$$x_0 < x_1 < x_2 < \dots < x_h = x.$$

The maximum number of h defined for all the above chains is denoted by $l[x]$, and is called the *length* of x . We also define the *length* $l[X]$ of X by

$$l[X] = \max\{l[x] \mid x \in X\}.$$

THEOREM 8. *If (X, τ) is a finite T_0 -space, then*

$$\text{deg} X = l[X].$$

PROOF. Let $A = [a_{ij}]$ be the triangular topogenous matrix. Set

$$d^p A = [a_{ij}(p)] \quad (p = 1, 2, \dots),$$

and let $\mu\{(a_j, a'_1, a'_2, \dots, a'_{p-1}, a_i) \mid a_j < a'_1 < a'_2 < \dots < a'_{p-1} < a_i\}$ be the number of the different chains of the form $a_j < a'_1 < a'_2 < \dots < a'_{p-1} < a_i$.

Then we shall prove by the induction that

$$(1) \quad a_{ij}(p) = \mu\{(a_j, a'_1, a'_2, \dots, a'_{p-1}, a_i) \mid a_j < a'_1 < a'_2 < \dots < a'_{p-1} < a_i\}.$$

In the case that $p=1$, we have evidently

$$a_{ij}(1) = \begin{cases} 1 & \text{for } a_j < a_i \\ 0 & \text{otherwise} \end{cases},$$

on the other hand,

$$(2) \quad \mu\{(a_j, a_i) \mid a_j < a_i\} = \begin{cases} 1 & \text{if } a_j < a_i \\ 0 & \text{otherwise} \end{cases},$$

Hence, we obtain (1) in the case that $p=1$.

Second, we assume that (1) is valid for p , and consider the case of $p+1$.

Remarking that

$$(3) \quad (dA)^{p+1} = (dA)^p(dA),$$

we have

$$\begin{aligned} a_{ij}(p+1) &= \sum_{k=1}^n a_{ik}(p) a_{kj}(1). \\ &= \sum_{k=1}^n [\mu\{(a_k, a'_1, \dots, a'_{p-1}, a_i) \mid a_k < a'_1 < \dots < a'_{p-1} < a_i\} \cdot \mu\{(a_j, a_k) \mid a_j < a_k\}] \end{aligned}$$

$$= \mu\{(a_j, a_k, a'_1, \dots, a'_{p-1}, a_i) \mid a_j < a_k < a'_1 < \dots < a'_{p-1} < a_i\}.$$

So, we obtain (1) in the case of $p+1$.

Hence, by the induction, (1) is established.

Now, let $\text{deg} X = m$, then

$$d^m A \neq 0 \quad \text{and} \quad d^{m+1} A = 0.$$

Hence by (1) there exists at least a chain of the length m , but no chains of the length $m+1$.

Therefore

$$\text{deg} X = l[X].$$

Reference

- [1] V. KRISHNAMURTHY: On the number of topologies on a finite set. Amer. Math. Monthly 73, (1966), 154-157.
- [2] P. ALEXANDROFF: Diskrete Räume. Mat. Sb. (N.S) Vol. 2, (1937), 501-518.