

## On Set Representations and Intersection Numbers of Some Graphs

著者	SAKAI Kouichi
journal or publication title	鹿児島大学理学部紀要=Reports of the Faculty of Science, Kagoshima University
volume	33
page range	39-46
URL	<a href="http://hdl.handle.net/10232/6248">http://hdl.handle.net/10232/6248</a>

# On Set Representations and Intersection Numbers of Some Graphs

著者	SAKAI Kouichi
journal or publication title	鹿児島大学理学部紀要=Reports of the Faculty of Science, Kagoshima University
volume	33
page range	39-46
URL	<a href="http://hdl.handle.net/10232/00010014">http://hdl.handle.net/10232/00010014</a>

# On Set Representations and Intersection Numbers of Some Graphs

Koukichi SAKAI \*

(Received August 11, 2000)

## Abstract

Let  $G$  be any simple and connected graph,  $i(G)$  be the intersection number of  $G$  in the sense of Mckee et al. [6], and  $\theta_1(G)$  be the minimum number of cliques  $\{Q_j\}$  of  $G$  by which the edge set  $E(G)$  is covered. In this paper using the fact that  $i(G) = \theta_1(G)$  we shall determine the intersection number for the graphs as follows: split graphs, the complete  $r$ -partite graphs, the  $(n-2)$ -regular graphs of order  $n$ , and the complementary graphs of the cycle  $C_n$  of order  $n$  respectively.

**Key words:** intersection number, edge clique cover, split graph, the complete  $r$ -partite graph, regular graph.

## 1 Introduction and Preliminary

Throughout this paper any graphs are assumed always to be finite, simple and connected. The terminology and notion concerning graphs follow Chartrand et al. [5] unless otherwise stated. Let  $G$  be a graph with the vertex set  $V(G)$  and the edge set  $E(G)$ , and put  $q(G) = |E(G)|$ . We begin with the definition of set representations and edge clique covers of  $G$ . A *set representation* of  $G$  is a mapping  $\phi$  of  $V(G)$  to the set of non-empty finite sets of positive integers with the following property:

(1.1) For any  $u, v \in V(G)$ ,  $u$  and  $v$  are adjacent if and only if  $\phi(u) \cap \phi(v) \neq \emptyset$ .

For any set representation  $\phi$  of  $G$ ,  $S(\phi) = \cup\{\phi(u); u \in V(G)\}$  is called the range of  $\phi$ , and  $|\phi| = |S(\phi)|$  is the rank of  $\phi$ . We note that any graphs have set representations. For example let  $E(G) = \{e_1, e_2, e_3, \dots, e_q\}$ , where  $q = q(G)$ , and for any  $u \in V(G)$  we put  $\phi(u) = \{j; e_j \text{ is incident with } u\}$ . Then this  $\phi$  is a representation of  $G$  with  $|\phi| = q(G)$ . The *intersection number*  $i(G)$  of  $G$  is the minimum number of  $|\phi|$  for any set representations  $\phi$  of  $G$ . Any representation  $\phi$  of  $G$  is said to be *minimal* if  $|\phi| = i(G)$ .

By the above example of a set representation of  $G$ , we have  $i(G) \leq q(G)$ . A subset  $W$  of  $V(G)$  is called an *independent set* of  $G$  if any distinct two vertices in  $W$  are not adjacent. For any independent set  $W$  and set representations  $\phi$  of  $G$ , the sets  $\phi(w)$ ,  $w \in W$ , are mutually disjoint by (1.1). So we have an estimation of  $i(G)$  as follows:

(1.2)  $\beta_0(G) \leq i(G) \leq q(G)$ ,

---

\* Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima 890-0065, Japan.

where the  $\beta_0(G)$  is the vertex independent number of  $G$ , *i.e.* the maximum cardinal number of independent sets of  $G$ .

Any complete subgraph of  $G$  is called a *clique*, and especially is *maxclique* if it is not properly contained in another cliques. A family of cliques  $\mathbf{F} = \{Q_j; j = 1, 2, \dots, m\}$  of  $G$  is called an *edge* [resp. *vertex*] *clique cover* of  $G$  if  $E(G) \subseteq \cup_{j=1}^m E(Q_j)$  [resp.  $V(G) \subseteq \cup_{j=1}^m V(Q_j)$ ]. For example the edge set  $E(G)$  and the set  $MC(G)$  of all maxcliques of  $G$  are edge clique covers of  $G$ .

Here we introduce the following three kinds of numbers for  $G$  defined by:

$$(1.3) \quad \theta_1(G) = \text{the minimum cardinal number of } |\mathbf{F}| \text{ for any edge clique covers } \mathbf{F} \text{ of } G,$$

$$(1.4) \quad \theta_0(G) = \text{the minimum cardinal number of } |\mathbf{F}| \text{ for any vertex clique covers } \mathbf{F} \text{ of } G,$$

$$(1.5) \quad \theta_m(G) = |MC(G)|.$$

We say that any edge [resp. vertex] clique cover  $\mathbf{F}$  of  $G$  is *minimal* if  $|\mathbf{F}| = \theta_1(G)$  [resp.  $\theta_0(G)$ ]. By the definitions above we have

$$(1.6) \quad \beta_0(G) \leq \theta_0(G) \leq \theta_1(G) \leq \theta_m(G).$$

The aim of this paper is to determine the intersection numbers for some graphs. In section 2 we state the relationship between set representations and edge clique covers of  $G$  following [6]. By the relationship we have  $i(G) = \theta_1(G)$ . We note also the conditions for  $G$  to be  $i(G) = q(G)$ ,  $i(G) = \theta_m(G)$  and  $i(G) = \beta_0(G)$  respectively.

In section 3, the relations of  $i(G)$  and  $\beta_0(G)$  are established for any split graphs  $G$ , and the intersection number of the complete  $r$ -partite graphs is given. The intersection numbers of the  $(n-2)$ -regular graph of order  $n$  and the complementary graph of the cycle  $C_n$  are determined respectively in the final section.

For any graph  $G$  we use the following notation:

$$(1.7) \quad N[u] = N(u) \cup \{u\} \text{ for } u \in V(G), \text{ where } N(u) \text{ is the neighborhood of } u.$$

$$(1.8) \quad E(G, u) \text{ is the set of all edges of } G \text{ incident with } u \in V(G).$$

$$(1.9) \quad \langle W \rangle = \text{the induced subgraph of } G \text{ from } W \subset V(G).$$

$$(1.10) \quad \langle W_1, W_2, \dots, W_k \rangle = \langle \cup_{j=1}^k W_j \rangle.$$

Throughout this paper any symbols of variables, say  $i, j, k, m, n$  and so on, denote any positive integers unless otherwise stated. For any  $m, n$  with  $m \leq n$ ,  $[m, n]$  is the set of the consecutive numbers from  $m$  to  $n$ , and especially  $[n] = [1, n]$ .

## 2 Key Lemma and its Consequences

First following [6] we state the relationship between set representations and edge clique covers of any graphs  $G$ . Let  $\phi$  be any set representation of  $G$ . For any  $k \in S(\phi)$  we define a subgraph  $Q_k$  by:

$$(2.1) \quad Q_k = \langle \{u \in V(G); k \in \phi(u)\} \rangle.$$

Then by (1.1) it is seen easily that  $Q_k$  is a clique and  $\mathbf{F}_\phi = \{Q_k; k \in S(\phi)\}$  is an edge clique cover of  $G$ .

Conversely for some  $m$  let  $\mathbf{F} = \{Q_j; j \in [m]\}$  be any edge clique cover of  $G$ . For any  $u \in V(G)$  we define a non-empty subset  $\phi(u)$  of  $[m]$  by:

$$(2.2) \quad \phi(u) = \{j \in [m]; u \in V(Q_j)\}.$$

Then  $\phi$  is a set representation of  $G$ , which is denoted by  $\phi_{\mathbf{F}}$ . We note that

$$(2.3) \quad |\phi| = |\mathbf{F}_{\phi}| \text{ and } |F| = |\phi_{\mathbf{F}}|.$$

Summarizing the above discussion we have our Key lemma.

**Key Lemma.** *Let  $G$  be any graph. Then*

- (1) *For any set representation  $\phi$  of  $G$  there corresponds an edge clique cover  $\mathbf{F}_{\phi}$  of  $G$  defined by (2.1) with  $|\phi| = |\mathbf{F}_{\phi}|$ ,*
- (2) *For any edge clique cover  $\mathbf{F}$  of  $G$  there corresponds a set representation  $\phi_{\mathbf{F}}$  of  $G$  defined by (2.2) with  $|\mathbf{F}| = |\phi_{\mathbf{F}}|$ .  $\square$*

Key lemma teaches us that the study on set representations of  $G$  is equivalent to the study on edge clique covers of  $G$ . The next is an immediate consequence of Key lemma (cf. [6], Slater [8]).

**Theorem 2.1.**  *$i(G) = \theta_1(G)$  for any graphs  $G$ .  $\square$*

From (1.2), (1.6) and Theorem 2.1 we have

$$(2.4) \quad \beta_0(G) \leq i(G) = \theta_1(G) \leq \theta_m(G) \leq q(G).$$

Here we collect some theorems concerning  $i(G)$ , which are immediate consequences of Key Lemma. If  $G$  is triangle-free, then  $MC(G) = E(G)$  and  $E(G)$  is only one edge clique cover of  $G$ . Hence we have

**Theorem 2.2.** *For any graph  $G$ ,  $i(G) = q(G)$  if and only if  $G$  is triangle-free.  $\square$*

An edge  $e$  of  $G$  is said to be *proper* if there is a unique maxclique  $Q$  such that  $e \in E(Q)$ . A maxclique is also said to be *proper* if it has at least one proper edge, and otherwise to be *non-proper*. If any maxcliques of  $G$  are proper, then  $MC(G)$  is the only one edge maxclique cover of  $G$ . In this case we have  $i(G) = \theta_m(G)$ . Conversely If  $G$  has a non-proper maxclique  $Q$ , then  $MC(G) \setminus \{Q\}$  is an edge maxclique cover of  $G$ . Therefore we have

**Theorem 2.3.** *For any graph  $G$ ,  $i(G) = \theta_m(G)$  if and only if every maxclique of  $G$  is proper.  $\square$*

On graphs  $G$  with  $i(G) = \theta_m(G)$  it is investigated in Wallis et al.[10], in which such graph is called a *maximal clique irreducible graph*. In the author's paper [7] some non-proper maxcliques in regular graphs are considered.

We recall that a subset  $S$  of  $V(G)$  is a *dominating set* of  $G$  if  $V(G) = \cup\{N[s]; s \in S\}$ .

**Theorem 2.4.** *For any graph  $G$  let  $W = \{v_1, v_2, \dots, v_m\}$  be an independent set of  $G$ , where  $m = \beta_0(G)$ , and put  $Z = V(G) \setminus W$ . Then the following conditions for  $G$  are mutually equivalent.*

- (1)  $i(G) = \beta_0(G)$ ,
- (2)  $W$  is a dominating set of  $G$ , and for any  $u, v \in Z$ ,  $u$  and  $v$  are adjacent if and only if  $N(u) \cap N(v) \cap W \neq \emptyset$ ,
- (3) For every  $v_j \in W$ ,  $Q_j := \langle N[v_j] \rangle$  is a maxclique and the family  $\{Q_j; j \in [m]\}$  is an edge maxclique cover of  $G$ .

**Proof.** It suffices to see (1)  $\Rightarrow$  (2). Let  $\phi$  be a set representation of  $G$  with  $|\phi| = \beta_0(G)$ . Then we may assume that  $S(\phi) = [m]$  and  $\phi(v_j) = \{j\}$  for  $j \in [m]$ . Hence for any  $u \in Z$

we have  $\phi(u) = \{k \in [m]; v_k \in N(u)\}$  and for any  $u, v \in Z, \phi(u) \cap \phi(v) \neq \emptyset$  if and only if  $N(u) \cap N(v) \cap W \neq \emptyset$  by (1.1). So (2) follows from (1). (3)  $\Rightarrow$  (1) is obvious from (2.4).  $\square$

In Brigham et al.[3] any graphs  $G$  with  $i(G) = \theta_0(G)$  are characterized in terms of vertex clique covers of  $G$  as follows.

**Theorem 2.5.** *The following conditions for any graph  $G$  are mutually equivalent:*

- (1)  $i(G) = \theta_0(G)$ ,
- (2) *Every minimal vertex clique cover of  $G$  is a minimal edge clique cover,*
- (3)  $\beta_0(G) = \beta_1(G)$ ,

where  $\beta_1(G)$  is the maximum number of edges of  $G$  having the property that no two are in the same clique.  $\square$

Combining Theorems 2.3 and 2.4 we have

**Theorem 2.6.** *Under the same notation as in Theorem 2.4, the following conditions for  $G$  are mutually equivalent:*

- (1)  $i(G) = \beta_0(G) = \theta_m(G)$ ,
- (2) *For every  $v_j \in W, Q_j := \langle N[v_j] \rangle$  is a maxclique and  $MC(G) = \{Q_j; j \in [m]\}$ .*  $\square$

**Examples 2.7.** The following are examples of graphs as in the above theorems:

- (1)  $i(G) = q(G)$  for  $G =$  any trees or bipartite graphs.
- (2)  $i(G) = \theta_m(G)$  for  $G =$  the wheel graphs  $W_n, n > 3$ , the interval graphs (cf. [10]), or 3-regular graphs (cf.[7], [9]).
- (3)  $i(G) = \beta_0(G) < \theta_m(G)$  for  $G =$  the 3-sun (Hajos-graph)  $S_3$  (cf. [2]).
- (4)  $i(G) = \beta_0(G) = \theta_m(G)$  for  $G =$  the complete graphs, or the intersection graphs  $\Omega(P([m]))$  for any  $m \geq 2$ . Where  $P([m])$  is the family of all non-empty subsets of  $[m]$ ,  $\Omega(P([m]))$  is the graph with the vertex set  $P([m])$  such that any distinct  $S_j, S_k \in P([m])$  are adjacent if and only if  $S_j \cap S_k \neq \emptyset$ .

In [7] any  $r$ -regular graph  $G$  with  $i(G) < \theta_m(G)$  is characterized for  $r = 4$  and 5. For example any 4-regular graph  $G, i(G) < \theta_m(G)$  if and only if  $G$  contains the 3-sun graphs as induced subgraphs.

**Theorem 2.8.** *For any given  $n > 1$  the set of the intersection numbers  $i(G)$  of graphs  $G$  of order  $n$  is the integer interval  $[\lfloor \frac{n^2}{4} \rfloor]$ .*

**Proof.** In Erdős et al.[4] it is proved that  $i(G) \leq \lfloor \frac{n^2}{4} \rfloor$  for any graph  $G$  of order  $n$ , and that  $i(G_0) = \lfloor \frac{n^2}{4} \rfloor$  for the complete bipartite graph  $G_0 = K(k, k)$  or  $K(k+1, k)$  according as even  $n = 2k$  or odd  $n = 2k+1$ . Note that any bipartite graph is triangle-free. So removing repeatedly a cycle edge from  $G_0$  until its spanning tree, for every  $k \in [n-1, \lfloor \frac{n^2}{4} \rfloor]$  we get a bipartite graph  $G$  with  $i(G) = k$ .

On the other hand, let  $p \in [2, n-1]$ , and define a graph  $G_p$  of order  $n$  as follows:  $V(G_p) = \{v_1, v_2, \dots, v_{p-1}, v_p, v_{p+1}, \dots, v_n\}$  such that  $Q := \langle v_p, v_{p+1}, \dots, v_n \rangle$  is the complete subgraph  $K_{n-p+1}$  and  $E(G_p) = E(Q) \cup \{v_j v_n; j \in [p-1]\}$ . Then  $i(G_p) = p$  by Theorem 2.4. Obviously  $i(K_n) = 1$ . This completes the proof.  $\square$

### 3 Intersection numbers of split graphs and the complete $r$ -partite graphs

Let  $G$  be any connected split graph, i.e.,  $V(G)$  is partitioned into two sets  $D$  and  $S$  such that  $\langle D \rangle$  is a clique and  $S$  is an independent set. Let  $k = |D|$  and  $s = |S|$ . Note that  $\langle N[s] \rangle$  is a clique for every  $s \in S$  and  $\beta_0(G)$  is equal to  $s$  or  $s + 1$ . Dividing the following three cases (a)-(c) we determine  $i(G)$ .

Case (a): There is a  $v \in D$  with  $\deg(v) = k - 1$ .

Then  $N[v] = D$  and  $W = S \cup \{v\}$  is the maximal independent set. Moreover the family  $\{\langle N[w] \rangle; w \in W\}$  of cliques is an edge clique cover of  $G$ . Hence  $i(G) = \beta_0(G) = s + 1$  by Theorem 2.4.

Case (b): For any distinct  $u, v \in D$  there exists an  $s \in S$  for which  $u, v \in N(s)$ .

Then  $S$  is the maximal independent subset of  $V(G)$  and the family  $\{\langle N[s] \rangle; s \in S\}$  is an edge clique cover of  $G$ . Hence  $i(G) = \beta_0(G) = s$ . In this case note that  $\deg(u) \geq k$  for all  $u \in D$ .

Case (c):  $\deg(u) \geq k$  for all  $u \in D$  and there is at least one edge of  $\langle D \rangle$  which is not covered by any cliques  $\langle N[s] \rangle, s \in S$ .

Then  $S$  is the maximal independent subset of  $V(G)$  and  $\{\langle D \rangle\} \cup \{\langle N[s] \rangle; s \in S\}$  is the minimal edge clique cover of  $G$ . Hence  $i(G) = s + 1$  and  $\beta_0(G) = s$ .

Summarizing we have

**Theorem 3.1.** *Let  $G$  be a connected split graph and let  $\mathbf{F} = \cup_{s \in S} E(\langle N[s] \rangle)$  under the above notation. Then the following holds:*

- (1)  $i(G) = \beta_0(G) = |S| + 1$  if and only if  $S$  is not a dominating set of  $G$ .
- (2)  $i(G) = \beta_0(G) = |S|$  if and only if  $E(\langle D \rangle) \subset \mathbf{F}$ .
- (3)  $i(G) = |S| + 1, \beta_0(G) = |S|$  if and only if  $S$  is a dominating set of  $G$  and  $E(\langle D \rangle) \setminus \mathbf{F} \neq \emptyset$ .

□

Next we consider the intersection number of the complete  $r$ -partite graphs with  $r \geq 2$ . For the complete  $r$ -partite graph  $G = K(m_1, m_2, \dots, m_r)$  assume  $m_1 \geq m_2 \geq m_3 \geq \dots \geq m_r \geq 1$  and denote by  $V_j(G)$  the  $j$ -th partite set with  $|V_j(G)| = m_j$  for  $j \in [r]$ .

**Lemma 3.2.**

- (1)  $i(K(m_1, m_2)) = m_1 m_2$ .
- (2)  $i(K(m_1, m_2, \dots, m_{r-1}, 1)) = i(K(m_1, m_2, \dots, m_{r-1}))$ .
- (3)  $i(K(m_1, m_2, \dots, m_{r-1}, m_r + 1)) = i(K(m_1, m_2, \dots, m_{r-1}, m_r)) + m_1$ .

**Proof.** (1) follows from Theorem 2.2. To see (2), we put  $G = K(m_1, m_2, \dots, m_{r-1}), G' = K(m_1, m_2, \dots, m_{r-1}, 1)$  with  $V_r(G') = \{v\}$ , and let  $\{Q_j; j \in [s]\}$  be a minimal edge clique cover of  $G$ , where  $s = i(G)$ . Then for every  $j \in [s], R_j := \langle v, Q_j \rangle$  is a maxclique of  $G'$  and the family  $\{R_j; j \in [s]\}$  is an edge maxclique cover of  $G'$ . Hence we have  $i(G') = i(G)$ , because  $N(v) = \cup_{j=1}^{r-1} V_j(G)$  and  $i(G) \leq i(G')$ . To see (3) put  $G = K(m_1, m_2, \dots, m_r), G' = K(m_1, m_2, \dots, m_r + 1)$  with  $V_r(G') = V_r(G) \cup \{v\}$ , and let  $\{Q_j; j \in [s]\}$  be a minimal edge clique cover of  $G$ , where  $s = i(G)$ . We regard  $G$  as a subgraph of  $G'$ . Since  $V_r(G')$  is an independent

set of  $G'$ , any clique covering the edges in  $E(G', v)$  does not cover any edges in  $E(G, w)$  for any  $w \in V_r(G)$ . Further since  $V_1(G')$  is an independent set of  $G'$ , there are at least  $m_1$  maxcliques of  $G'$  which covers  $E(G', v)$ . Actually we can find a family  $\{Q_j; j \in [s+1, s+m_1]\}$  of maxcliques of  $G'$  which covers  $E(G', v)$ . Therefore  $\{Q_j; j \in [s+m_1]\}$  is a minimal maxclique cover of  $G'$  and  $i(G') = i(G) + m_1$ .  $\square$

The next result is derived inductively from the above lemma.

**Theorem 3.3.**  $i(K(m_1, m_2, \dots, m_r)) = m_1(\sum_{j=2}^r m_j - r + 2)$ .  $\square$

## 4 Intersection numbers of some regular graphs

First we consider a minimal set representation of the  $(n-2)$ -regular graph  $G_n$  of order even  $n = 2m$ . Since  $G_{2m}$  is the complementary graph of the 1-regular graph  $mK_2$ , we label  $V(G_{2m}) = \{u_j, v_j; j \in [m]\}$  such that for every  $j \in [m]$ ,  $u_j$  [resp.  $v_j$ ] is adjacent to all another vertices except  $v_j$  [resp.  $u_j$ ]. As  $G_4 = C_4$ , we have  $i(G_4) = 4$  from Theorem 2.2. In what follows let  $m > 2$ . Since there are many maxcliques in  $G_{2m}$ , precisely  $\theta_m(G_{2m}) = 2^m$ , in order to determine  $i(G_{2m})$  we consider not edge clique covers but set representations of  $G_{2m}$ . For some  $t$ , the construction of any set representation  $\phi$  of  $G_{2m}$  with  $|\phi| = t$  is reduce to give a family  $\mathbf{F}_m = \{S_j; j \in [m]\}$  of mutually distinct subsets in  $[t]$  with the following the properties:

$$(4.1) \quad S_j \cap S_k \neq \emptyset, \text{ neither } S_j \subset S_k \text{ nor } S_k \subset S_j \text{ for any distinct } j, k \in [m],$$

$$(4.2) \quad |S_j| \leq \lfloor \frac{t}{2} \rfloor \text{ for any } j \in [m].$$

Indeed for such  $\phi$ , we may assume without loss of generality that for every  $j \in [m]$ ,  $S_j := \phi(u_j)$  is a non-empty subset of  $[t]$  and the family  $\{S_j; j \in [m]\}$  satisfies (4.1) and (4.2). Conversely for any  $\mathbf{F}_m = \{S_j; j \in [m]\}$  in the above, we put  $\phi(u_j) = S_j$  and  $\phi(v_j) = [t] \setminus S_j$  for  $j \in [m]$ . Then for any distinct  $j, k \in [m]$ ,  $\phi(v_j) \cap \phi(u_k)$  and  $\phi(v_j) \cap \phi(v_k)$  are non-empty by (4.1) and (4.2) respectively. Hence  $\phi$  is a set representation of  $G_{2m}$  with  $|\phi| = t$ . Here we denote  $\phi(\mathbf{F}_m)$  by the set representation  $\phi$  defined from  $\mathbf{F}_m$ . Any family  $\mathbf{F} = \{S_j\}$  of subsets of  $[t]$  is called an *intersecting Sperner family* if it satisfies the condition (4.1). The determination of  $i(G_{2m})$  is to find the smallest positive integer  $t$  such that in  $[t]$  there exists an intersecting Sperner family  $\mathbf{F}$ ,  $|\mathbf{F}| = m$ , with (4.2). We note that from any non-empty subfamily  $\mathbf{F}'$  with  $|\mathbf{F}'| = m'$  of any intersecting Sperner family  $\mathbf{F}$ ,  $|\mathbf{F}| = m$ , with (4.2) there corresponds a set representation  $\phi(\mathbf{F}')$  of  $G_{2m'}$ . So our problem is reduced to a combinatorial problem to find the largest cardinal number of any intersecting Sperner family with (4.2) in  $[t]$  for any given  $t$ . The answer for this problem is derived from some results stated in Bollobás [1].

For any given  $t > 1$  let  $X_t = [t]$  and use the notation concerning family of subsets of  $X_t$  as follows:

$$X_t(\leq r) = \{S; S \subset X_t, |S| \leq r\} \text{ for any } r \text{ with } 1 \leq r < t,$$

$$X_t(r) = \{S; S \subset X_t, |S| = r\} \text{ for any } r \text{ with } 1 \leq r < t,$$

$$X_t(\{j\}) = \{S; j \in S \subset X_t\} \text{ for any fixed } j \in X_t,$$

$$X_t(r, \{j\}) = X_t(r) \cap X_t(\{j\}).$$

Under these notation we have the next lemma, which is due to [1, Theorem 13.2].

**Lemma 4.1.** *Let  $\mathbf{F}$  be any intersecting Sperner family in  $X_t(\leq \lfloor \frac{t}{2} \rfloor)$ . Then*

$$(4.3) \quad \sum_{A \in \mathbf{F}} \binom{t-1}{|A|-1}^{-1} \leq 1.$$

*Especially the equality holds in (4.3) if and only if  $\mathbf{F} = X_t(r, \{j\})$  for some  $r < \frac{t}{2}$ ,  $j \in X_t$  or  $t$  is even and  $\mathbf{F}$  contains precisely one of each pair  $\{A, X_t \setminus A\}$  in any  $A \in X_t(\frac{t}{2})$ .  $\square$*

From Lemma 4.1 it follows that the largest cardinal number  $M(t)$  of intersecting Sperner families in  $X_t(\leq \frac{t}{2})$  is given as follows:

$$(4.4) \quad M(t) = \binom{t-1}{p-1}, \text{ where } p = \lfloor \frac{t}{2} \rfloor,$$

and  $\mathbf{F}_t := X_t(p, \{1\})$  is a maximal intersecting Sperner family in  $X_t$  with  $|\mathbf{F}_t| = M(t)$ . Therefore for any  $m$  with  $2 \leq m \leq M(t)$ ,  $G_{2m}$  has a set representation  $\phi$  with  $|\phi| = t$ , and for any  $m$  with  $m > M(t)$ ,  $i(G_{2m}) > t$ . Consequently we have

**Theorem 4.2.** *Under the notation (4.4) for any  $m > 2$ ,  $i(G_{2m}) = t$ , where  $t$  is determined by the inequalities:  $M(t-1) < m \leq M(t)$ .  $\square$*

For examples  $i(G_6) = 4$ ,  $i(G_8) = 5$ , and  $i(G_{2m}) = 6$  for  $4 < m \leq 10$ .

Here we consider the intersection number of the complementary graph  $G_n$  of the cycle  $C_n$  with  $n \geq 5$ , which is a  $(n-3)$ -regular graph of order  $n$ . The vertex set  $\{u_1, u_2, u_3, \dots, u_n\}$  of  $G_n$  and  $C_n$  are labeled as  $u_j u_{j+1} \in E(C_n)$  for any  $j \in [n]$ . Now for convenience we use the label  $j$  of  $u_j$  in the sense of modulo  $n$ , say  $u_{n+1}$  should be understood as  $u_1$ . So  $N(u_j) = \{u_{j+k}; k \in [2, n-2]\}$  for any  $u_j$ . We note that for any subset  $W$  of  $V(G_n)$ ,  $\langle W \rangle$  is a clique in  $G_n$  if and only if  $W$  is an independent set in  $C_n$ . So enumerating the maximal independent sets in  $C_n$  for the case  $n = \text{odd}$ , we have

**Lemma 4.3.** *For the case  $n = 2m - 1$ ,  $m \geq 3$ ,  $MC(G_n)$  is the family  $\{Q_j; j \in [n]\}$  given as follows:  $Q_j = \langle W_j \rangle$ , where*

$$W_j = \{u_j, u_{j+2}, u_{j+4}, \dots, u_{j+2m-4}\} \text{ for any } j \in [n].$$

*In this case for every  $j \in [n]$  the edge  $u_j u_{j+2m-4}$  is proper.  $\square$*

Let  $n = 2m$ ,  $m \geq 3$ . Then we note that  $G_{n-1} = G_n - \{v_{2m}\} - \{u_1 u_{2m-1}\}$ . So using this fact we can get  $MC(G_n)$  from  $MC(G_{n-1})$  by a slight modification. Under the notation in Lemma 4.3, we add  $\{v_{2m-1}\}$  to  $W_1$ , add  $\{v_{2m}\}$  to  $W_2$ , and for  $W_3$  add  $\{v_{2m}\}$  and delete  $\{v_{2m-1}\}$ . Therefore we have

**Lemma 4.4.** *For the case  $n = 2m$ ,  $m \geq 3$ ,  $MC(G_n)$  is the family  $\{Q_j; j \in [n-1]\}$  given as follows:  $Q_j = \langle W_j \rangle$  for any  $j \in [n]$ , where*

$$W_1 = \{u_1, u_3, u_5, \dots, u_{2m-3}, u_{2m-1}\}$$

$$W_2 = \{u_2, u_4, u_6, \dots, u_{2m-2}, u_{2m}\}$$

$$W_3 = \{u_3, u_5, u_7, \dots, u_{2m-3}, u_{2m}\}$$

$$W_j = \{u_j, u_{j+2}, u_{j+4}, \dots, u_{j+2m-4}\} \text{ for any } j \in [4, n-1].$$

*In this case the edges  $u_1 u_{2m-1}, u_2 u_{2m}, u_3 u_{2m}$  and the edges  $u_j u_{j+2m-4}$  for every  $j \in [4, n-1]$  are proper.  $\square$*

From Lemmas 4.3-4.4 and Theorem 2.3 we have

**Theorem 4.5.** *For the complementary graph  $G_n$  of the cycle  $C_n$  with  $n \geq 5$ ,  $i(G_n) = \theta_m(G_n)$ , and  $i(G_n) = n$  if  $n$  is odd and  $i(G_n) = n - 1$  if  $n$  is even.  $\square$*

### References

- [1] B. Bollobás: *Combinatorics*, Cambridge University Press, 1998.
- [2] A.Brandstädt, Van Ban Le, and J.P. Spinrad: *Graph classes. A survey*, SIAM Monographs on Discrete Mathematics and Applications, 1999.
- [3] R.C. Brigham and R.D. Dutton: *On clique covers and independence numbers of graphs*, Discrete Math., **44** (1963) 139-144.
- [4] P. Erdős, A.W. Goodman, and L.Pósa.: *The representation of a graph by set intersections*, Canad. J. Math., **18** (1966),106-112.
- [5] L. Chartrand and L. Lesniak: *Graphs & Digraphs*, Chapman & Hall, 1996.
- [6] T.A.McKee and F.R.McMorris: *Topics in intersection graph theory*, SIAM Monographs on Discrete Mathematics and Applications, 1999.
- [7] K. Sakai: *A note on maxcliques covered by another maxcliques in regular graphs*, Rep. Fac. Sci. Kagoshima Univ. **33**(2000), 35-38.
- [8] P.J. Slater: *A note on pseudointersection graphs*, Res. Nat. Bur. Standards Section B **80**(1976),441-445.
- [9] M.Tsuchiya: *On antichain intersection numbers, total cliques covers and regular graphs*, Discrete Math.**127** (1994), 305-318.
- [10] W.D. Wallis and Guo-Hui Zhang: *On maximal clique irreducible graphs*, J.Combin. Math. Combin. Comput., **8**(1990), 187-193.