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## COMPACT $T_1$ -SPACE AND THE FAMILY OF FINITE TOPOLOGICAL SPACES

By

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### § 1. Introduction.

We first note that given a finite open covering of a topological space, there is a finite  $T_0$ -space corresponds to the covering. And if we consider all finite open coverings of the space, we have a family of finite  $T_0$ -spaces. From the family we can obtain an inverse system by defining an order relation on the family. These shall be given in §2. In §3 we shall show that any  $T_0$ -space can be embedded in the inverse limit space of the associated inverse system. Moreover in §3 it will be proved that any compact  $T_1$ -space is homeomorphic to a subspace in the inverse limit space which will be called the maximal inverse limit space here.

### § 2. The family of finite $T_0$ -spaces.

Let  $(X, \tau)$  be any  $T_0$ -space and suppose  $\{\mathcal{A}_\alpha | \alpha \in M\}$  is the family of all finite open covering of  $(X, \tau)$ , where  $\mathcal{A}_\alpha$  is an open covering of  $X$  for all  $\alpha \in M$ . For  $\mathcal{A}_\alpha$  let  $\tau_\alpha$  be the topology on  $X$  for which  $\mathcal{A}_\alpha$  is a subbase. Then we have the family  $\{(X, \tau_\alpha) | \alpha \in M\}$  of topological spaces.

Now let us consider a topological space  $(X, \tau_\alpha)$  of the family. We define the following relation  $\sim$  on  $X$  with respect to  $\tau_\alpha$ : for  $x, y \in X$ , let  $x \sim y$  mean that  $G \in \tau_\alpha$  and  $G \cap (x \cup y) \neq \emptyset$  imply  $G \supset (x \cup y)$ . Then the relation  $\sim$  is a equivalence relation.

In fact,  $x \sim x$  and  $x \sim y \Rightarrow y \sim x$  are immediate consequence of its definition.

To prove that  $\sim$  is transitive, let  $x \sim y$  and  $y \sim z$ , and suppose that  $G \in \tau_\alpha$  and  $G \cap (x \cup z) \neq \emptyset$ . If  $G \ni x$ , then  $G \ni y$  since  $y \sim x$ . Hence  $G \ni z$ , so that  $G \supset (x \cup z)$ . If  $G \ni z$ , then  $G \ni y$  since  $y \sim z$ . Hence  $G \ni x$  since  $x \sim y$ . Hence  $G \supset (x \cup z)$ . Thus  $x \sim z$ , whence  $\sim$  is transitive.

By the equivalence relation  $\sim$  we may obtain the quotient space  $(X, \tau_\alpha)/\sim$  of  $(X, \tau_\alpha)$  with respect to  $\sim$ , which we denote by  $(Y_\alpha, \sigma_\alpha)$  (or merely  $Y_\alpha$ ). Then the space  $(Y_\alpha, \sigma_\alpha)$  is a finite  $T_0$ -space.

In fact, let  $p: (X, \tau_\alpha) \rightarrow (Y_\alpha, \sigma_\alpha)$  be the natural projection, and  $y_1, y_2 \in Y_\alpha$ ,  $y_1 \neq y_2$ . Then  $p^{-1}(y_1) \cap p^{-1}(y_2) = \emptyset$  since  $y_1 \neq y_2$ , so taking  $x_i \in p^{-1}(y_i)$  ( $i=1, 2$ ),  $x_1$  is not  $\sim$  related to  $x_2$ . Hence there exists an open set  $G \in \tau_\alpha$  such that either  $x_1 \in G$  and  $x_2 \notin G$  or

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$x_1 \notin G$  and  $x_2 \in G$ . Now suppose that  $x_1 \in G$  and  $x_2 \notin G$ . Then  $y_1 = p(x_1) \in p(G)$  and  $y_2 = p(x_2) \notin p(G)$ . On the other hand,  $p(G)$  is open in  $(Y_\alpha, \sigma_\alpha)$  since  $p^{-1}(p(G)) = G$ . Therefore  $(Y_\alpha, \sigma_\alpha)$  is a finite  $T_0$ -space.

From the above observations it follows that there exists the family  $\{(Y_\alpha, \sigma_\alpha) | \alpha \in A\}$  of finite  $T_0$ -spaces associated with a given topological space  $(X, \tau)$ . Now we shall show that the family  $\{(Y_\alpha, \sigma_\alpha) | \alpha \in A\}$  can be made into an inverse system in a natural way.

We first define the order relation on  $A$  as follows: for  $\alpha, \beta \in A$ ,  $\beta \leq \alpha$  if and only if  $\tau_\beta \subset \tau_\alpha$ . Then  $(A, \leq)$  is a directed set. Because for  $\alpha, \beta \in A$ , taking the topology  $\tau_\gamma$  for which  $\Delta_\alpha \cup \Delta_\beta$  is a subbase, clearly  $\tau_\alpha \subset \tau_\gamma$  and  $\tau_\beta \subset \tau_\gamma$ . Hence  $\alpha \leq \gamma$  and  $\beta \leq \gamma$ .

We next define a mapping  $f_{\alpha\beta}: Y_\beta \rightarrow Y_\alpha$  for each pair  $\alpha, \beta \in A$  such that  $\alpha \leq \beta$ . Let  $i_\alpha: (X, \tau) \rightarrow (X, \tau_\alpha)$  be the identity mapping and  $p_\alpha: (X, \tau_\alpha) \rightarrow (Y_\alpha, \sigma_\alpha)$  be the natural projection. Then putting

$$f_\alpha = p_\alpha \cdot i_\alpha,$$

$f_\alpha: (X, \tau) \rightarrow Y_\alpha$  is clearly continuous onto. We now define  $f_{\alpha\beta}: Y_\beta \rightarrow Y_\alpha$  by the following manner: for  $y_\beta \in Y_\beta$  we take  $x \in X$  such that  $f_\beta(x) = y_\beta$  and put

$$f_{\alpha\beta}(y_\beta) = f_\alpha(x),$$

that is

$$f_{\alpha\beta}(f_\beta(x)) = f_\alpha(x).$$

This mapping is well defined. In fact, suppose that  $f_\beta(x_1) = f_\beta(x_2)$ , then there exists the open set  $G_\beta$  in  $(X, \tau_\beta)$  such that  $G_\beta$  is the minimal neighborhood of  $x_1$  and also of  $x_2$ . Hence  $x_1 \sim x_2$  that is  $U \in \tau_\beta$  and  $U \cap (x_1 \cup x_2) \neq \emptyset$  imply  $U \supset (x_1 \cup x_2)$ . But  $\tau_\alpha \subset \tau_\beta$  since  $\alpha < \beta$ . Accordingly, if  $V \in \tau_\alpha$  and  $V \cap (x_1 \cup x_2) \neq \emptyset$ , then  $V \in \tau_\beta$  and  $V \cap (x_1 \cup x_2) \neq \emptyset$ , so that  $V \supset (x_1 \cup x_2)$ . Hence  $x_1 \sim x_2$ , that is  $f_\alpha(x_1) = f_\alpha(x_2)$ . Thus  $f_{\alpha\beta}(f_\beta(x_1)) = f_{\alpha\beta}(f_\beta(x_2))$ , which shows that  $f_{\alpha\beta}$  is well defined.

Further it follows immediately from the definition that these mappings satisfy

(i)  $f_{\alpha\alpha}$  is the identity mapping for each  $\alpha \in A$

and

ii)  $f_{\beta\alpha} f_{\beta\gamma} = f_{\alpha\gamma}$  whenever  $\alpha < \beta < \gamma$ .

In summary, we have seen that  $\{Y_\alpha, f_{\alpha\beta}\}$  is the inverse system of finite  $T_0$ -spaces.

### §3. Embedding of a $T_0$ -space.

Let  $(X, \tau)$  be a  $T_0$ -space and let suppose  $\{Y_\alpha | \alpha \in A\}$  be the inverse system of finite  $T_0$ -spaces associated with  $(X, \tau)$ , described above. And we denote by  $Y_\infty$  the inverse limit space of this system. Then we have the following theorem.

**THEOREM 1.** *A  $T_0$ -space  $(X, \tau)$  can be embedded in  $Y_\infty$ .*

**PROOF.** Define  $f: X \rightarrow Y_\infty \subset \prod Y_\alpha$  by

$$f(x) = \{f_\alpha(x)\}_{\alpha \in A}.$$

Since  $\{f_\alpha(x)\} \in Y_\infty$ , we have  $f(X) \subset Y_\infty$ .

We shall show that  $f$  is embedding.

(i)  $f$  is continuous. This is obvious since  $f_\alpha$  is continuous for all  $\alpha \in A$ .

(ii)  $f$  is one to one. To prove this, let  $x, y \in X$  such that  $x \neq y$ . Since  $(X, \tau)$  is a  $T_0$ -space, there is an open neighborhood  $U$  of at least one, say  $x$  which does not contain the other  $y$ . Let now

$$\tau_x = \{\phi, U, X\},$$

then  $\alpha \in A$ . Clearly

$$f_\alpha(x) \neq f_\alpha(y).$$

in the space  $Y_\alpha$ , whence

$$\{f_\alpha(x)\} \neq \{f_\alpha(y)\},$$

that is

$$f(x) \neq f(y).$$

(iii)  $f$  is open. To prove this, it suffices to show that  $\{f_\alpha\}_{\alpha \in A}$  distinguishes points and closed sets. Suppose that  $F$  is closed in  $X$  and  $x \notin F$ . Since  $X$  is  $T_0$  there is an open nbd  $V$  of  $x$  such that  $V \cap F = \phi$ . We now take the topology

$$\tau_y = \{\phi, V, X\}.$$

and consider the space  $Y_\gamma$ , then clearly

$$f_\gamma(x) \notin \overline{f_\gamma(F)} = \overline{f_\gamma(F)}.$$

Thus (i), (ii) and (iii) have been established. Therefore by the embedding lemma  $f$  is embedding. [1].

#### § 4. Embedding of a compact $T_1$ -space.

We first observe that every finite  $T_0$ -space can be made into a partially ordered set by defining a suitable order relation on this space. [2].

Let  $Y = \{y_1, y_2, \dots, y_n\}$  be a finite topological space and  $\{U_1, U_2, \dots, U_n\}$  be the system of minimal basic neighborhoods, where  $U_i$  is the minimal neighborhood of  $y_i$ .

We define an order relation  $\leq$  on  $Y$ , by saying

$$y_1 \leq y_j \text{ whenever } U_i \subset U_j.$$

Then it follows that  $Y$  is a  $T_0$ -space if and only if  $(Y, \leq)$  is a partially ordered set.

Next let  $(X, \tau)$  be a topological space,  $\{Y_\alpha, f_{\alpha\beta}\}$  be its associated inverse system and  $Y_\infty$  be the inverse limit space of the family.

We define an ordering on the product space  $\prod_{\alpha \in A} Y_\alpha$  as follows: for  $\{x_\alpha\}, \{y_\alpha\} \in \prod_{\alpha \in A} Y_\alpha$  let  $\{x_\alpha\} \leq \{y_\alpha\}$  mean that  $x_\alpha \leq y_\alpha$  in  $(Y_\alpha, \leq)$  for all  $\alpha \in A$ . Then  $(\prod_{\alpha \in A} Y_\alpha, \leq)$  is a partially ordered set.

Here the subspace

$$M(Y_\infty) = \{\{z_\alpha\} \in Y_\infty \mid \{z_\alpha\} \text{ is maximal in } Y_\infty\}$$

will be called the *maximal inverse limit space* of the inverse system  $\{Y_\alpha, f_{\alpha\beta}\}$ .

**THEOREM 2.** *A compact  $T_1$ -space  $(X, \tau)$  is homeomorphic to  $M(Y_\infty)$ .*

**PROOF.** Let  $(X, \tau)$  be a compact  $T_1$ -space. The notation of Theorem 1 will be used. Let  $f: X \rightarrow Y_\infty$  be as in Theorem 1. Then it suffices to show that

(1)  $f(x) \in M(Y_\infty)$  for all  $x \in X$ .

and

(2)  $f: X \rightarrow M(Y_\infty)$  is onto.

(1) Suppose that there exists  $x \in X$  such that  $f(x) \notin M(Y_\infty)$ . Since  $(X, \tau)$  is  $T_1$ ,  $\{x\}$  is closed in  $(X, \tau)$ , so that  $U = X - \{x\}$  is open. Then taking the topology

$$\tau_\beta = \{\phi, U, X\},$$

we consider the finite  $T_0$ -space  $Y_\beta$ . Since  $f(x) \notin M(Y_\infty)$ ,  $f(x) = \{f_\alpha(x)\}$  is not maximal in  $Y_\infty$ , whence there exists  $\{z_\alpha\} \in Y_\infty$  such that  $\{f_\alpha(x)\} < \{z_\alpha\}$ , that is, there exists  $Y_\gamma$  such that  $f_\gamma(x) < z_\gamma$ .

We now choose the topology  $\tau_\xi$  generated by the subbase  $\tau_\beta \cup \tau_\xi$ , then we have

$$\beta, \gamma \leq \xi.$$

Since  $\{f_\alpha(x)\} < \{z_\alpha\}$  it follows that  $f_\xi(x) \leq z_\xi$ , hence  $f_\xi(x) < z_\xi$  or  $f_\xi(x) = z_\xi$ .

Suppose  $f_\xi(x) < z_\xi$ . Now there exists  $y \in X$  such that  $z_\xi = f_\xi(y)$ , then  $x \neq y$  since  $f_\xi(x) \neq z_\xi$ . Hence  $f_\beta(x) \neq f_\beta(y)$ , so that

$$f_\beta(y) < f_\beta(x).$$

Since  $z_\beta = f_\beta(y)$ , we have

$$z_\beta < f_\beta(x),$$

which contradicts  $\{f_\alpha(x)\} < \{z_\alpha\}$ .

Next suppose  $f_\xi(x) = z_\xi$ . Then  $f_{\gamma\xi}(f_\xi(x)) = f_\gamma(x)$  and  $f_{\gamma\xi}(z_\xi) = z_\gamma$ , whence

$$f_\gamma(x) = z_\gamma,$$

which contradicts  $f_\gamma(x) < z_\gamma$ .

Therefore

$$f(X) \subset M(Y_\infty).$$

(2) If  $f(X) = M(Y_\infty)$ , then there is  $\{z_\alpha\} \in M(Y_\infty)$  such that  $\{f_\alpha(x)\} \neq \{z_\alpha\}$  for all  $x \in X$ . Since  $\{z_\alpha\}$  is maximal, for each  $x \in X$  there exists  $\beta = \beta(x) \in \mathcal{A}$  such that

$$f_\beta(x) \cong z_\beta.$$

Now let  $U_\beta$  be the minimal neighborhood of  $x$  in  $(X, \tau_\beta)$ , then there exists  $y \in X$  such that  $z_\beta = f_\beta(y)$  and  $U_\beta \not\ni y$ .

Taking the topology

$$\tau_\gamma = \{\phi, U, X\}$$

and the finite  $T_0$ -space  $Y_\gamma$ ,

$$\gamma < \beta.$$

Also, since  $U_\beta \ni y$ ,  $f_\gamma(x) < f_\gamma(y)$  and  $f_\gamma(y) = z_\gamma$ . Hence

$$f_\gamma(x) < z_\gamma.$$

Now since  $\{U_{\beta(x)} \mid x \in X\}$  is an open covering of  $X$  and  $X$  is compact, this covering has a finite subcovering, which we denote by

$$D = \{U_{\beta_1}(x_1), U_{\beta_2}(x_2), \dots, U_{\beta_n}(x_n)\}.$$

Then let  $\tau_\delta$  be the topology on  $X$  generated by  $D$ , and consider the space  $Y_\delta$ . Then

$$\gamma_i \leq \delta \quad (i = 1, 2, \dots, n).$$

For  $z_\delta \in Y_\delta$ , take  $y \in X$  such that  $f_\delta(y) = z_\delta$ . Since  $D$  is a covering of  $X$ , there exists  $U_{\beta_k}(x_k) \in D$  such that  $y \in U_{\beta_k}(x_k)$ . Hence

$$f_{\gamma_k}(x_k) \geq f_{\gamma_k}(y)$$

in  $Y_{\gamma_k}$ , also  $\gamma_k \leq \delta$ ,  $z_{\gamma_k} = f_{\gamma_k}(y)$ , whence

$$f_{\gamma_k}(x_k) \geq z_{\gamma_k}.$$

But this a contradiction.

### References

- [1] John L. KELLEY: General topology. Van Nostrand, Princeton, (1955).
- [2] P. ALEXANDROFF: Diskrete Räume. Mat. Sb. (N.S) Vol. 2, (1937), 501-518.