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SOME BIVARIATE NORMAL TESTS OF COMPOSITE HYPOTHESES

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1. Introduction

Suppose (X, Y) has a bivariate normal distribution with a known variance matrix, which we assume without loss of generality, to have a unit variance and a correlation coefficient ρ, and a unknown mean vector (θ_1,θ_2). We consider the problems of testing the null hypothesis H_0: $\theta_1=0$ or $\theta_2=0$ against the alternative K_0: $\theta_1\neq 0$ and $\theta_2\neq 0$ (§ 2), and of testing the null hypothesis H_1: $\theta_1=0$ or $\theta_2=0$ or $\theta_1+\theta_2=0$ against the alternative K_1: $\theta_1\neq 0$ and $\theta_2\neq 0$ and $\theta_1+\theta_2\neq 0$ (§ 3). The purpose of this paper is to give the likelihood ratio test of these testing hypothesis problems.

Suppose we have samples from three normal distributions with different means μ_1, μ_2, μ_3 and a known common variance. Then the problem of testing the null hypothesis H_0 against the alternative K_0 includes the problem of testing the null hypothesis that means are equal only in one adjacent pair, namely, H_0': $\mu_1=\mu_2$ or $\mu_2=\mu_3$ against the alternative K_0': $\mu_1\neq\mu_2$ and $\mu_2\neq\mu_3$, and the problem of testing the null hypothesis H_1 against the alternative K_1 includes the problem of testing the null hypothesis that means are equal only in one pair, namely, H_1': $\mu_1=\mu_2$ or $\mu_2=\mu_3$ or $\mu_3=\mu_1$ against the alternative K_1': $\mu_1\neq\mu_2$ and $\mu_2\neq\mu_3$ and $\mu_3\neq\mu_1$.

2. A bivariate normal test I

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with the unknown mean vector $\theta=(\theta_1,\theta_2)$ and the known variance matrix which has a unit variance and a correlation coefficient ρ, and $(\bar{X}, \bar{Y})$ be a sample mean vector. In this section we consider the problem of testing the null hypothesis H_0: $\theta_1=0$ or $\theta_2=0$ against the alternative K_0: $\theta_1\neq 0$ and $\theta_2\neq 0$.

At first we shall derive the maximum likelihood estimates (MLE) $\hat{\theta}$ of θ under H_0 and $H_0\cup K_0$ respectively. To do this let us consider the following transformation in the two dimensional Euclidean space R^2 similar to the one used in [1]:

$$\xi = x\,, \qquad \eta = (1-\rho^2)^{-1/2}(\rho x - y) \tag{1}$$

or

$$x = \xi\,, \qquad y = \rho\xi - (1-\rho^2)^{1/2}\eta\,. \tag{2}$$

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(ξ, η), the random vector corresponding to (X, Y), is distributed as a bivariate normal with a mean

$$\phi = (\phi_1, \phi_2) = \{\theta_1, (1-\rho^2)^{-1/2}(\rho\theta_1-\theta_2)\} \tag{3}$$

and a common variance 1 and a covariance 0. H_0 and K_0 are transformed to H_0': $\phi_1=0$ or $\phi_2=\rho(1-\rho^2)^{-1/2}\phi_1$ and K_0': $\phi_1\neq 0$ and $\phi_2\neq\rho(1-\rho^2)^{-1/2}\phi_1$ respectively. The following factorization of the likelihood is convenient to derive the MLE.

$$L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right]\exp\left[-\frac{n}{2}(\bar{x}-\theta_1)^2\right]$$

$$\times\exp\left[-\frac{n}{2(1-\rho^2)}\{\bar{y}-\rho\bar{x}-(\theta_2-\rho\theta_1)\}^2\right] \tag{4}$$

$$= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right]\exp\left[-\frac{n}{2}(\bar{y}-\theta_2)^2\right]$$

$$\times\exp\left[-\frac{n}{2(1-\rho^2)}\{\bar{x}-\rho\bar{y}-(\theta_1-\rho\theta_2)\}^2\right]$$

where

$$Q(x,y) = \sum_{i=1}^{n}\{(x_i-\bar{x})^2-2\rho(x_i-\bar{x})(y_i-\bar{y})+(y_i-\bar{y})^2\}/(1-\rho^2)\,.$$

Thus we have the MLE $\hat{\phi}$ of ϕ under H_0' as follows.

If $|\bar{\xi}| \geqq |\sqrt{1-\rho^2}\bar{\eta}-\rho\bar{\xi}|$, $\hat{\phi}_1=\sqrt{1-\rho^2}(\sqrt{1-\rho^2}\bar{\xi}+\rho\bar{\eta})$ and $\hat{\phi}_2=\rho(\sqrt{1-\rho^2}\bar{\xi}+\rho\bar{\eta})$ (5)

and

if $|\bar{\xi}| < |\sqrt{1-\rho^2}\bar{\eta}-\rho\bar{\xi}|$, $\hat{\phi}_1=0$ and $\hat{\phi}_2=\bar{\eta}$. (6)

Transforming back to the original variables, the MLE $\hat{\theta}$ of θ under H_0 and its maximum likelihood, $\underset{H_0}{\text{Max}}\, L(\theta_1,\theta_2)$, are given as follows.

If $|\bar{X}| \geqq |\bar{Y}|$, $\hat{\theta}_1=\bar{X}-\rho\bar{Y}$, $\hat{\theta}_2=0$ and

$$\underset{H_0}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right]\exp\left[-\frac{n}{2}\bar{y}^2\right] \tag{7}$$

and

if $|\bar{X}| < |\bar{Y}|$, $\hat{\theta}_1=0$, $\hat{\theta}_2=\bar{Y}-\rho\bar{X}$ and

$$\underset{H_0}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right]\exp\left[-\frac{n}{2}\bar{x}^2\right]. \tag{8}$$

The MLE $\hat{\theta}$ of θ under $H_0 \cup K_0$ and its maximum likelihood, $\underset{H_0\cup K_0}{\text{Max}}\, L(\theta_1,\theta_2)$, can be easily obtained as follows.

$$\hat{\theta}_1=\bar{X},\quad \hat{\theta}_2=\bar{Y}\quad\text{and}\quad \underset{H_0\cup K_0}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right]. \tag{9}$$

Therefore the likelihood ratio test can be derived and the rigion of rejection is given by the following simple form:

$$\begin{cases} n\bar{Y}^2 \geqq B_0^2 & \text{if} \quad |\bar{X}| \geqq |\bar{Y}|\,, \\ n\bar{X}^2 \geqq B_0^2 & \text{if} \quad |\bar{X}| < |\bar{Y}| \end{cases} \tag{10}$$

where B_0 is chosen so that the probability of (10) when the null hypothesis is true is equal to the significance level α.

After simple calculations, it can be found that B_0 satisfies the following relation.

$$\alpha = 2Q(B_0) \tag{11}$$

where

$$Q(m) = \int_m^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}u^2\right] du\,.$$

Therefore the desired B_0 can be found easily from the table of a univariate normal distribution [2].

3. A bivariate normal test II

In this section we consider the problem of testing the null hypothesis H_1: $\theta_1=0$ or $\theta_2=0$ or $\theta_1+\theta_2=0$ against the alternative K_1: $\theta_1 \neq 0$ and $\theta_2 \neq 0$ and $\theta_1+\theta_2 \neq 0$. Making use of the same transformation (1) or (2) and applying the method similar to the one used in the previous section, the MLE $\hat{\theta}$ of θ under H_1 and its maximum likelihood, $\underset{H_1}{\text{Max}}\, L(\theta_1,\theta_2)$, are given as follows.

If $-(1+\sqrt{2(1+\rho)})^{-1}\bar{X} \leqq \bar{Y} \leqq \bar{X}$ or $\bar{X} \leqq \bar{Y} \leqq -(1+\sqrt{2(1+\rho)})^{-1}\bar{X}$,

$\hat{\theta}_1 = \bar{X} - \rho\bar{Y}$, $\hat{\theta}_2 = 0$ and

$$\underset{H_1}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right] \exp\left[-\frac{n}{2}\bar{y}^2\right], \tag{12}$$

if $\bar{X} < \bar{Y}$, $-(1+\sqrt{2(1+\rho)})\bar{X} \leqq \bar{Y}$ or $\bar{Y} < \bar{X}$, $\bar{Y} \leqq -(1+\sqrt{2(1+\rho)})\bar{X}$,

$\hat{\theta}_1 = 0$, $\hat{\theta}_2 = \bar{Y} - \rho\bar{X}$ and

$$\underset{H_1}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x,y)\right] \exp\left[-\frac{n}{2}\bar{x}^2\right] \tag{13}$$

and

if $-(1+\sqrt{2(1+\rho)})^{-1}\bar{X} < \bar{Y} < -(1+\sqrt{2(1+\rho)})\bar{X}$ or

$-(1+\sqrt{2(1+\rho)})\bar{X} < \bar{Y} < -(1+\sqrt{2(1+\rho)})^{-1}\bar{X}$,

$$\hat{\theta}_1 = \frac{\bar{X}-\bar{Y}}{2}, \quad \hat{\theta}_2 = \frac{\bar{Y}-\bar{X}}{2}, \quad \text{and} \quad \underset{H_1}{\text{Max}}\, L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n$$

$$\times \exp\left[-\frac{1}{2}Q(x,y)\right] \exp\left[-\frac{n}{4(1+\rho)}(\bar{X}+\bar{Y})^2\right]. \tag{14}$$

The MLE $\hat{\theta}$ of θ under $H_1 \cup K_1$ and its maximum likelihood, $\underset{H_1 \cup K_1}{\mathrm{Max}}\, L(\theta_1, \theta_2)$, are the same one that we obtained in (9) in the previous section.

Therefore the likelihood ratio test can be derived and the rigion of rejection is given by the following form:

$$\begin{cases} \sqrt{n}\,\mathrm{Min}\,(|\bar{X}|, |\bar{Y}|) \geqq B_1 & \text{if } \bar{X} \text{ and } \bar{Y} \text{ are of the same sign,} \\ \sqrt{n}\,(\bar{X}+\bar{Y}) \leqq -\sqrt{2(1+\rho)}\,B_1, \quad \sqrt{n}\,\bar{X} \geqq B_1 & \\ \text{or} & \text{if } \bar{X} \geqq 0, \quad \bar{Y} \leqq 0, \\ \sqrt{n}\,(\bar{X}+\bar{Y}) \geqq \sqrt{2(1+\rho)}\,B_1, \quad \sqrt{n}\,\bar{Y} \leqq -B_1 & \\ \sqrt{n}\,(\bar{X}+\bar{Y}) \leqq -\sqrt{2(1+\rho)}\,B_1, \quad \sqrt{n}\,\bar{Y} \geqq B_1 & \\ \text{or} & \text{if } \bar{X} \leqq 0, \quad \bar{Y} \geqq 0 \\ \sqrt{n}\,(\bar{X}+\bar{Y}) \geqq \sqrt{2(1+\rho)}\,B_1, \quad \sqrt{n}\,\bar{X} \leqq -B_1 & \end{cases} \tag{15}$$

where B_1 is chosen so that the probability of (15) when the null hypothesis is true is equal to the significance level α.

After some calculations, it can be found that B_1 satisfies the following relation.

$$\alpha = 2Q(B_1)\,. \tag{16}$$

References

[1] Kudô, A. and Fujisawa, H. (1964). A bivariate normal test with two-sided alternative, Memoirs of the Faculty of Science, Kyushu University, Ser. A, 18, 104–108.

[2] Yamauti, Z. (1972). Statistical Tables and Formulas with Computer Applications JSA-1972. Japanese Standards Association.