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§ 1. Introduction.

We propose here an axiom system for nonstandard set theory, which can be used to formalize nonstandard mathematics. A theory with the axiom system, which we write **NST**, is an extension of *internal set theory* **IST** which Nelson [2] has given. The theory **NST** deals with external sets directly while **IST** does not. The axiom system of the theory **NST** is similar to that of a theory $\mathfrak{N}\mathfrak{S}_2$ which Hrbacek [1] has given. The differences between the two are in the axiom schema of saturation and the axiom of standardization (the axiom of transfer in [1]). In §3 it is proved that **NST** is a conservative extension of **ZFC** (Zermelo-Fraenkel set theory with the axiom of choice).

§2. Axioms.

We add new unary predicates S and I to the theory **ZFC** formalized in a language having a binary predicate $\in$. Thus we obtain a nonstandard extension **NST** of **ZFC**. Boldface types $\boldsymbol{a}$, $\boldsymbol{A}$, $\cdots$ denote variables of **NST**. We consider that they range over external sets. $S(\boldsymbol{a})$ reads: $\boldsymbol{a}$ is a standard set. Variables ranging over standard sets are denoted by lightface letters a, $A, \cdots$; intuitively, the standard sets can be identified with the members of the "universe of discourse" of **ZFC**. $I(\boldsymbol{a})$ reads: $\boldsymbol{a}$ is an internal set. Variables ranging over internal sets are denoted by Greek letters α, $\beta, \cdots$.

If ϕ is a formula of **ZFC**, ${}^S\phi$ (${}^I\phi$, respectively) denotes a formula obtained by replacing all variables of ϕ by variables ranging over standard sets (internal sets, respectively).

The axioms of **NST** are the following [A. 1]–[A. 12].

[A. 1] ${}^S\phi$ is an axiom of **NST** whenever the sentence ϕ is an axiom of **ZFC**.

[A. 2] $(\forall a)\ I(a)$.

All standard sets are internal.

[A. 3] $(\forall \alpha)(\forall \boldsymbol{b})\,[\boldsymbol{b} \in \alpha \rightarrow I(\boldsymbol{b})\,]$.

The class of the internal sets is transitive.

[A. 4] (*Transfer Principle*)

Let $\phi(k_1, \cdots, k_n)$ be a formula of **ZFC** with free variables $k_1, \cdots, k_n$ and no other free variables. Then

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$$(\forall a_1,\cdots,a_n)\,[{}^I\phi(a_1,\cdots,a_n)\equiv {}^S\phi(a_1,\cdots,a_n)\,].$$

We define

$\boldsymbol{a}$ is finite

$\equiv(\exists m;$ natural number$)\,(\exists \boldsymbol{f})\,[\boldsymbol{f}\colon \boldsymbol{a}\to m$ (1:1, onto)$]$.

[A. 5] (*The Axiom Schema of Saturation*)

Let $\Psi(\boldsymbol{a})$ be a formula of **NST** with a free variable $\boldsymbol{a}$ and possibly other free variables; let $\Omega(\boldsymbol{a}, \boldsymbol{b})$ be a formula of **NST** with free variables $\boldsymbol{a}, \boldsymbol{b}$ and possibly other free variables; and let $\phi\,(k_1, k_2, l_1\cdots, l_n)$ be a formula of **ZFC** with free variables k_1, k_2, $l_1,\cdots, l_n$ and no other free variables. Then

$$\left.\begin{array}{l}(\forall\beta)\,[\Psi(\beta)\to(\exists a)\,\Omega(a,\beta)\,]\\ \wedge(\forall a)(\forall\beta)(\forall\gamma)\,[\Omega(a,\beta)\wedge\Omega(a,\gamma)\to\beta=\gamma]\end{array}\right\}\ (\mathrm{SS})$$

$$\to(\forall\xi_1,\cdots,\xi_n)$$

$$\begin{bmatrix}(\forall\delta)\,[\delta \text{ is finite}\wedge(\forall\alpha\in\delta)\Psi(\alpha)\to(\exists\beta)(\forall\alpha\in\delta)\,{}^I\phi(\alpha,\beta,\xi_1,\cdots,\xi_n)\,]\\ \to(\exists\beta)(\forall\alpha)\,[\Psi(\alpha)\to{}^I\phi(\alpha,\beta,\xi_1,\cdots,\xi_n)\,]\end{bmatrix}.$$

A formula Ψ is said to be a SS-formula if there is a formula Ω such that the sentence (SS) is a theorem of **NST**. For example, the predicate S is a SS-formula.

[A. 5E] (*The Axiom Schema of Enlarging*)

Let $\phi(k_1, k_2, l_1,\cdots, l_n)$ be a formula of **ZFC** with free variables k_1, k_2, $l_1,\cdots, l_n$ and no other free variables. Then

$$(\forall x_1,\cdots,x_n)$$

$$\begin{bmatrix}(\forall d)\,[d \text{ is finite}\to(\exists b)(\forall a\in d)\,{}^S\phi(a,b,x_1,\cdots,x_n)\,]\\ \to(\exists\beta)(\forall a)\,{}^I\phi(a,\beta,x_1,\cdots,x_n)\end{bmatrix}.$$

The axiom schema [A.5E] is weaker than [A.5].

[A. 6] (*The Axiom of Standardization*)

$$(\forall \boldsymbol{A})\,[\,(\exists S)\boldsymbol{A}\subset S\to(\exists a)(\forall x)\,[x\in\boldsymbol{A}\equiv x\in a]\,].$$

The standard set a having the same standard elements as $\boldsymbol{A}$ is denoted by $^*\boldsymbol{A}$; $^*\boldsymbol{A}$ is called the standard kernel of $\boldsymbol{A}$.

[A. 7] (*The Axiom of Extensionality*)

$$(\forall \boldsymbol{A},\boldsymbol{B})\,[\boldsymbol{A}=\boldsymbol{B}\equiv(\forall\boldsymbol{x})\,[\boldsymbol{x}\in\boldsymbol{A}\equiv\boldsymbol{x}\in\boldsymbol{B}]\,].$$

[A. 8] (*The Axiom of Pairing*)

$$(\forall \boldsymbol{A},\boldsymbol{B})(\exists\boldsymbol{C})(\forall\boldsymbol{x})\,[\boldsymbol{x}\in\boldsymbol{C}\equiv\boldsymbol{x}=\boldsymbol{A}\vee\boldsymbol{x}=\boldsymbol{B}].$$

[A. 9] (*The Axiom of Union*)

$$(\forall \boldsymbol{A})(\exists\boldsymbol{B})(\forall\boldsymbol{x})\,[\boldsymbol{x}\in\boldsymbol{B}\equiv(\exists\boldsymbol{y})\,[\boldsymbol{x}\in\boldsymbol{y}\wedge\boldsymbol{y}\in\boldsymbol{A}]\,].$$

[A. 10] (*The Axiom Schema of Comprehension*)

Let $\Phi(\boldsymbol{x})$ be a formula of **NST** with a free variable $\boldsymbol{x}$ and possibly other free variables. Then

$$(\forall A)(\exists B)(\forall \boldsymbol{x})\,[\boldsymbol{x} \in \boldsymbol{B} \equiv \boldsymbol{x} \in A \wedge \Phi(\boldsymbol{x})\,]\,.$$

[A. 11] (*The Axiom of Power Set*)

$$(\forall A)(\exists B)(\forall \boldsymbol{x})\,[\boldsymbol{x} \in \boldsymbol{B} \equiv \boldsymbol{x} \subset A]\,.$$

[A. 12] (*Well Ordering Principle*)

$$(\forall A)(\exists B)\,[B \text{ wellorders } A]\,.$$

§ 3. The conservation theorem.

The following theorem shows that **NST** is a conservative extension of **ZFC**. A process of extension is based on an idea in [1], and our proof is more elementary.

Theorem. *Let ψ be a sentence of ZFC. If ${}^S\psi$ is a theorem of NST, then ψ is a theorem of ZFC.*

Proof. Only finitely many of axioms from [A. 1], say ${}^S\psi_1, \cdots, {}^S\psi_h$, and axioms from [A.2]–[A.12] are used in the proof of ${}^S\psi$ within **NST**. By reflection principle, there is a set R such that any subset of an element of R is an element of R and such that

$$(\psi \equiv \psi^R) \wedge \psi_1^R \wedge \cdots \wedge \psi_h^R\,,$$

where ψ^R and others are the relativizations of ψ and others to R, respectively. Let J be an infinite set, and let $\mathscr{F}$ be an ultrafilter on J. Put $V_0 = R^J \times \{0\}$ and define a one-to-one mapping ζ of R into V_0 by

$$\zeta(a) = (\bar{a},0) \quad (a \in R)\,, \qquad \bar{a}(j) = a \quad (j \in J)\,.$$

Let i_0 and e_0 denote binary relations in V_0 such that

$$((p,0)\,,\ (q,0)) \in i_0 \equiv \{j \in J\colon\ p(j) = q(j)\} \in \mathscr{F} \quad (p, q \in R^J)$$

and

$$((p,0)\,,\ (q,0)) \in e_0 \equiv \{j \in J\colon\ p(j) \in q(j)\} \in \mathscr{F} \quad (p, q \in R^J)\,,$$

respectively. We extend V_0 inductively by

$$V_{n+1} = V_0 \cup (P(V_n) \times \{1\}\,) \text{ (for each nonnegative integer } n)$$

and

$$V = \bigcup_{n=0}^{\infty} V_n\,,$$

where $P(V_n)$ is the power set of V_n. Then we have

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots \subset V$$

and

$$V_0 \cap (\bigcup_{n=1}^{\infty} P(V_n) \times \{1\}) = 0\,.$$

Furthermore, we proceed by induction. Suppose that $i_0,\cdots, i_m, e_0,\cdots, e_m$ have been defined so that they satisfy the following conditions (1) and (2).

(1) Let n be an integer such that $0 \leqq n \leqq m$. Then

$$i_n \subset V_n \times V_n, \qquad e_n \subset V_n \times V_n ;$$

$$(\forall a_1, a_2, b \in V_n)\,[(a_1, a_2) \in i_n \wedge (a_1, b) \in e_n \rightarrow (a_2, b) \in e_n] ;$$

$$(\forall a, b \in V_n)\,[(a, b) \in i_n \equiv (\forall c \in V_n)\,[(c, a) \in e_n \equiv (c, b) \in e_n]] .$$

(2) Let n be an integer such that $1 \leqq n \leqq m$. Then

$$i_n \cap (V_{n-1} \times V_{n-1}) = i_{n-1}, \qquad e_n \cap (V_{n-1} \times V_{n-1}) = e_{n-1} ;$$

for $a \in V_{n-1}$ and $b = (z,1)\,(z \in P(V_{n-1}))$,

$$(a, b) \in e_n \equiv (\exists c \in V_{n-1})\,[(a, c) \in i_{n-1} \wedge c \in z] ;$$

for $a, b \in V_n$,

$$(a, b) \in e_n \equiv (\exists c \in V_{n-1})\,[(a, c) \in i_n \wedge (c, b) \in e_n] .$$

Define e'_{m+1} as the union of $(V_m \times V_0) \cap e_m$ and

$$\{(a,(z, 1)) : a \in V_m \wedge z \in P(V_m) \wedge (\exists c \in V_m)\,[(a, c) \in i_m \wedge c \in z]\} .$$

Moreover, we define

$$i_{m+1} = \{(a, b) \in V_{m+1} \times V_{m+1} : (\forall c \in V_m)\,[(c, a) \in e'_{m+1} \equiv (c, b) \in e'_{m+1}]\}$$

and

$$e_{m+1} = \{(a, b) \in V_{m+1} \times V_{m+1} : (\exists c \in V_m)\,[(a, c) \in i_{m+1} \wedge (c, b) \in e'_{m+1}]\}.$$

It follows that $i_0,\cdots, i_m, i_{m+1}, e_0,\cdots, e_m, e_{m+1}$ satisfy the conditions obtained from (1) and (2) by replacing m by $m+1$. We have thus defined by induction binary relations i_n and e_n for every nonnegative integer n. Let

$$i = \bigcup_{n=0}^{\infty} i_n, \quad e = \bigcup_{n=0}^{\infty} e_n .$$

Then we have

$$i \subset V \times V, \quad i \cap (V_n \times V_n) = i_n (n \geqq 0) ;$$

$$e \subset V \times V, \quad e \cap (V_n \times V_n) = e_n (n \geqq 0) ;$$

and

(3) for $a_1, a_2, b \in V$,

$$(a_1, a_2) \in i \wedge (a_1, b) \in e \rightarrow (a_2, b) \in e ;$$

(4) for $a, b \in V$,

$$(a, b) \in i \equiv (\forall c \in V)\,[(c, a) \in e \equiv (c, b) \in e] ;$$

(5) for $a \in V$ and $b \in V - V_0 (b = (z, 1), \; z \in \bigcup_{n=0}^{\infty} P(V_n))$,

$$(a, b) \in e \equiv (\exists c \in V)\,[(a, c) \in i \wedge c \in z] ;$$

(6) for $a \in V$ and $b \in V_n (n \geqq 1)$,

$$(a, b) \in e \equiv (\exists c \in V_{n-1}) [(a, c) \in i \wedge (c, b) \in e] .$$

Let U be the quotient set of V with respect to the equivalence relation i. We write η for the natural mapping of V onto U. Let $X = \eta[\zeta[R]]$ and $Y = \eta[V_0]$. Then $X \subset Y \subset U$. By (3) and (4), there is a binary relation E in U such that

$$(\eta(a), \eta(b)) \in E \equiv (a, b) \in e \quad (a, b \in V) .$$

We claim that U with the interpretations

$$\begin{array}{lll} (x, y) \in E & \text{for} & \boldsymbol{x} \in \boldsymbol{y} , \\ x \in X & \text{for} & S(\boldsymbol{x}) , \\ x \in Y & \text{for} & I(\boldsymbol{x}) \end{array}$$

satisfies the axioms ${}^S\psi_1, \cdots, {}^S\psi_h$ and [A.2]–[A.12]. For a formula Φ of **NST**, let $\tau(\Phi)$ be a formula of **ZFC** obtained from Φ by the preceding interpretations.

From $\psi_1{}^R, \cdots, \psi_h{}^R$ we have $\tau({}^S\psi_1), \cdots, \tau({}^S\psi_h)$.

It is obvious that U satisfies [A.2].

If ϕ is a formula of **ZFC** and $p_1, \cdots, p_n \in R^J$, then Łoś's theorem asserts that

$$\begin{aligned} &\tau({}^I\phi)(\eta((p_1, 0)), \cdots, \eta((p_n, 0))) \\ &\equiv \{j \in J : \phi^R(p_1(j), \cdots, p_n(j))\} \in \mathscr{F} . \end{aligned}$$

In particular, if $x_1, \cdots, x_n \in X$, then

$$\tau({}^I\phi)(x_1, \cdots, x_n) \equiv \tau({}^S\phi)(x_1, \cdots, x_n) .$$

This shows that U satisfies [A. 4].

Let $\mathscr{F}$ be a $|R|$-good ultrafilter, where $|R|$ is the cardinal number of R. If ϕ is a formula of **ZFC** and Q is a subset of Y such that $|Q| \leqq |R|$, then

$$(\forall y_1, \cdots, y_n \in Y) \begin{bmatrix} (\forall d)[d \text{ is finite} \wedge d \subset Q \rightarrow (\exists b \in Y)(\forall a \in d)\, \tau({}^I\phi)(a, b, y_1, \cdots, y_n)] \\ \rightarrow (\exists b \in Y)(\forall a \in Q)\, \tau({}^I\phi)(a, b, y_1, \cdots, y_n) \end{bmatrix}$$

(see, for example, Saito [3, pp. 74–76]). This implies that U satisfies [A. 5].

Since any subset of an element of R is an element of R, it follows that U satisfies [A.6].

The remaining axioms follow from (3), (4), (5) and (6). This establishes the claim.

Now the proof of ${}^S\psi$ from ${}^S\psi_1, \cdots, {}^S\psi_h$ and [A.2]–[A.12] gives a proof of $\tau({}^S\psi)$ from $\tau({}^S\psi_1), \cdots, \tau({}^S\psi_h)$ and the interpretations of [A.2]–[A.12]. The sentence ψ^R follows from $\tau({}^S\psi)$, and so we have ψ. This gives a proof of ψ within **ZFC**.

References

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