ERROR BOUNDS FOR SPLINE INTERPOLATION

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Abstract

Let s be a cubic spline, with equally spaced knots on [0, 1] interpolating to a given function y at the knots. The parameters which determine s are used to give more accurate approximations to y and its derivatives than those obtained from s, with very little additional effort to compute, at any point $t \in [0, 1]$. Extensions to quintic spline are possible. A selection of numerical results is presented in Tables 1–6.

1. Introduction

Let s be a cubic spline on [0, 1], with equally spaced knots t_i ($i=0, 1,\cdots,n$) and use the notations $m_i=s'(t_i)$ $M_i=s''(t_i)$. It is well known that if $y \in C^4[0, 1]$ and s satisfies the appropriate end conditions then

$$\max|s^{(r)}(t)-y^{(r)}(t)| = O(h^{4-r}) \qquad r=0, 1, 2\,.$$

It is also known that if $y \in C^4[0, 1]$, for a variety of end conditions,

$$m_i = y_i' + O(h^4) \qquad i=0,1,\cdots,n$$

$$(M_{i+1}+10M_i+M_{i-1})/12 = y_i'' + O(h^4) \qquad i=1, 2,\cdots, n-1 \ ([1])\,.$$

The main results of this paper are contained in the following theorems.

THEOREM 1 (cf. [2]). *Let s be an interpolatory cubic spline function which agrees with the function $y \in C^8[0, 1]$ at the equally spaced knots and satisfies the end conditions:*

$$M_0+\lambda_1 M_1 = c_1 \text{ and } M_n+\lambda_2 M_{n-1} = c_2 \qquad (\lambda_i \neq 2+\sqrt{3})\,.$$

Then we have the asymptotic expansions in the interval bounded away from the end points $t=0, 1$:

$$M_i = y_i'' - (h^2/12)y_i^{(4)} + (h^4/360)y_i^{(6)} + O(h^8)$$

$$m_i = y_i' - (h^4/180)y_i^{(5)} + O(h^6)\,.$$

Also,

$$(1) \qquad (-M_{i+2}+34M_{i+1}+294M_i+34M_{i-1}-M_{i-2})/360 = y_i'' + O(h^6)$$

$$(2) \qquad (m_{i+2}-4m_{i+1}+186m_i-4m_{i-1}+m_{i-2})/180 = y_i' + O(h^6)\,.$$

COROLLARY. *Let s_5 be the piecewise quintic polynomial induced by s such that*

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$$s_5(t_i)=y(t_i)$$
$$s_5'(t_i)=(m_{i+2}-4m_{i+1}+186m_i-4m_{i-1}+m_{i-2})/180$$
$$s_5''(t_i)=(-M_{i+2}+34M_{i+1}+294M_i+34M_{i-1}-M_{i-2})/360.$$

Then for any t bounded away from the end points

$$s_5^{(r)}(t)-y^{(r)}(t)=O(h^{6-r}) \qquad r=0,1,2\,.$$

In an exactly analogous manner as in Theorem 1, we have

THEOREM 2. *Let* $y\in C^9[0,1]$ *and the hypotheses of Theorem 1 hold. Then we have*

$$(3)\qquad (-m_{i+3}+9.5m_{i+2}-29m_{i+1}+671m_i-29m_{i-1}+9.5m_{i-2}-m_{i-3})/630 = y_i'+O(h^8)$$

$$(4)\qquad (-M_{i+3}-9M_{i+2}+75M_{i+1}-75M_{i-1}+9M_{i-2}-M_{i-3})/(120h) = y_i^{(3)}+O(h^6)\,.$$

Using (1), (3) and (4), we have

COROLLARY. *Let* s_7 *be the piecewise polynomial of degree 7 induced by s such that*

$$s_7(t_i)=y(t_i)$$
$$s_7'(t_i)=(-m_{i+3}+9.5m_{i+2}-29m_{i+1}+671m_i-29m_{i-1}+9.5m_{i-2}-m_{i-3})/630$$
$$s_7''(t_i)=(-M_{i+2}+34M_{i+1}+294M_i+34M_{i-1}-M_{i-2})/360$$
$$s_7^{(3)}(t_i)=(M_{i+3}-9M_{i+2}+75M_{i+1}-75M_{i-1}+9M_{i-2}-M_{i-3})/(120h)\,.$$

Then for any t bounded away from the end points

$$s_7^{(r)}(t)-y^{(r)}(t)=O(h^{8-r}) \qquad r=0,1,2\,.$$

2. Proof of Theorem 1

Before we procced with analysis, we shall require the following lemmas 1–3. Let A be the matrix of order $n+1$:

$$A=\begin{pmatrix} 1 & \lambda_1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 4 & 1 & \cdot & \cdot & \cdot \\ 0 & 1 & 4 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda_2 & 1 \end{pmatrix}$$

LEMMA 1 (cf. [2]). *If* $\lambda_i\neq 2+\sqrt{3}$, *A is nonsingular for sufficiently large n.*

PROOF. Let us consider the homogeneous system $A\xi=0$. Setting $\theta=-2-\sqrt{3}$ and $p_i(t)=1+\lambda_i t$, we have

$$\xi_i=a\theta^i+b\theta^{-i} \qquad i=0,1,\cdots,n$$

where

$$\begin{bmatrix} p_1(\theta) & p_1(1/\theta) \\ \theta^n p_2(1/\theta) & p_2(\theta)/\theta^n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $p_i\,(1/\theta) \neq 0$, we have the desired result.

LEMMA 2. *If* $\lambda_i \neq 2+\sqrt{3}$, *we have*

$$\|A^{-1}\|_\infty \leq C \qquad \textit{for sufficiently large } n,$$

where C *is a generic constant independent of* n.

PROOF. For any vector ξ, we consider $A\xi=\eta$. Let the sequence $\{a_i\}$ be given by $a_0=0$, $a_1=1$, $a_{i+1}-4a_i+a_{i-1}=0$. Then if $a_{i+1}/a_i \neq \lambda_1$ we have

$$\xi_i = -(a_i-a_{i-1}\lambda_1)/(a_{i+1}-a_i\lambda_1)\,\xi_{i+1}+\sum_{j=0} c_{i,j}\,\eta_j \qquad \text{for some } c_{i,j}$$

where

$$\lim\,(a_i-a_{i-1}\lambda_1)/(a_{i+1}-a_i\lambda_1) = 1/(2+\sqrt{3}) < 1\,.$$

Including the case when $a_{i+1}/a_i=\lambda_1$ for some i, the appropriate submatrix of A is diagonally dominant, from which follows the desired result.

LEMMA 3. *Let* $A^{-1}=(a_{i,j}^{-1})$. *If* $\lambda_i \neq 2+\sqrt{3}$, *we hove*

$$|a_{i,0}^{-1}| \leq C/(2+\sqrt{3})^i \qquad \textit{for suffciently large } n.$$

PROOF. Setting $\theta=-2-\sqrt{3}$, $a_{i,0}^{-1}=a\theta^i+b\theta^{-i}$ $\quad i=0, 1, \cdots, n$

$$\begin{bmatrix} p_1(\theta) & p_1(1/\theta) \\ \theta^n p_2(1/\theta) & p_2(\theta)/\theta^n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus we have

$$a \approx -p_2(\theta)/\{p_1(1/\theta)\,p_2(1/\theta)\theta^{2n}\}\,, \quad b \approx -1/p_1(1/\theta)\,.$$

To prove Theorem 1, let

$$\mu_j = M_j - y_j'' + (h^2/12)y_j^{(4)} - (h^4/360)y_j^{(6)}.$$

Then we have

$$\mu_0+\lambda_1\mu_1 = c_1-\{y_0''-(h^2/12)\,y_0^{(4)}+(h^4/360)\,y_0^{(6)}\}-\lambda_1\{y_1''-(h^2/12)\,y_1^{(4)}+\cdots\}\,.$$

$$\mu_{j+1}+4\mu_j+\mu_{j-1} = O(h^6).$$

$$\mu_n+\lambda_2\mu_{n-1} = c_2-\{y_n''-(h^2/12)\,y_n^{(4)}+(h^4/360)\,y_n^{(6)}\}-\cdots.$$

Combining Lemmas 2 and 3 gives us

$$M_j = y_j''-(h^2/12)\,y_j^{(4)}+(h^4/360)\,y_j^{(6)}+O(h^{2+m}) \qquad m = 0, 1, 2, 3, 4$$
$$\text{for} \quad (m+2-r)\,p \leq j \leq n-(m+2-r)\,p$$

where $p = [-\log(h)/\log(2+\sqrt{3})\,]$ and $r\,(0\leq r\leq m+2)$ is the integer such that $c_1-\{y_0''-(h^2/12)\,y_0^{(4)}+(h^4/360)\,y_0^{(6)}\}-\lambda_1\{y_1''-(h^2/12).$

$$y_1^{(4)}+(h^4/360)\,y_1^{(6)}\} = O(h^r), \cdots.$$

By repeated use of consistency relations, it is possible to rewrite the end condition $\Delta^r M_0=0$ $(r\neq 2)$ as follows:

$$M_0+d_r M_1=\cdots \qquad (d_r\to 2+\sqrt{3})\,.$$

It is well known that the choice of end conditions plays a critical role on the quality of spline approximations. In using the formulas (1), (2) and (3) the end conditions $\Delta^r M_0=\nabla^r M_n=0$ $(r=5,\ 7)$ would give rise to the better approximations.

3. Extensions to quintic spline approximation

In this section we shall consider the quintic spline interpolation under the following end conditions:

$$\text{(5)}\qquad \begin{aligned} &M_0+\alpha_1 M_1+\beta_1 M_2=c_0\,,\quad M_0+\gamma_1 M_1+\delta_1 M_2+\eta_1 M_3=c_1\\ &M_n+\gamma_2 M_{n-1}+\delta_2 M_{n-2}+\eta_2 M_{n-3}=c_{n-1},\quad M_n+\alpha_2 M_{n-1}+\beta_2 M_{n-2}=c_n\,. \end{aligned}$$

Letting θ and $\kappa(|\theta|>|\kappa|>1)$ be the roots of the quartic polynomial $t^4+26t^3+66t^2+26t+1=0$ and $p_i(t)=1+\alpha_i t+\beta_i t^2$, $q_i(t)=1+\gamma_i t+\delta_i t^2+\eta_i t^3$, we have the following theorem.

THEOREM 3. *Let s be an interpolatory quintic spline function which agrees with the function $y\in C^8[0,\ 1]$ at the equally spaced knots and satisfies the end conditions* (5). *If $p_i(1/\theta)\ q_i(1/\kappa)-p_i(1/\kappa)\ q_i(1/\theta)\neq 0$, we have in the interval bounded away from the end points*

$$M_i=y_i''+(h^4/720)\,y_i^{(6)}+O(h^6)$$

$$m_i=y_i'+(h^6/5040)\,y_i^{(7)}+O(h^8)$$

from which follow

$$\text{(5)}\qquad (-M_{i+2}+4M_{i+1}+714M_i+4M_{i-1}-M_{i-2})/720=y_i''+O(h^6)$$

$$\text{(6)}\qquad (-m_{i+3}+6m_{i+2}-15m_{i+1}+5060m_i-15m_{i-1}+6m_{i-2}-m_{i-3})/5040=y_i'+O(h^8)\,.$$

To prove this Theorem we shall require the following lemmas 4–6.

LEMMA 4. *Let the hypotheses of Theorem* 3 *hold. Then the following matrix of order $n+1$ is nonsingular*:

$$B=\begin{pmatrix} 1 & \alpha_1 & \beta_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \gamma_1 & \delta_1 & \eta_1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 26 & 66 & 26 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 26 & 66 & 26 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \eta_2 & \delta_2 & \gamma_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_2 & \alpha_2 & 1 \end{pmatrix}$$

PROOF. Setting $B\xi=0$, we have

$$\xi_i = a\theta^i + b\theta^{-i} + c\kappa^i + d\kappa^{-i} \qquad i = 0, 1, \cdots, n$$

where

$$\begin{pmatrix} p_1(\theta) & p_1(1/\theta) & p_1(\kappa) & p_1(1/\kappa) \\ q_1(\theta) & q_1(1/\theta) & q_1(\kappa) & q_1(1/\kappa) \\ \theta^n q_2(1/\theta) & q_2(\theta)/\theta^n & \kappa^n q_2(1/\kappa) & q_2(\kappa)/\kappa^n \\ \theta^n p_2(1/\theta) & p_2(\theta)/\theta^n & \kappa^n p_2(1/\kappa) & p_2(\kappa)/\kappa^n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the determinant of the coefficient matrix is

$$\theta^n \kappa^n [\prod_j \{p_j(1/\theta)\, q_j(1/\kappa) - p_j(1/\kappa)\, q_j(1/\theta)\}] + \cdots,$$

we have $a=b=c=d=0$ for sufficiently large n, from which follows the desired result.

LEMMA 5. *Let the hypotheses of Theorem 3 hold. Then*

$$\|B^{-1}\|_\infty \le C \qquad \textit{for sufficiently large } n.$$

PROOF. Let us consider the matrix B_p of order p:

$$B_p = \begin{pmatrix} 1 & \alpha_1 & \beta_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \gamma_1 & \delta_1 & \eta_1 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 26 & 66 & 26 & 26 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & & & & & & & & & & \\ \cdot & & & & & & & & & & & \\ \cdot & & & & & & & & & & & \\ \cdot & & & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & 0 & 1 & 26 & 66 & 26 \\ \cdot & \cdot & & & & & & \cdot & \cdot & 0 & 1 & 26 & 66 \end{pmatrix}$$

Setting $p(t)=66+26t+t^2$ and $q(t)=26+66t+26t^2+t^3$, we have

$$p(1/\theta)q(1/\kappa) - p(1/\kappa)\, q(1/\theta) = 26(\kappa-\theta) \neq 0$$

from which, by Lemma 4, B_p is nonsingular for sufficiently large p. Here we consider the system: $B\xi=\eta$.

Since B_p is nonsingular, $\xi_p=\kappa_p\,\xi_{p+1}+l_p\,\xi_{p+2}+\sum d_{p,i}\,\eta_j$ for $p\ge p_0$ and some $d_{i,j}$ provided that p_0 and n are sufficiently large. Thus we have only to show $|\mathit{lim}\ \kappa_p|+|\mathit{lim}\ l_p|<1$. Since κ_p and l_p are independent of η, let $\eta=0$ to compute these values.

Then $\quad \xi_i = a\theta^i + b\theta^{-i} + c\kappa^i + d\kappa^{-i} \qquad i = 0, 1, \cdots, n$

where

$$\begin{pmatrix} p_1(\theta) & p_1(1/\theta) & p_1(\kappa) & p_1(1/\kappa) \\ q_1(\theta) & q_1(1/\theta) & q_1(\kappa) & q_1(1/\kappa) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence we have $b=r_{1,1}a+r_{1,2}c$, $d=r_{2,1}a+r_{2,2}c$ for some $r_{i,j}$, from which follows

$$\xi_i = a(\theta^i + r_{1,1}/\theta^i + r_{2,1}/\kappa^i) + c(\kappa^i + r_{1,2}/\theta^i + r_{2,2}/\kappa^i).$$

Thus we have two equations with respect to κ_p and l_p:

$$(\theta^{p+1}+r_{1,1}/\theta^{p+1}+r_{2,1}/\kappa^{p+1})\,k_p+(\theta^{p+2}+r_{1,1}/\theta^{p+2}+r_{2,1}/\kappa^{p+2})\,l_p$$
$$=\theta^p+r_{1,1}/\theta^p+r_{2,1}/\kappa^p$$
$$(\kappa^{p+1}+r_{1,2}/\theta^{p+1}+r_{2,2}/\kappa^{p+1})\,\kappa_p+(\kappa^{p+2}+r_{1,2}/\theta^{p+2}+r_{2,2}/\kappa^{p+2})\,l_p$$
$$=\kappa^p+r_{1,2}/\theta^p+r_{2,2}/\kappa^p\,.$$

Since $\kappa_p \to 1/\theta+1/\kappa$ and $l_p \to -1/\theta\kappa$, we have the desired result.

LEMMA 6. *Let* $B^{-1}=(b_{i,j}^{-1})$. *Then we have*

$$|b_{i,0}^{-1}|,\ |b_{i,1}^{-1}| \leq C/|\kappa|^i \qquad \textit{for suffciently large } n.$$

PROOF. $b_{i,0}^{-1}$ is represented in the form:

$$b_{i,0}{}^{-1}=a\theta^i+b\theta^{-i}+c\kappa^i+d\kappa^{-i}$$

where

$$\begin{pmatrix} p_1(\theta) & p_1(1/\theta) & p_1(\kappa) & p_1(1/\kappa) \\ q_1(\theta) & q_1(1/\theta) & q_1(\kappa) & q_1(1/\kappa) \\ \theta^n q_2(1/\theta) & q_2(\theta)/\theta^n & \kappa^n q_2(1/\kappa) & q_2(\kappa)/\kappa^n \\ \theta^n p_2(1/\theta) & p_2(\theta)/\theta^n & \kappa^n p_3(1/\kappa) & p_2(\kappa)/\kappa^n \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence we have $a \approx O(1/|\theta\kappa|^n)$, $b \approx O(1)$, $c \approx O(1/|\kappa|^{2n})$, $d \approx O(1)$. Replacing (1, 0, 0, 0)′ by (0, 1, 0, 0)′ we have the estimate for $b_{i,1}^{-1}$.

4. Numerical Illustration

In this section we shall consider the applciation of the stated method by the sample functions under the various end conditions. For the cubic spline, let us impose the following ones:

(1) $M_0=M_n=0$, (2) $\Delta^5 M_0=\nabla^5 M_n=0$ and (3) $M_0=y''(0)$, $M_n=y''(1)$.

S_5 can be used to give better orders of approximations to y than those obtained from s!

Table 1.1 (e^t, $n=32$)

	(1)		(2)		(3)	
5/64	3.20(−6)	4.49(− 6)	−2.68(−9)	−2.29(−14)	−2.59(−9)	−3.65(−10)
17/64	−2.05(−9)	1.66(− 9)	−3.24(−9)	2.58(−14)	−3.24(−9)	−1.10(−13)
31/64	−4.03(−9)	−1.27(−14)	−4.03(−9)	3.31(−14)	−4.03(−9)	3.22(−14)
47/64	−1.95(−9)	4.52(− 9)	−5.17(−9)	4.13(−14)	−5.18(−9)	−3.26(−14)
59/64	8.71(−6)	1.22(− 5)	−6.24(−9)	1.53(−13)	−6.95(−9)	−9.33(−10)

Table 1.2 ($\log(1+t)$, $n=32$)

	(1)		(2)		(3)	
5/64	−3.20(−9)	−4.49(− 6)	1.10(−5)	−1.84(−11)	1.25(−8)	2.19(− 9)
17/64	−4.60(−9)	−1.66(− 9)	3.06(−9)	−4.22(−13)	5.79(−9)	3.95(−13)
31/64	3.06(−9)	−2.44(−14)	2.82(−9)	−1.78(−13)	3.06(−9)	−1.78(−13)
47/64	1.35(−9)	−1.62(−10)	1.64(−9)	−7.50(−13)	1.64(−9)	−2.44(−14)
59/64	−8.01(−9)	−1.12(− 6)	1.09(−9)	2.62(−13)	1.19(−9)	1.37(−10)

Table 1.3 $(1/\{1+25(2t-1)^2\},\ n=32)$

	(1)		(2)		(3)	
5/64	2.34(−6)	3.76(−6)	−3.54(−7)	−1.78(−8)	−3.58(−7)	−2.21(−8)
17/64	−4.47(−6)	1.43(−6)	−4.47(−6)	1.42(−7)	−4.77(−6)	1.42(−7)
31/64	−6.47(−4)	−1.91(−4)	−6.47(−4)	−1.91(−4)	−6.47(−4)	−1.91(−4)

Table 1.4 $(\sin t,\ n=32)$

	(1)		(2)		(3)	
5/64	−1.94(−10)	−1.62(−15)	−1.94(−10)	3.91(−14)	−1.94(−10)	−1.62(−15)
17/64	−6.52(−10)	−5.38(−14)	−6.52(−10)	−5.37(−15)	−6.52(−10)	−5.38(−15)
31/64	−1.16(− 9)	−4.67(−13)	−1.16(− 9)	−9.53(−15)	−1.16(− 9)	−4.67(−14)
47/64	−2.66(− 9)	−1.40(− 9)	−2.66(− 9)	−1.37(−14)	−2.66(− 9)	−1.40(− 9)
59/64	−2.70(− 6)	−3.78(− 6)	−2.70(− 9)	−4.10(−14)	−2.70(− 6)	−3.78(− 6)

S_5' can be used to give better orders of approximations to y' than those obtained from s'!

Table 2.1 $(e^t,\ n=32)$

	(1)		(2)		(3)	
1/8	4.65(−5)	5.88(−5)	−6.00(−9)	1.13(−12)	−9.79(−9)	−4.54(− 9)
3/8	−6.67(−9)	1.48(−9)	−7.71(−9)	2.15(−12)	−7.71(−9)	2.03(−12)
5/8	−1.33(−8)	−4.03(−9)	−9.90(−9)	2.77(−12)	−9.90(−8)	3.09(−12)
7/8	−1.26(−4)	−1.52(−4)	−1.27(−8)	2.27(−12)	−2.42(−9)	1.23(− 8)

Table 2.2 $(\log(1+t),\ n=32)$

	(1)		(2)		(3)	
1/8	−4.65(−5)	−5.58(− 5)	−7.05(−8)	2.50(−10)	−4.77(−8)	2.77(− 8)
3/8	−2.71(−8)	−1.37(− 9)	−2.58(−8)	1.15(−10)	−2.58(−8)	1.15(−10)
5/8	−1.09(−8)	4.06(−10)	−1.12(−8)	3.56(−11)	−1.12(−8)	3.55(−11)
7/8	1.16(−5)	1.35(− 5)	−5.48(−9)	9.28(−12)	−6.90(−9)	−1.69(− 9)

Table 2.3 $(1/\{1+25(2t-1)^2\},\ n=32)$

	(1)		(2)		(3)	
1/8	2.09(−5)	4.80(−5)	−1.83(−5)	1.06(−6)	−1.83(−5)	1.00(−6)
2/8	−1.24(−4)	4.08(−6)	−1.24(−4)	3.84(−6)	−1.24(−4)	3.84(−6)
3/8	2.58(−3)	−1.06(−3)	2.58(−3)	−1.06(−3)	2.58(−3)	−1.06(−3)

Table 2.4 $(\sin t,\ n=32)$

	(1)		(2)		(3)	
1/8	−4.65(−5)	−5.58(− 5)	−7.05(−8)	2.50(−10)	−4.77(−8)	2.77(− 8)
3/8	−2.71(−8)	−1.37(− 9)	−2.58(−8)	1.15(−10)	−2.58(−8)	1.15(−10)
5/8	−1.09(−8)	5.66(−10)	−1.12(−8)	3.56(−11)	−1.12(−8)	3.55(−11)
7/8	1.16(−5)	1.37(− 5)	−5.48(−9)	9.28(−12)	−6.90(−9)	−1.69(− 9)

S_5'' can be used to give better orders of approximations to y'' than those obtained from s''!

Table 3.1 (e^t, $n = 32$)

	(1)		(2)		(3)	
1/8	−5.25(−3)	−2.06(−3)	−9.22(−5)	1.99(−11) −3.24(−13)*	−9.18(−5)	1.68(− 7)
3/8	−1.19(−4)	−5.48(−8)	−1.18(−4)	−4.01(−13)	−1.18(−4)	4.06(−12)
5/8	−1.52(−4)	−1.49(−8)	−1.52(−4)	−9.30(−14)	−1.52(−4)	1.20(−11)
7/8	−1.42(−2)	−5.06(−3)	−1.95(−4)	−4.78(−11) −6.52(−13)*	−1.94(−4)	4.56(− 7)

Table 3.2 ($\log(1+t)$, $n = 32$)

	(1)		(2)		(3)	
1/8	5.46(−3)	2.06(−3)	3.05(−4)	8.14(− 9) 3.07(−10)*	3.02(−4)	−1.01(− 6)
3/8	1.37(−4)	5.48(−8)	1.37(−4)	2.41(−11)	1.37(−4)	−2.02(−12)
5/8	7.00(−5)	1.37(−8)	7.00(−4)	6.44(−12)	7.00(−4)	4.77(−12)
7/8	1.33(−3)	5.15(−4)	3.95(−5)	−1.38(−10) 5.25(−13)*	3.95(−5)	−6.29(− 8)

Table 3.3 ($1/\{1+25(2t-1)^2\}$, $n = 32$)

	(1)		(2)		(3)	
1/8	−2.56(−2)	−1.73(−3)	−2.12(−2)	6.17(−6)	−2.12(−2)	8.96(−6)
2/8	−1.36(−1)	6.25(−5)	−1.30(−1)	7.10(−5)	−1.30(−1)	7.10(−5)
3/8	3.70(−1)	5.79(−3)	3.70(−1)	5.79(−3)	3.70(−1)	5.79(−3)
4/8	−1.89(+1)	5.75(−1)	−1.89(+1)	5.75(−1)	−1.89(+1)	5.75(−1)

Table 3.4 ($\sin t$, $n = 32$)

	(1)		(2)		(3)	
1/8	−1.01(−5)	−1.06(−14)	−1.01(−5)	−1.87(−11)	−1.01(−5)	−1.06(−10)
3/8	−2.98(−5)	1.23(−12)	−2.98(−5)	6.58(−15)	−2.98(−5)	7.17(−15)
5/8	−4.75(−5)	4.61(− 8)	−4.67(−5)	6.57(−14)	−4.67(−5)	3.82(−12)
7/8	4.28(−3)	1.74(− 3)	−6.25(−5)	1.15(−11)	−6.21(−5)	1.41(− 7)

($*\cdots \Delta^7 M_0 = \nabla^7 M_n = 0$)

Tables 4 contain the errors in s'_r and s' to $\mathbf{e}^t$ under the end conditions:

$$\Delta^7 M_0 = \nabla^7 M_n = 0 .$$

Table 4.1 $(n=32)$

	s'-y'	s'_5-y'	s'_7-y'
1/8	−6.00(−9)	1.67(−12)	−2.00(−15)
2/8	−6.80(−9)	1.90(−12)	4.44(−16)
3/8	−7.71(−9)	2.15(−12)	6.66(−16)
4/8	−8.73(−9)	2.43(−12)	−5.33(−15)
5/8	−9.90(−9)	2.77(−12)	4.44(−15)
6/8	−1.12(−8)	3.13(−12)	−1.55(−15)
6/8	−1.27(−8)	3.55(−12)	2.00(−15)

Table 4.2 $(n=16)$

	s-y	s_5-y	s_7-y
7/32	−4.94(−8)	1.61(−12)	4.44(−15)
13/32	−5.96(−8)	1.94(−12)	−1.55(−15)
25/32	−8.67(−8)	2.81(−12)	−1.18(−14)

Table 5 contains the errors in s and s_7 to Chebyshev and Legendre polynomials with degree 20 under the end conditions:

$$\Delta^7 M_0 = \nabla^7 M_n = 0 .$$

Table 5 $(n=64)$

	Chebyshev		Legendre	
	s-y	s_7-y	s-y	s_7-y
21/128	−2.65(−5)	−7.18(−10)	5.11(−6)	1.56(−10)
41/128	−3.11(−5)	−9.50(−10)	−6.02(−6)	−2.18(−10)
61/128	4.01(−5)	1.71(− 9)	7.53(−5)	4.26(−10)
81/128	−4.57(−5)	−5.61(− 9)	−6.47(−5)	−1.52(− 9)
101/128	−8.55(−5)	5.87(− 8)	−3.34(−5)	1.75(− 8)

For the quintic spline, it is possible to rewrite the end condition $m_0 = y'_0$ as follows:

$$hy'_0 = -(37y_0+54y_1-9y_2-8y_3)/12$$
$$+ h^2(-23M_0+354M_1+20M_2+8M_3)/240 \quad ([3]) .$$

Table 6 gives us the errors for the quintic spline interpolating to the sample functions under the end conditions:

$$\Delta^7 M_0 = \nabla^7 M_n = 0 \text{ and } m_0 = y'_0 , \ m_n = y'_n .$$

Let $\bar{s}'$ and $\bar{s}''$ denote the values modified by (5) and (6).

Table 6.1 (e^t, $n=32$)

	s''-y''	$\bar{s}''$-y''	s'-y'	$\bar{s}'$-y'
1/8	1.49(−9)	−1.37(−11)	4.48(−13)	2.43(−13)
2/8	1.70(−9)	−1.29(−12)	2.57(−13)	1.98(−14)
3/8	1.93(−9)	−1.08(−12)	2.85(−13)	1.55(−14)
4/8	2.18(−9)	−4.60(−13)	3.15(−13)	9.55(−15)
5/8	2.47(−9)	−8.72(−13)	3.63(−13)	1.73(−14)
6/8	2.80(−9)	−6.02(−13)	4.11(−13)	1.95(−14)
7/8	3.18(−9)	4.72(−12)	5.62(−13)	1.21(−13)

Table 6.2 ($\log(1+t)$, $n=32$)

	s''-y''	$\bar{s}''$-y''	s'-y'	$\bar{s}'$-y'
1/8	−7.58(−8)	2.99(− 9)	2.09(−11)	−3.89(−11)
2/8	−4.14(−8)	4.84(−10)	2.64(−11)	−1.77(−12)
3/8	−2.34(−8)	1.97(−10)	1.42(−11)	−2.44(−13)
4/8	−1.39(−8)	9.70(−11)	7.74(−12)	−8.69(−14)
5/8	−8.60(−9)	5.11(−11)	4.42(−12)	−4.75(−14)
6/8	−5.52(−9)	2.77(−11)	2.62(−12)	−3.16(−14)
7/8	−3.67(−9)	−9.36(−13)	1.31(−12)	−3.41(−13)

Table 6.3 ($1/\{1+25(2t-1)^2\}$, $n=32$)

	s''-y''	$\bar{s}''$-y''	s'-y'	$\bar{s}'$-y'
1/8	1.20(−4)	4.54(−5)	2.76(− 6)	2.64(− 6)
2/8	3.36(−3)	2.88(−3)	5.29(− 5)	5.44(− 5)
3/8	5.46(−2)	8.63(−2)	1.34(− 3)	1.42(− 3)
4/8	−1.22(+0)	−7.61(−1)	1.33(−14)	1.33(−14)

Table 6.4 ($\sin t$, $n=32$)

	s''-y''	$\bar{s}''$-y''	s'-y'	$\bar{s}'y'$
1/8	−1.65(−10)	−6.84(−13)	−1.82(−13)	9.58(−16)
2/8	−3.28(−10)	−1.44(−13)	−1.78(−13)	1.18(−15)
3/8	−4.58(−10)	−2.01(−13)	−1.69(−13)	3.12(−15)
4/8	−6.35(−10)	−2.83(−13)	−1.64(−13)	−1.71(−15)
5/8	−7.75(−10)	−2.77(−13)	−1.59(−13)	−9.13(−15)
6/8	−9.03(−10)	−2.06(−13)	−1.42(−13)	−7.09(−15)
7/8	−1.02(− 9)	−1.76(−12)	−1.57(−13)	−3.93(−14)

References

[1] T. LUCAS, *Error bounds for interpolating cubic splines under various end conditions*, SIAM J. Numer. Anal. **11** (1974), 191–198.

[2] G. BEHFOROOZ and N. PAPAMICHAEL, *Improved orders of approximation derived from interpolatory cubic splines*, BIT **19** (1979), 19–26.

[3] W. HOSKINS and G. MCMASTER, *Multipoint boundary expansions for spline interpolation*, Proceeding of the second Manitoba conference on Numerical Mathematics. Utilitas Mathematica Publishing Incorporated. Winnipeg, 1972.