ON THE PATHS IN GENERALIZED BERWALD SPACES

ON THE PATHS IN GENERALIZED BERWALD SPACES

ON THE PATHS IN GENERALIZED BERWALD SPACES

By

Tadashi Aikou* and Masao Hashiguchi**

(Received September 30, 1981)

Abstract

In the present paper we consider whether the paths in generalized Berwald spaces can coincide with the geodesics.

§0. Introduction.

Let a Finsler space be a generalized Berwald space with respect to a generalized Cartan connection ([10], [1]). Then, in addition to the geodesics of the space, there are defined the paths with respect to the generalized Cartan connection.

The purpose of the present paper is to consider whether the paths can coincide with the geodesics. We shall first treat the problem in a general Finsler space (§2), and then it will be shown that, if the space becomes a generalized Berwald space with respect to a generalized Cartan connection satisfying the condition of the problem, the space is nothing but a Berwald space, and all such generalized Cartan connections will be determined (§3). Applying the results to the case of Wagner spaces, we shall recognize that a Berwald space cannot become a non-trivial Wagner space satisfying the condition of the problem (§4). On the other hand, a Randers change of Finsler metrics is generally not projective ([7], [2]). Lastly we shall consider whether a Wagner connection can be introduced in the Finsler space with the changed metric in such a way that the paths coincide with the geodesics of the Finsler space with the original metric (§5).

Throughout the present paper the terminology and notations are referred to Matsumoto [4, 5, 6] and Hashiguchi [1]. As to generalized Berwald spaces, for convenience' sake, we shall summerize in §1 from [1], [8] the definitions necessary for the discussion.

The authors wish to express their sincere gratitude to Professor Dr. M. Matsumoto and Professor Dr. Y. Ichijyō for the invaluable suggestions and encouragement. This interesting problem was proposed by Prof. Matsumoto at the Symposium on Finsler Geometry (held at Naruto, Japan, 6–8 Oct., 1980) for Wagner spaces, and then the

* Kagoshima-chuo High School, Kagoshima, Japan.
** Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

authors' attention was drawn by Prof. Ichijyō to treat the problem in generalized Berwald spaces.

§1. Generalized Berwald spaces.

We are concerned with an n-dimensional Finsler space $F^n=(M, L)$, where $L(x, y)$ is the fundamental function, and x denotes a point of the underlying manifold M, and $y=\dot{x}$ denotes a supporting element. Then the fundamental tensor field is given by $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$.

For a given skew-symmetric and $(0)p$-homogeneous Finsler tensor field $T_j{}^i{}_k$, there exists a unique Finsler connection $C\Gamma(T)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ satisfying the following four axioms:

(C1) $C\Gamma(T)$ is metrical: $g_{ij|k}=0$, $g_{ij}|_k=0$,

(C2) The deflection tensor field D vanishes: $N^i{}_k=y^j F_j{}^i{}_k$,

(C3) The $(h)h$-torsion tensor field T is the given $T_j{}^i{}_k$: $F_j{}^i{}_k-F_k{}^i{}_j=T_j{}^i{}_k$,

(C4) The $(v)v$-torsion tensor field S^1 vanishes: $C_j{}^i{}_k=C_k{}^i{}_j$.

$C\Gamma(T)$ is called a *generalized Cartan connection* and its coefficients are given by

$$
(1.1)\quad
\begin{cases}
F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - g^{ih}C_{jkm}(C_h{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_h) + C_j{}^i{}_m(C_k{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_k) \\
\qquad + C_k{}^i{}_m(C_j{}^m{}_r A_0{}^r{}_0 - A_0{}^m{}_j) + A_j{}^i{}_k\,, \\
N^i{}_k = G^i{}_k - C_k{}^i{}_r A_0{}^r{}_0 + A_0{}^i{}_k\,, \\
C_j{}^i{}_k = (g^{ir}\dot{\partial}_j g_{kr})/2\,,
\end{cases}
$$

where $A_j{}^i{}_k=(T_j{}^i{}_k-T^i{}_{jk}+T_{jk}{}^i)/2$, and $(\Gamma^*{}_j{}^i{}_k, G^i{}_k, C_j{}^i{}_k)$ are the coefficients of the Cartan connection $C\Gamma=C\Gamma(0)$, and the subscript 0 means the contraction by y.

A Finsler space is called a *generalized Berwald space* if there is possible to introduce a generalized Cartan connection $C\Gamma(T)$ in such a way that the coefficients $F_j{}^i{}_k$ are functions of position only.

Also, for a given $(0)p$-homogeneous covariant Finsler vector field s_j, there exists a unique Finsler connection $W\Gamma(s)=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ satisfying (C1), (C2), (C4) and

(C3*) $W\Gamma(s)$ is semi-symmetric with respect to the given s_j:

$$
F_j{}^i{}_k - F_k{}^i{}_j = \delta_j{}^i s_k - \delta_k{}^i s_j\,.
$$

$W\Gamma(s)$ is called a *Wagner connection*, and its coefficients are given by

$$
(1.2)\quad
\begin{cases}
F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + L^2(S_j{}^i{}_{kl}+C_j{}^i{}_m C_k{}^m{}_l)s^l \\
\qquad + (y^i C_{jkl}-y_j C_k{}^i{}_l-y_k C_j{}^i{}_l)s^l + C_j{}^i{}_k s_0 + g_{jk}s^i - \delta_k{}^i s_j\,, \\
N^i{}_k = G^i{}_k - L^2 C_k{}^i{}_l s^l + y_k s^i - \delta_k{}^i s_0\,, \\
C_j{}^i{}_k = (g^{ir}\dot{\partial}_j g_{kr})/2\,,
\end{cases}
$$

where $s^l=g^{lm}s_m$, $y_j=g_{jr}y^r$, and $S_j{}^i{}_{kl}$ are the coefficients of the v-curvature tensor field S^2 of $C\Gamma$.

A Finsler space is called a *Wagner space* if there is possible to introduce a Wagner connection $W\Gamma(s)$ in such a way that the coefficients $F_j{}^i{}_k$ are functions of position only.

On the other hand, in his recent paper [8], Matsumoto generalized Okada's axioms [9], which determine the Berwald connection $B\Gamma$, and gave the notion of a *generalized Berwald connection* $B\Gamma(T)$. Let $T_j{}^i{}_k$ be a skew-symmetric and (0)p-homogeneous Finsler tensor field satisfying the condition

$$(1.3) \qquad y^r(\dot{\partial}_k T_j{}^i{}_r - \dot{\partial}_j T_k{}^i{}_r) = 0 .$$

Then $B\Gamma(T) = (F_j{}^i{}_k, N^i{}_k, 0)$ is a Finsler connection determined by the following four axioms:

(B1) $L_{|i} = \partial_i L - (\dot{\partial}_r L) N^r{}_i = 0$,

(B2) The deflection tensor field D vanishes: $N^i{}_k = y^j F_j{}^i{}_k$,

(B3) The $(v)hv$-torsion tensor field P^1 vanishes: $F_j{}^i{}_k = \dot{\partial}_j N^i{}_k$,

(B4) The $(h)h$-torsion tensor field T is the given $T_j{}^i{}_k$: $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$.

The coefficients of $B\Gamma(T)$ are given by

$$(1.4) \qquad \begin{cases} F_j{}^i{}_k = G_j{}^i{}_k - (\dot{\partial}_j \dot{\partial}_k T^i{}_{00} + \dot{\partial}_j T_k{}^i{}_0)/2 , \\ N^i{}_k = G^i{}_k - (\dot{\partial}_k T^i{}_{00} + T_k{}^i{}_0)/2 , \end{cases}$$

where $(G_j{}^i{}_k, G^i{}_k, 0)$ are the coefficients of $B\Gamma = B\Gamma(0)$.

Contrary to the case of $C\Gamma(T)$, $T_j{}^i{}_k$ in $B\Gamma(T)$ is not necessarily given arbitrarily. It must satisfy the condition (1.3). It is noted, however, that (1.3) holds good if $T_j{}^i{}_k$ depend on position only, and then

Proposition 1.1. *Let $T_j{}^i{}_k$ be a skew-symmetric and (0)p-homogeneous tensor field. If $T_j{}^i{}_k = T_j{}^i{}_k(x)$, then $C\Gamma(T) = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ and $B\Gamma(T) = (\bar{F}_j{}^i{}_k, \bar{N}^i{}_k, 0)$ are defined. $F_j{}^i{}_k$ are functions of position only, if and only if $\bar{F}_j{}^i{}_k$ are functions of position only. In this case $\bar{F}_j{}^i{}_k = F_j{}^i{}_k$.*

Thus the notion of a generalized Berwald space can be defined in terms of $B\Gamma(T)$. Since the notion of a Berwald space was defined in terms of $B\Gamma$, the above result of Matsumoto is very satisfactory to the establishment of the notion of a generalized Berwald space.

§2. The paths with respect to a generalized Cartan connection.

In a Finsler space F^n with a generalized Cartan connection $C\Gamma(T)$, let us consider whether the paths with respect to $C\Gamma(T)$ coincide with the geodesics of F^n.

The geodesics of F^n and the paths with respect to $C\Gamma(T) = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ are respectively given by

$$(2.1) \qquad \frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = \lambda \frac{dx^i}{dt} ,$$

$$(2.2) \qquad \frac{d^2 x^i}{dt^2} + F_j{}^i{}_k\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \mu \frac{dx^i}{dt} ,$$

where $G^i = G_0{}^i{}_0/2$.

As is well-known in the projective theory, (2.1) and (2.2) give the same curves if and only if there exists a scalar field p such that

$$(2.3) \qquad\qquad F_0{}^i{}_0 = 2G^i + py^i \ .$$

Since from (1.1) we have $F_0{}^i{}_0 = 2G^i + T_{00}{}^i$, (2.3) is equivalent to $T_{00}{}^i = py^i$. Contracting this by y_i, we get $p = 0$, that is, $T_{00}{}^i = 0$, or equivalently $F_0{}^i{}_0 = 2G^i$. Thus we have

Theorem 2.1. *Let* $C\Gamma(T) = (F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ *be a generalized Cartan connection defined in a Finsler space* F^n. *The paths with respect to* $C\Gamma(T)$ *coincide with the geodesics of* F^n, *if and only if it holds good*

$$(2.4) \qquad\qquad T_r{}^s{}_k y^r y_s = 0 \ ,$$

or equivalently

$$(2.5) \qquad\qquad F_j{}^i{}_k y^j y^k = 2G^i \ .$$

If we consider a Finsler space F^n with a generalized Berwald connection $B\Gamma(T) = (F_j{}^i{}_k,\ N^i{}_k,\ 0)$ in place of $C\Gamma(T)$, we get $F_0{}^i{}_0 = 2G^i + T_{00}{}^i$ from (1.4) and the (0)p-homogeneity of $T_j{}^i{}_k$. The paths with respect to $B\Gamma(T)$ are given in the same way as (2.2), and we have

Theorem 2.2. *Let* $B\Gamma(T) = (F_j{}^i{}_k,\ N^i{}_k,\ 0)$ *be a generalized Berwald connection defined in a Finsler space* F^n. *The paths with respect to* $B\Gamma(T)$ *coincide with the geodesics of* F^n, *if and only if it holds* (2.4), *or equivalently* (2.5).

In the following we proceed the discussion in terms of $C\Gamma(T)$.

Especially, we consider the case of $T_j{}^i{}_k = T_j{}^i{}_k(x)$. Differentiating (2.4) by y^i and y^j successively, we get

$$(2.6) \qquad\qquad 2T_r{}^s{}_k C_{ijs} y^r + T_j{}^s{}_k g_{si} + T_i{}^s{}_k g_{sj} = 0 \ .$$

Conversely, contracting (2.6) by $y^i y^j$, we get (2.4). If we use the *Q-tensor* defined by

$$(2.7) \qquad\qquad Q_{ijk}{}^h = 2C_{ijk} y^h + g_{ik}\delta_j{}^h + g_{jk}\delta_i{}^h \ ,$$

(2.6) is written in the form

$$(2.8) \qquad\qquad Q_{ijs}{}^r T_r{}^s{}_k = 0 \ .$$

The *Q*-tensor is defined by Ichijyō [3], and has the following meaning.

Proposition 2.1. *Let* $C\Gamma(T) = (F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ *be a generalized Cartan connection, and let* $D_j{}^i{}_k$ *be a Finsler tensor field of type* (1, 2). *Then,* $(F_j{}^i{}_k - D_j{}^i{}_k,\ N^i{}_k - D_0{}^i{}_k,\ C_j{}^i{}_k)$ *is a generalized Cartan connection if and only if* $Q_{ijs}{}^r D_r{}^s{}_k = 0$.

On the other hand, if we contract (2.6) by $y^i g^{jk}$, we get

$$(2.9) \qquad\qquad T_i{}^s{}_s = 0 \ .$$

Thus we have

Theorem 2.3. *Let $C\Gamma(T)$ be a generalized Cartan connection defined in a Finsler space F^n. In the case of $T_j{}^i{}_k = T_j{}^i{}_k(x)$, the paths with respect to $C\Gamma(T)$ coincide with the geodesics of F^n if and only if (2.8) is satisfied. Then (2.9) holds good.*

§3. The paths in generalized Berwald spaces.

Now, let a Finsler space F^n be a generalized Berwald space with respect to a generalized Cartan connection $C\Gamma(T) = (F_j{}^i{}_k(x),\ N^i{}_k,\ C_j{}^i{}_k)$. If the paths with respect to $C\Gamma(T)$ coincide with the geodesics of F^n, (2.5) holds good. Differentiating by y^j and y^k successively, we get

(3.1) $$F_j{}^i{}_k + F_k{}^i{}_j = 2G_j{}^i{}_k.$$

Thus the coefficients $G_j{}^i{}_k$ of the Berwald connection $B\Gamma$ of F^n depend on position only. Hence, the space is just a Berwald space.

Since $\Gamma^*{}_j{}^i{}_k = G_j{}^i{}_k$ in a Berwald space, (3.1) is also written as

(3.1′) $$F_j{}^i{}_k + F_k{}^i{}_j = 2\Gamma^*{}_j{}^i{}_k.$$

If we solve from $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$, we get

(3.2) $$F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + T_j{}^i{}_k/2,$$

and then

(3.3) $$N^i{}_k = G^i{}_k + T_0{}^i{}_k/2,$$

where $T_j{}^i{}_k$ satisfy (2.8).

Conversely, in a Berwald space the Finsler connection $(F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ given by (3.2), (3.3) for some skew-symmetric and $(0)p$-homogeneous tensor field $T_j{}^i{}_k(x)$ satisfying (2.8) is a generalized Cartan connection, whose coefficients $F_j{}^i{}_k$ depend on position only, and the paths with respect to the connection coincide with the geodesics. Thus we have proved

Theorem 3.1. *Let F^n be a generalized Berwald space with respect to a generalized Cartan connection $C\Gamma(T) = (F_j{}^i{}_k(x),\ N^i{}_k,\ C_j{}^i{}_k)$. If the paths with respect to $C\Gamma(T)$ coincide with the geodesics of F^n, F^n is a Berwald space, and $C\Gamma(T)$ is given by (3.2), (3.3) for $T_j{}^i{}_k(x)$ satisfying (2.8).*

Conversely, in a Berwald space F^n, let $T_j{}^i{}_k(x)$ be a skew-symmetric and $(0)p$-homogeneous tensor field satisfying (2.8). Then, the Berwald space F^n becomes a generalized Berwald space with respect to the generalized Cartan connection $C\Gamma(T)$ given by (3.2), (3.3), and the paths with respect to $C\Gamma(T)$ coincide with the geodesics of F^n.

It depends on the existence of the non-trivial solution $T_j{}^i{}_k(x)$ of (2.8) whether a Berwald space may become various generalized Berwald spaces in which the paths coincide with the geodesics.

§4. The paths with respect to a Wagner connection.

In this section, we consider a Finsler space $F^n=(M, L)$ with a Wagner connection $W\Gamma(s)$. Since the $(h)h$-torsion tensor field is given by $T_j{}^i{}_k=\delta_j{}^i s_k-\delta_k{}^i s_j$, (2.4) becomes $L^2 s_k-s_0 y_k=0$, that is,

$$(4.1) \qquad\qquad s^i = pl^i ,$$

where $l^i=y^i/L$. Corresponding to Theorem 2.1 we have

Theorem 4.1. *Let $W\Gamma(s)$ be a Wagner connection defined in a Finsler space F^n. The paths with respect to $W\Gamma(s)$ coincide with the geodesics of F^n if and only if s^i is proportional to the supporting element y^i.*

Substituting in (1.2) from (4.1), we get

$$(4.2) \qquad \begin{cases} F_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + p\,(LC_j{}^i{}_k + g_{jk}l^i-\delta_k{}^i l_j) , \\ N^i{}_k = G^i{}_k - ph^i{}_k , \end{cases}$$

where $l_j=\dot\partial_j L=g_{jr}l^r$, $h^i{}_k=\delta_k{}^i-l^i l_k$, and p is some Finsler scalar field.

Conversely, a Finsler connection given by (4.2) is a Wagner connection $W\Gamma(pl)$, with respect to which the paths coincide with the geodesics. Thus we have

Theorem 4.2. *In a Finsler space $F^n=(M, L)$, all Wagner connections $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$, with respect to which the paths coincide with the geodesics of F^n, are given by (4.2), where p is an arbitrary $(0)p$-homogeneous Finsler scalar field.*

Especially, we consider the case of $s_j=s_j(x)$. Since $T_j{}^i{}_k$ depend on position only, (2.9) implies $s_j=0$. Thus we have corresponding to Theorem 2.3 and Theorem 3.1 respectively

Theorem 4.3. *Let $W\Gamma(s)$ be a Wagner connection defined in a Finsler space F^n. In the case of $s_j=s_j(x)$, if the paths with respect to $W\Gamma(s)$ coincide with the geodesics of F^n, s_j vanishes, and the $W\Gamma(s)$ becomes the Cartan connection.*

Theorem 4.4. *Let a Finsler space F^n be a Wagner space with respect to a Wagner connection $W\Gamma(s)$. If the paths with respect to $W\Gamma(s)$ coincide with the geodesics of F^n, F^n is a Berwald space. Moreover, a Berwald space cannot become a non-trivial Wagner space in such a way the paths coincide with the geodesics of F^n.*

§5. Randers changes of metrics and Wagner connections.

Let $F^n=(M, L)$ be a Finsler space. If we consider a change of the metrics $L\to\bar L$, we get another Finsler space $\bar F^n=(M, \bar L)$ on the common underlying manifold M. If any geodesic of F^n is also a geodesic of $\bar F^n$, and the inverse is true, the change is called *projective*. For example, as is well known, a *Randers change* $L\to\bar L=L+b_i(x)y^i$ (b_i is a covariant vector field on M) is projective if and only if $b_i(x)$ is a gradient vector field: $b_i(x)=\partial_i b(x)$. Therefore, the geodesics are not invariant by general Randers changes.

So, we shall consider whether the geodesics of $F^n=(M, L)$ can be represented as the paths with respect to some Wagner connection $W\bar{\Gamma}(s)=(\bar{F}_{j}{}^{i}{}_{k}, \bar{N}^{i}{}_{k}, \bar{C}_{j}{}^{i}{}_{k})$ in $\bar{F}^n=(M, \bar{L})$.

From (1.2) we get

$$(5.1) \qquad \bar{F}_0{}^i{}_0 = \bar{N}^i{}_0 = 2\bar{G}^i+\bar{L}^2\bar{s}^i-s_0 y^i ,$$

where $\bar{G}^i{}_k$ is the non-linear connection of the Cartan connection in $\bar{F}^n$, and $\bar{s}^i=\bar{g}^{ir}s_r$. On the other hand, it is known ([7], [2]) that $\bar{G}^i$ are given by

$$(5.2) \qquad \bar{G}^i = G^i+Lg^{ij}b_{[jk]}y^k+L\bar{L}^{-1}(b_{j|k}y^jy^k/2-Lg^{hj}b_hb_{[jk]}y^k)\, l^i ,$$

where $b_{[jk]}=(\partial_k b_j-\partial_j b_k)/2$, and $b_{j|k}$ is the h-covariant derivative with respect to the Cartan connection $C\Gamma$ in F^n. Therefore we get

$$\bar{F}_0{}^i{}_0 = 2G^i+2Lg^{ij}b_{[jk]}y^k+L\bar{L}^{-1}(b_{j|k}y^jy^k-2Lg^{hj}b_hb_{[jk]}y^k)\, l^i+\bar{L}^2\bar{s}^i-s_0 y^i .$$

So, the paths with respect to $W\bar{\Gamma}(s)$ in $\bar{F}^n$ coincide with the geodesics of F^n, if and only if there exists a Finsler scalar field P such that

$$2Lg^{ij}b_{[jk]}y^k+\bar{L}^2\bar{s}^i = Py^i ,$$

which is equivalent to

$$(5.3) \qquad s_h = \bar{L}^{-2}(P\bar{y}_h-2Lg^{ij}\bar{g}_{ih}b_{[jk]}y^k) .$$

Thus we have

Theorem 5.1. *Let $\bar{F}^n=(M, \bar{L})$ be the Finsler space obtained by a Randers change $L\to\bar{L}=L+b_i(x)\, y^i$ from a Finsler space $F^n=(M, L)$. There exists in $\bar{F}^n$ a Wagner connection $W\bar{\Gamma}(s)$ such that the paths with respect to $W\bar{\Gamma}(s)$ coincide with the geodesics of the original Finsler space F^n. All such connections $W\bar{\Gamma}(s)$ are given by s_h of (5.3), where P is an arbitrary $(1)p$-homogeneous Finsler scalar field.*

It is a future problem whether $\bar{F}^n=(M, \bar{L})$ becomes a Wagner space by some vector field $s_j(x)$ satisfying (5.3).

References

[1] HASHIGUCHI, M., *On Wagner's generalized Berwald space,* J. Korean Math. Soc. **12** (1975), 51–61.

[2] HASHIGUCHI, M. and Y. ICHIJYŌ, *Randers spaces with rectilinear geodesics,* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **13** (1980), 33–40.

[3] ICHIJYŌ, Y., *On the G-connections and motions on a {V, G}-manifold,* J. Math. Tokushima Univ. **14** (1980), 11–23.

[4] MATSUMOTO, M., *The theory of Finsler connections,* Publ. Study Group of Geom. **5**, Depart. Math., Okayama Univ., 1970.

[5] MATSUMOTO, M., *Metrical differential geometry,* Kiso Sugaku Sensho **14**, Shokabo, Tokyo, 1975 (in Japanese).

[6] MATSUMOTO, M., *Foundation of Finsler geometry and special Finsler spaces,* to appear.

[7] MATSUMOTO, M., *On Finsler spaces with Randers' metric and special forms of important tensors,* J. Math. Kyoto Univ. **14** (1974), 477–498.

[8] MATSUMOTO, M., *Berwald connections with (h)h-torsion and generalized Berwald spaces,* Tensor, N.S. **35** (1981), 223–229.

[9]   OKADA, T., *Minkowskian product of Finsler spaces and Berwald connection*, to appear.
[10] WAGNER, V., *On generalized Berwald spaces*, C.R. (Doklady) Acad. Sci. URSS (N.S.) **39** (1943), 3–5.