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A NONSTANDARD REPRESENTATION OF FOURIER TRANSFORMS OF CONTINUOUS FUNCTIONS

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Abstract

We give a nonstandard representation of Fourier transforms of continuous functions on compact abelian groups. Applying the representation, we prove the Bochner-Weil theorem on compact abelian groups.

Introduction

Let G be a compact abelian group with dual Γ. We denote $C(G)$ the space of all continuous functions on G and define the Fourier transform $\hat{f}$ of $f \in C(G)$ by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \quad (\gamma \in \Gamma).$$

In 1, we represent G and Γ by $*$-finite abelian groups and give a nonstandard representation of functions in $C(G)$ and their Fourier transforms. In [2, 6.5 Theorem], Luxemburg has given a nonstandard proof of the Bochner's theorem. Using our representation, we give a nonstandard proof of the Bochner-Weil theorem on compact abelian groups in 2. It is simpler than the proof in [2].

Throughout the present paper, we adopt a nonstandard set theory NST in [1] to express nonstandard analysis by axiomatic method. For example, every standard infinite set has nonstandard elements.

1. A nonstandard representation

Theorem 1. *Let G be a standard compact abelian group with dual* Γ. *Then there*

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exist an internal closed subgroup H of G and a $*$-finite subgroup D of G/H satisfying the following (i) - (vi) :

(i) *Let Λ be the annihilator of H. Then $^\circ\Gamma \subset \Lambda$, where $^\circ\Gamma$ is the set of all standard elements of Γ.*

(ii) *Let Δ be the dual of D and $\Theta : \Lambda \to \Delta$ be the natural mapping. Then the restriction of Θ to $^\circ\Gamma$ is one-to-one.*

(iii) *Let $\xi : G \to G/H$ be the natural mapping. Then for every $x \in G$ there is $y \in G$ such that $\xi(y) \in D$ and $x \approx y$.*

(iv) *Suppose that $f \in C(G)$ is standard. Define the internal function $\Phi(f) \in C(G/H)$ by*

$$\Phi(f)(\xi(x)) = \int_H f(x+h)\,dh \quad (\xi(x) \in G/H)$$

and let $\Psi(f)$ be the restriction of $\Phi(f)$ to D. If $x \in G$ and $\xi(x) \in D$, then $f(x) \approx \Psi(f)(\xi(x))$.

(v) *If $f \in C(G)$ is standard, then*

$$\int_G f(x)\,dx \approx \frac{1}{|D|} \sum_{t \in D} \Psi(f)(t),$$

where $|D|$ is the cardinal of D.

(vi) *If $f \in C(G)$ and $\gamma \in \Gamma$ are standard, then*

$$\hat{f}(\gamma) \approx \widehat{\Psi(f)}(\Theta(\gamma)).$$

Proof. Let $\{F_j\}$ be any finite set of standard finite subsets of Γ. Then there is a standard finitely generated subgroup Λ of Γ such that $F_j \subset \Lambda$ for each j. By the saturation principle, there exists an internal finitely generated subgroup Λ of Γ such that $F \subset \Lambda$ for all standard finite subset F of Γ. Let H be the annihilator of Λ. Then $H \subset U$ for every neighborhood U of 0 in G. Since the internal group G/H is topologically isomorphic to the direct sum of a $*$-finite dimensional torus group and a $*$-finite abelian group, it follows from the transfer principle that for any finite set of neighborhoods V_j of 0 in G/H, $g_j \in C(G/H)$, $\gamma_j \in \Lambda$, and standard $\varepsilon_j > 0$ ($j=1, 2, \cdots, n$), there exists a $*$-finite subgroup D of G/H such that

$$\forall t \in G/H\ \exists w \in D\quad t-w \in U_j\,; \qquad \cdots(1)$$

$$\gamma_j \in A(\Lambda, D) \to \gamma_j = 0\,; \qquad \cdots(2)$$

$$\left| \int_{G/H} g_j(t)\ dt - \frac{1}{|D|} \sum_{t \in D} g_j(t) \right| < \varepsilon_j\,; \qquad \cdots(3)$$

fro each j ($j=1,2,\cdots,n$), where $A(\Lambda, D)$ is the annihilator of D in Λ. Put

$$M = \{\xi[U] : U \text{ is a standard neighborhood of } 0 \text{ in } G\}$$

and

$$K = \{\Phi(f) : f \text{ is a standard function in } C(G)\}.$$

Since M and K are S-size, it follows that for any finite sets of $V_j \in M$, $g_j \in K$, standard $\gamma_j \in \Gamma$, and standard $\varepsilon_j > 0\,(j=1,2,\cdots,n)$, there exists a $*$-finite subgroup D of G/H satisfying (1)-(3)for each j=1, 2, $\cdots$, n. The saturation principle implies that there exists a $*$-finite subgroup D of G/H such that for any $V \in M$, $g \in K$, standard $\gamma \in \Gamma$, and standard $\varepsilon > 0$,

$$\forall t \in G/H \ \exists w \in D \ \ t-w \in V\,; \qquad \cdots(4)$$

$$\gamma \in A(\Lambda, D) \rightarrow \gamma = 0\,; \qquad \cdots(5)$$

$$\left| \int_{G/H} g(t)\,dt - \frac{1}{|D|} \sum_{t \in D} g(t) \right| < \varepsilon \qquad \cdots(6)$$

Let $\Delta = \Lambda / A(\Lambda, D)$. Then Δ is the dual of D. From (5), we have ${}^{\circ}\Gamma \cap A(\Lambda, D) = \{0\}$ and so (ii) follows. By (4) and the saturation principle, there is $w \in D$ such that for any standard neighborhood U of 0 in G, $t-w \in \xi\,[U]$. Thus we obtain (iii). From the continuity of g, we obtain (iv). Let f be a standard function in C(G). From (6) and the fact that

$$\int_G f(x)\,dx = \int_{G/H} \Phi(f)(t)\,dt,$$

we obtain (v). Suppose that $f \in C(G)$ and $\gamma \in \Gamma$ are standard. Then it follows from (v) that

$$\begin{aligned} \hat{f}(\gamma) &= \int_G f(x)\,\overline{(x, \gamma)}\,dx \approx \frac{1}{|D|} \sum_{t \in D} \Psi(f\bar{\gamma})(t) \\ &= \frac{1}{|D|} \sum_{t \in D} \Psi(f)(t)\,\overline{(t, \Theta(\gamma))} = \widehat{\Psi(f)}(\Theta(\gamma)). \end{aligned}$$

This completes the proof of the theorem.

2. An application to positive-definite functions

Applying Theorem 1, we obtain a nonstandard proof of the Bochner Weil theorem.

Theorem 2 (**Bochner-Weil**). *Let G be a compact abelian group with dual* Γ *and f be a continuous positive-definite function on G. Then* $\hat{f}(\gamma) \geq 0$ *for all* $\gamma \in \Gamma$ *and* $\sum_{\gamma \in \Gamma} \hat{f}(\gamma)$ *is finite.*

Proof. First let D be a standard finite abelian group with dual Δ and f be a standard positive definite function on D. It follows form the definition of $\hat{f}$ that

$$\hat{f}(\delta)=\frac{1}{|D|}\sum_{x\in D} f(x)(-x,\ \delta)=\frac{1}{|D|}\sum_{x\in D} f(x-y)(-x+y,\delta) \qquad (\delta\in\Delta, y\in D).$$

Summing over $y\in D$, we have

$$|D|\,\hat{f}(\delta)=\frac{1}{|D|}\sum_{x\in D}\sum_{y\in D} f(x-y)(-x,\delta)\overline{(-y,\xi)}\geq 0 \qquad (\delta\in\Delta).$$

Also we have $\sum_{\delta\in\Delta}\hat{f}(\delta)=f(0)$.

Now suppose that G is a standard compact abelian group and f is a standard continuous positive-definite function on G. Then there exist an internal closed subgroup H of G and a $*$-finite subgroup $\mathbb{D}$ of G/H as in Theorem 1. If g is an internal function in C(G/H), then we have

$$\int_{G/H}\int_{G/H} g(u)\,\overline{g(v)}\,\Phi(f)(u-v)\,du\,dv = \int_G\int_G g(\xi(x))\,\overline{g(\xi(y))}\,f(x-y)\,dx\,dy\geq 0.$$

This shows that $\Phi(f)$ is internally positive-definite on G/H, and so is $\Psi(f)$ on D. By the transfer principle, we have $\widehat{\Psi(f)}(\delta)\geq 0$ for all $\delta\in\Delta$ and $\sum_{\delta\in\Delta}\widehat{\Psi(f)}(\delta)=\Psi(f)(0)$. If $\gamma\in\Gamma$ is standard, then it follows from (vi) in Theorem 1 that

$$\hat{f}(\gamma)\approx\widehat{\Psi(f)}(\theta(\gamma))$$

Since the right hand side is non-negative, it follows that $\hat{f}(\gamma)\geq 0$ for all standard $\gamma\in\Gamma$. Let $\{\gamma_1,\cdots,\gamma_m\}$ be any standard finite subset of Γ. Since $\Theta(\gamma_1),\cdots,\Theta(\gamma_m)$ are distinct by (ii) in Theorem 1, it follows that

$$\sum_{j=1}^{m}\widehat{\Psi(f)}(\Theta(\gamma_j))\leq\sum_{\delta\in\Delta}\widehat{\Psi(f)}(\delta)=\Psi(f)(0).$$

Taking the standard parts, we have

$$\sum_{j=1}^{m}\hat{f}(\gamma_j)\leq f(0)$$

This implies that $\sum_{\gamma\in\Gamma}\hat{f}(\gamma)$ is finite. The proof of the theorem is complete.

References

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