Buffon's short needle on the sphere

Yukinao ISOKAWA *
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Abstract

We study Buffon's short needle problem on the 2-dimensional sphere. We throw a short needle on a grid of circles of latitudes, and find the probability p_S that it intersects at least one circle. We prove that this probability p_S is strictly smaller than the probability p_E of the classical Buffon's short needle problem in the 2-dimensional Euclidean plane. Moreover we give an asymptotic expansion of the probability p_S as the number of grids n tends large. This expansion roughly tells that p_S can be approximated by p_E fairly well even if n is relatively small.

1 Introduction

In a memoir submitted to the Académie des Sciences in 1733, Buffon gave birth to the field of geometric probability. In that paper (not to be published until 1777) he introduced the classical problem, which bears his name, of finding the probability that a needle thrown at random on a grid of evenly spaced parallel lines will touch a line.

Buffon's original needle problem has flourished in many directions. The needle has been lengthen and bent (Kendall and Moran(1963), Ramaley(1969), Diaconis(1976), Santaló(1976)); intersection probabilities for much more general families of "needles" and "grids" have been described (Solomon(1978)); the needles have been thrown in higher than 2 dimensional Euclidean spaces (Santaló(1976)); the grids has been modified to improve statistics which estimate π (Schuster(1974), Perlman and Wichura (1975)) and inverse problems to the original one have been studied (Detemple and Robertson (1980), Robertson and Siegel (1986)).

In spite of such flourish of studies on Buffon's problem, the present author has seldom seen needles thrown in non-Euclidean planes, in particular, on the sphere. Only one exception that the present author has found is Peter and

*Faculty of Education, Kagoshima University

Tanasi (1984). Thus it seems that there remain some unsolved problems about the needle on the sphere, and in this paper we study one such problem on the 2-dimensional sphere.

Before we study a needle on the sphere, we need to decide on what kind of grid a needle is thrown, because no grid of parallel lines exist on the sphere. Peter and Tanasi (1984) investigated a needle thrown on a grid of circles of longitudes. In contrast to their study, in this paper, we throw a needle on a grid of circles of latitudes.

Now we give a precise formulation of our problem. Let $\mathbf{S}^2$ be the sphere with unit radius. Denoting by $E(u)$ the circle of latitude u, we consider a family of $(2n+1)$ circles $\{E(u_i) : i = -n, \ldots, -1, 0, 1, \ldots, n\}$ where $u_i = i \cdot \pi/(2(n+1))$. In other words, we consider a grid of $(2n+1)$ equidistant curves with a common spherical distance $D_n = \pi/(2(n+1))$ apart. Note that $E(u_0)$ denotes for the equator of $\mathbf{S}^2$. On this grid of equidistant curves, we throw a needle with length L at random. Our problem is to find a probability $p_S(D_n, L)$ that the needle intersects at least one of the equidistant curves.

In this paper in order to avoid some complexity that results from a needle possibly intersecting more than one equidistant curves, we assume that our needle is short enough. To be precise we assume that $L \leq D_n$.

2 An expression for the Buffon's short needle probability

To state an answer to the Buffon's short needle problem on the sphere, we introduce a function

$$Q(u, L) = \arcsin\left(\sin\frac{L}{2}\sec u\right) - \sin u \cdot \arcsin\left(\tan\frac{L}{2}\tan u\right). \tag{2.1}$$

Theorem 1 *Assume that $L \leq D_n$, where n is a non-negative integer. Then the Buffon's short needle probability $p_S(D_n, L)$ can be given by*

$$\frac{4}{\pi}\left\{\frac{1}{2}Q(0, L) + \sum_{i=1}^{n} Q(iD_n, L)\right\}.$$

In particular, letting $n = 0$ in Theorem 1 and noting that $Q(0, L) = L/2$, we have the following corollary.

Corollary *The probability that a needle intersects the equator is equal to* L/π.

In order to prove Theorem 1, we represent the sphere $\mathbf{S}^2$ by the unit sphere of the 3-dimensional Euclidean space whose center lies at the origin, i.e., $X^2 + Y^2 + Z^2 = 1$. We may assume that the equator $E(u_0)$ is represented by a great circle $X^2 + Y^2 = 1, Z = 0$ in the XY-plane. Then $E(u)$ can be represented by a small circle which is an intersection of a plane $Z = \sin u$ with the unit sphere. Let O denote the intersection point of the equator $E(u_0)$ with the ZX-plane, and U the intersection point of $E(u)$ with the same plane. Now we introduce a function

$$f(x,u) = \arccos\left(\frac{\cos L \sin x - \sin u}{\sin L \cos x}\right) , \tag{2.2}$$

which is well-defined if $u - L < x < u + L$.

Lemma 2.1 *Suppose that one of the endpoints of a needle drops at* P *which lies on the* ZX*-plane and has the latitude* x*, and the other endpoint drops at* Q *on the equidistant curve* $E(u)$*. Assume that* $u - L < x < u + L$*. Then, the angle* OPQ *which we denote by* θ *is given by* $f(x)$*.*

Proof. Denote the longitude of Q by ϕ. Then the Cartesian coordinates of three points P, Q, and U are given by

$$\begin{pmatrix} \cos x \\ 0 \\ \sin x \end{pmatrix}, \begin{pmatrix} \cos u \cos\phi \\ \cos u \sin\phi \\ \sin u \end{pmatrix}, \text{ and } \begin{pmatrix} \cos u \\ 0 \\ \sin u \end{pmatrix}$$

respectively. Accordingly, if we denote the spherical distance UQ by y, we have

$$\begin{cases} \cos L = \cos x \cos u \cos\phi + \sin x \sin u \\ \cos y = \cos^2 u \cos\phi + \sin^2 u \end{cases} .$$

Then, eliminating ϕ from these expressions, we get

$$\cos y = \frac{\cos u(\cos L - \sin x \sin u)}{\cos x} + \sin^2 u . \tag{2.3}$$

Now, using the cosine formula of spherical geometry, we have

$$\cos y = \cos(x-u)\cos L + \sin(x-u)\sin L \cos\theta . \tag{2.4}$$

From (2.3) and (2.4) we can deduce the desired expression (2.2). Thus the proof of the lemma is completed.

As we see later, in order to compute the probability p_S, we need to evaluate an indefinite integral

$$F(x,u) = \int f(x,u) \cos x \, dx . \tag{2.5}$$

Lemma 2.2

$$\begin{aligned} F(x,u) &= \sin x \cdot f(x,u) + \sin u \cdot \arcsin\left(\frac{\sin x - \cos L \sin u}{\sin L \cos u}\right) \\ &\quad - \arccos\left(\frac{\cos L - \sin x \sin u}{\cos x \cos u}\right) \end{aligned}$$

Proof. By differentiation we can easily check that $\dfrac{\partial}{\partial x}F(x,u) = f(x,u)\cos x$.

Now we prove Theorem 1.

Proof of Theorem 1. Without loss of generality, we may assume that one of the endpoints of a needle, P, drops on the northern hemisphere. Then the latitude x of P is distributed according to the probability density $\cos x$. Consequently, using Lemma 2.1, we have

$$p_S(D_n, L) = \sum_{i=0}^{n} \int_{u_i}^{u_i+L} \frac{f(x,u_i)}{\pi} \cos x \; dx + \sum_{i=1}^{n} \int_{u_i-L}^{u_i} \left(1 - \frac{f(x,u_i)}{\pi}\right) \cos x \; dx \; .$$

Since, by Lemma 2.2,

$$\text{(2.6)} \qquad F(u+L,u) = \frac{\pi}{2} \sin u \quad \text{and} \quad F(u-L,u) = \pi \sin(u-L) - \frac{\pi}{2} \sin u \; ,$$

we can see that

$$\int_{u_i}^{u_i+L} f(x,u_i) \; \cos x \; dx = \int_{u_i-L}^{u_i} (\pi - f(x,u_i)) \; \cos x \; dx \; .$$

Thus we have
(2.7)

$$p_S(D_n, L) = \frac{1}{\pi}\left[(F(L,0) - F(0,0)) + 2\sum_{i=1}^{n}(F(u_i+L,u_i) - F(u_i,u_i))\right] \; .$$

Again, by Lemma 2.1, we have

$$\text{(2.8)} \qquad \begin{aligned} F(u,u) = &\ \frac{\pi}{2}\sin u + 2\sin u \cdot \arcsin\left(\frac{1-\cos L}{\sin L}\tan u\right) \\ &- \arccos\left(\cos L - (1-\cos L)\tan^2 u\right) \; . \end{aligned}$$

Therefore, substitution of (2.6) and (2.8) into (2.7) establishes the theorem.

3 Comparison of the probability p_S with the Buffon needle probability in the Euclidean plane

In this section we will compare $p_S(D_n, L)$ with the classical Buffon needle probability in the Euclidean plane,

$$p_E(D_n, L) = \frac{2L}{\pi D_n} .$$

We start our investigation from the Euler-Maclaurin formula. Let us put

$$J_n = J_n(L) = \int_0^{nD_n} Q(u, L)\, du + \frac{D_n}{2} Q(nD_n, L) . \tag{3.1}$$

Lemma 3.1

$$D_n \left[\frac{1}{2} Q(0, L) + \sum_{i=1}^{n} Q(iD_n, L) \right] < J_n(L)$$

Proof. The Euler-Maclaurin formula asserts that there exists a number θ such that $0 < \theta < 1$ and

$$\frac{1}{2} Q(0, L) + \sum_{i=1}^{n} Q(iD_n, L) = \frac{1}{D_n} \cdot J_n(L) + R_1 ,$$

where, B_1 denoting the 1st Bernoullian number,

$$R_1 = \frac{B_1}{2} D_n^2 \sum_{i=0}^{n-1} \frac{\partial^2 Q}{\partial u^2}((i+\theta)D_n, L) .$$

Therefore the next Lemma 3.2 immediately establishes the present lemma.

Lemma 3.2 *The function $Q(u, L)$ is a strictly decreasing and concave function of u.*

Proof. By an elementary calculus we have

$$\frac{\partial Q}{\partial u}(u, L) = -\cos u \cdot \arcsin\left(\tan\frac{L}{2} \tan u \right) \tag{3.2}$$

and

$$\frac{\partial^2 Q}{\partial u^2}(u, L) = \sin u \cdot \arcsin\left(\tan\frac{L}{2} \tan u \right) - \frac{\sin^2 \frac{L}{2}}{\sqrt{\cos^2 \frac{L}{2} - \sin^2 u}} . \tag{3.3}$$

From (3.2) it immediately follows that Q is a strictly decreasing function of u. In order to show the concavity of Q, we put $a = \cot\frac{L}{2}$ and $t = \tan u$. Then $\dfrac{\partial^2 Q}{\partial u^2}$ can be written as

$$\frac{1+t^2}{t\sqrt{a^2-t^2}} - \arcsin\frac{t}{a}\ ,$$

which we denote by $g_1(t)$. Obviously the function $g_1(t)$ is well-defined for $0 < t < a$.
Since

$$g_1'(t) = \frac{t^4+2t^2-a^2}{t^2(a^2-t^2)^{3/2}}\ ,$$

the function g_1 has its minimum at $t = \sqrt{\sqrt{a^2+1}-1}$ and its minimum is equal to

$$\frac{(a^2+1)^{1/4}}{\sqrt{a^2+1}-1} - \arcsin\left(\sqrt{\frac{\sqrt{a^2+1}-1}{a^2}}\right)\ .$$

Now, letting $b = \sqrt{\dfrac{\sqrt{a^2+1}-1}{a^2}}$, we can rewrite the minimum of g_1 as

$$\frac{b\sqrt{1-b^2}}{1-2b^2} - \arcsin b\ ,$$

which we denote by $g_2(b)$. The function $g_2(b)$ is defined for $0 < b < \sqrt{\sqrt{2}-1}$. Since $g_2'(b) = \dfrac{4b\sqrt{1-b^2}}{(1-2b^2)^2}$, g_2 is strictly increasing. Accordingly we see that $g_2(b) > g_2(0) = 0$. Therefore the minimum of g_1 is positive, which implies that Q is a concave function of u.

Now we study an upper estimate for $J_n(L)$. Let us introduce a function

$$j(t) = \int_0^{nD_n} q(u,t)\,du + \frac{D_n}{2}\,q(nD_n,t)\ , \tag{3.4}$$

where

$$q(u,t) = \sqrt{1-t\ \sin^2 u}\ . \tag{3.5}$$

Furthermore we put

$$t(L) = \sec^2\frac{L}{2}\ . \tag{3.6}$$

Lemma 3.3

$$J_n(L) < \frac{L}{2}$$

Proof. We can easily check that

$$\frac{\partial Q}{\partial L} = \frac{1}{2}\, q(u, t(L)) \ .$$

Hence

$$\frac{d}{dL} J_n(L) = \frac{1}{2}\, j(t(L)) \ .$$

To say in other words,

$$J_n(L) = \frac{1}{2} \int_0^L j(t(w))\, dw \ . \tag{3.7}$$

Since

$$\frac{d}{dt} q(u,t) = -\frac{\sin^2 u}{2\sqrt{1 - t\sin^2 u}} < 0 \ ,$$

we can see that $j(t)$ is a decreasing function of t. Moreover, we have

$$j(1) = \sin nD_n + \frac{D_n}{2} \cos nD_n = \cos D_n + \frac{D_n}{2} \sin D_n < 1 \ .$$

Consequently, from $t(L) > 1$, we can deduce $j(t(L)) < 1$. Therefore, by (3.7), we obtain $J_n(L) < L/2$, which completes the proof.

Combining Lemma 3.1 and Lemma 3.3, we obtain the following theorem.

Theorem 2 *Assume that* $L \leq D_n$. *Then, for all non-negative integer* n,

$$p_S(D_n, L) < p_E(D_n, L) = \frac{2L}{\pi D_n},$$

4 Asymptotic behaviour of the probability p_S

In this section we study an asymptotic behaviour of the Buffon needle probability as n tends to the infinity. Our starting point is again the Euler-Maclaurin formula, which asserts that

$$\begin{aligned} &\frac{1}{2} Q(0, L) + \sum_{i=1}^{n} Q(iD_n, L) \\ &= \frac{1}{D_n} \cdot J_n + \frac{B_1}{2}\, D_n \left\{ \frac{\partial Q}{\partial u}(nD_n, L) - \frac{\partial Q}{\partial u}(0, L) \right\} + K_n \ , \end{aligned} \tag{4.1}$$

where

$$J_n = \int_0^{nD_n} Q(u, L)\, du + \frac{D_n}{2}\, Q(nD_n, L) \tag{4.2}$$

and

$$K_n = -\frac{D_n^4}{24}\int_0^1 \phi_4(t)\sum_{i=0}^{n-1}\frac{\partial^4 Q}{\partial u^4}((i+t)D_n, L)\ dt\ . \tag{4.3}$$

(In the above B_1 stands for the 1-st Bernoullian number, and ϕ_4 the 4-th Bernullian polynomial.)

First we study an asymptotic behaviour of J_n. In this study we need to evaluate definite integrals

$$C_m = \int_0^{nD_n} \tan^{2m} u\ \cos u\ du \tag{4.4}$$

for $m \geq 0$. Furthermore, in this study, we need to use functions

$$g(x) = \sum_{m=2}^{\infty}\frac{(2m-3)!!}{2^m\ m!}\ \frac{x^{m-1}}{m-1} \tag{4.5}$$

and

$$G(x) = \int_0^x \sqrt{x}g(x)dx\ . \tag{4.6}$$

In the following Lemma 4.1 and Lemma 4.2, we prepare certain preliminary results for these quantities (4.4), (4.5), and (4.6).

Lemma 4.1

$$C_0 = \cos D_n\ ,\ C_1 = -\cos D_n + \log\cot\frac{D_n}{2}\ ,$$

and for $m \geq 2$,

$$C_m = \frac{1}{2m-2}\cdot\frac{\cos^{2m-1} D_n}{\sin^{2m-2} D_n} - \frac{2m-1}{2m-2}C_{m-1}\ . \tag{4.7}$$

Proof. We can easily compute C_0 and C_1. For $m \geq 2$, changing variable as $x = \sin u$, we have

$$C_m = \int_0^{\sin nD_n}\frac{x^{2m}}{(1-x^2)^m}\ dx\ .$$

Then integration by parts leads to the desired recurrence relation (4.7).

Lemma 4.2

$$g(x) = \frac{1}{2} + \log 2 - \frac{1-\sqrt{1-x}}{x} - \frac{1}{2}\log\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} - \frac{1}{2}\log x\ ,$$

and

$$
\begin{aligned}
G(x) &= -2\sqrt{x} + \frac{4}{3}\sqrt{x}\sqrt{1-x} + \frac{2}{3}\arcsin\sqrt{x} + \frac{5+6\log 2}{9}x^{\frac{3}{2}} \\
&\quad -\frac{1}{3}x^{\frac{3}{2}}\log\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} - \frac{1}{3}x^{\frac{3}{2}}\log x \,.
\end{aligned}
$$

Proof. This lemma can be proved by an elementary calculus. Thus we omit the proof.

Lemma 4.3 *Assume that* $\epsilon < \frac{1}{2}D_n^2$. *Then*

$$
\begin{aligned}
j(1+\epsilon) &= \cos D_n + \frac{1}{2}\epsilon\left(\cos D_n - \log\cot\frac{D_n}{2}\right) \\
&\quad -\frac{1}{2}\epsilon\cos D_n\, g\left(\epsilon\ \cot^2 D_n\right) + \frac{D_n^2}{2}\sqrt{1-\epsilon\cot^2 D_n} + O\left(D_n^4\log\frac{1}{D_n}\right) \,.
\end{aligned}
$$

Proof. Recall the definition (3.4) in the previous section of the function j. Expanding $q(u, 1+\epsilon)$ defined by (3.5) into a Maclaurin series, we have

$$
q(u,1+\epsilon) = \cos u\ \sqrt{1-\epsilon\tan^2 u} = \cos u\left\{1 + \sum_{m=1}^{\infty}\binom{\frac{1}{2}}{m}\left(-\epsilon\tan^2 u\right)^m\right\} \,.
$$

Note that $\epsilon\tan^2 u < \frac{1}{2}$ because $\epsilon < \frac{1}{2}D_n^2$ and $u \leq nD_n$. Consequently the infinite series in the above converges uniformly in u, and we get

$$
\int_0^{nD_n} q(u,1+\epsilon)du = C_0 + \sum_{m=1}^{\infty}\binom{\frac{1}{2}}{m}(-\epsilon)^m C_m \,.
$$

Using Lemma 4.1, we have

$$
\begin{aligned}
\sum_{m=2}^{\infty}\binom{\frac{1}{2}}{m}(-\epsilon)^m C_m &= -\sum_{m=2}^{\infty}\frac{(2m-3)!!}{2^m\ m!}\epsilon^m C_m \\
&= -\sum_{m=2}^{\infty}\frac{(2m-3)!!}{2^m\ m!}\epsilon^m\cdot\frac{1}{2m-2}\ \frac{\cos^{2m-1}D_n}{\sin^{2m-2}D_n} \\
&\quad +\sum_{m=2}^{\infty}\frac{(2m-3)!!}{2^m\ m!}\epsilon^m\cdot\frac{2m-1}{2m-2}\ C_{m-1} \,,
\end{aligned}
$$

which we write as $(-S_1 + S_2)$.
Now we evaluate S_1 and S_2. Using Lemma 4.2, we can express

$$
S_1 = \frac{1}{2}\ \epsilon\ \cos D_n\ g\left(\epsilon\ \frac{\cos^2 D_n}{\sin^2 D_n}\right) \tag{4.8}
$$

On the other hand, since from (4.7) it follows that

$$C_{m-1}=\frac{1}{2m-4}\,\frac{\cos^{2m-3}D_n}{\sin^{2m-4}D_n}-\frac{2m-3}{2m-4}\,C_{m-2}<\frac{1}{2m-4}\,\frac{1}{\sin^{2m-4}D_n}$$

for $m\geq 3$, we have

$$S_2<\frac{3}{16}\epsilon^2C_1+\sum_{m=3}^{\infty}\frac{(2m-3)!!}{2^m\ m!}\left(\frac{D_n^2}{2}\right)^m\cdot\frac{2m-1}{2m-2}\,\frac{1}{2m-4}\,\frac{1}{\sin^{2m-4}D_n}\ .$$

Hence

$$S_2=O\left(D_n^4\log\frac{1}{D_n}\right)\ . \tag{4.9}$$

Therefore, using (4.8) and (4.9), and noting that

$$\frac{D_n}{2}\ q(nD_n,1+\epsilon)=\frac{D_n^2}{2}\sqrt{1-\epsilon\cot^2 D_n}\ ,$$

we have completed the proof of the lemma.

Now we define a function

$$\epsilon(w)=\tan^2\frac{w}{2}$$

and show the following lemma which gives estimates for various integrals concerning $\epsilon(w)$.

Lemma 4.4

(a)
$$\int_0^L \epsilon(w)\ dw=\frac{L^3}{12}+O\left(L^5\right)$$

(b)
$$\int_0^L \epsilon(w)g(\epsilon(w)\cot^2 D_n)\ dw=\tan^3 D_n\ G\left(\frac{L^2}{4}\cot^2 D_n\right)+O\left(L^5\right)$$

(c)
$$\int_0^L \sqrt{1-\epsilon(w)\cot^2 D_n}\ dw$$
$$=\tan D_n\ \arcsin\left(\frac{L}{2}\cot D_n\right)+\frac{L}{2}\sqrt{1-\frac{L^2}{4}\cot^2 D_n}+O\left(L^5\right)\ .$$

Proof. Since $\epsilon(w)=w^2/4+O\left(w^4\right)$, we can easily see (a).
Now we put

$$M_0=\max_{0\leq x\leq 1/2}|g(x)|\ \text{ and }M_1=\max_{0\leq x\leq 1/2}|g'(x)|\ .$$

Then, noting that $\epsilon(w)\cot^2 D_n \le 1/2$ for $0 \le w \le L$, and using the mean value theorem, we have

$$\begin{aligned}
&\left|\int_0^L \epsilon(w)\ g\left(\epsilon(w)\cot^2 D_n\right)\ dw - \int_0^L \frac{w^2}{4} g\left(\frac{w^2}{4}\cot^2 D_n\right)\ dw\right| \\
\le\ & \int_0^L \left|\epsilon(w) - \frac{w^2}{4}\right|\ g\left(\frac{w^2}{4}\cot^2 D_n\right)\ dw \\
& + \int_0^L \frac{w^2}{4}\left|g\left(\epsilon(w)\cot^2 D_n\right) - g\left(\frac{w^2}{4}\cot^2 D_n\right)\right|\ dw \\
\le\ & M_0 \int_0^L \left|\epsilon(w) - \frac{w^2}{4}\right|\ dw + M_1 \int_0^L \epsilon(w)\ \left|\epsilon(w) - \frac{w^2}{4}\right|\ \cot^2 D_n\ dw \\
=\ & O\left(L^5\right)
\end{aligned}$$

Accordingly, changing variable as $x = \frac{w^2}{4}\cot^2 D_n$, we get (b).
Finally, in a similar way to that for the derivation of (b), we can show (c). Thus the proof of the lemma is completed.

Combining Lemma 4.3 and Lemma 4.4, we obtain an asymptotic behaviour of J_n as follows

Proposition 4.5 *Assume that $b = L/(2D_n)$ be a constant. Then, as n tends to the infinity,*

$$\begin{aligned}
J_n \ =\ & \frac{L}{2} + D_n^3\left\{\frac{1-12\log 2}{36}b^3 - \frac{1}{12}\ b\sqrt{1-b^2} + \frac{1}{12}\arcsin b\right. \\
& \left. + \frac{b^3}{6}\ \log\left(1+\sqrt{1-b^2}\right) + \frac{b^3}{6}\log D_n\right\} + O\left(D_n^5 \log\frac{1}{D_n}\right)
\end{aligned}$$

Proof. Recall the relation (3.7) of the previous section, that is,

$$J_n = \frac{1}{2}\int_0^L j(1+\epsilon(w))\ dw\ .$$

Using Lemma 4.3, we have

$$\begin{aligned}
J_n \ =\ & \frac{L}{2}\cos D_n + \frac{1}{4}\left(\cos D_n - \log\cot\frac{D_n}{2}\right)\int_0^L \epsilon(w)\ dw \\
& - \frac{1}{4}\cos D_n \int_0^L \epsilon(w) g(\epsilon(w)\cot^2 D_n)\ dw \\
& + \frac{D_n}{4}\ \sin D_n \int_0^L \sqrt{1-\epsilon(w)\cot^2 D_n}\ dw \\
& + O\left(L D_n^4 \log\frac{1}{D_n}\right)\ .
\end{aligned}$$

Then, using Lemma 4.4, we get

$$\begin{aligned} J_n(L) &= \frac{L}{2}\cos D_n + \frac{L^3}{48}\left(\cos D_n - \log\cot\frac{D_n}{2}\right) - \frac{1}{4}\cos D_n \cdot \tan^3 D_n\ G\left(\frac{L^2}{4}\cot^2 I\right. \\ &\quad + \frac{D_n}{4}\sin D_n\left(\tan D_n \arcsin\left(\frac{L}{2}\cot D_n\right) + \frac{L}{2}\sqrt{1-\frac{L^2}{4}\cot^2 D_n}\right) \\ &\quad + O\left(D_n^5 \log\frac{1}{D_n}\right) . \end{aligned}$$

Hence follows the desired expression.

Now we study an asymptotic behaviour of K_n. By differentiation we can see

$$\frac{\partial^4 Q}{\partial u^4}(u,L) = -k_1(u) + k_2(u) - k_3(u) - 3k_4(u) ,$$

where

$$k_1(u) = \sin u \cdot \arcsin\left(\tan\frac{L}{2}\tan u\right) , \qquad k_2(u) = \frac{\sin\frac{L}{2}}{\sqrt{\cos^2\frac{L}{2} - \sin^2 u}}$$

$$k_3(u) = \sec^2 u\ k_2(u)^3 , \text{ and } k_4(u) = \frac{\sin^2 u}{\sin^2\frac{L}{2}}\ k_2(u)^5 .$$

Thus, putting

$$K_{n,j} = \int_0^1 \phi_4(t) \sum_{i=0}^{n-1} k_j\left((i+t)D_n\right)\ dt$$

for $j = 1, 2, 3, 4$, we have

$$K_n = -\frac{D_n^4}{24}\left(-K_{n,1} + K_{n,2} - K_{n,3} - 3K_{n,4}\right) .$$

Our aim is to derive an asymptotic expression for $D_n^2 \cdot K_n$ as n tends large.

As the following lemma shows, both $K_{n,1}$ and $K_{n,2}$ make only a negligible contribution to K_n.

Lemma 4.6

$$K_{n,1} = O\left(D_n^{-1}\right) \quad \text{and} \quad K_{n,2} = O\left(D_n^{-1}\right) .$$

Proof. It is easy to see that $k_1(u) \le k_1(nD_n) = O(1)$ and $k_2(u) \le 1$ for $u \le nD_n$. Hence the conclusion follows immediately.

In order to study asymptotic behaviours of $K_{n,3}$ and $K_{n,4}$, we will approximate the functions k_3 and k_4 by suitable functions. For this purpose we define functions

$$k_1(x) = \frac{\sin\frac{L}{2}}{\sqrt{\sin^2 x - \sin^2\frac{L}{2}}} \quad \text{and} \quad \tilde{k}_1(x) = \frac{\frac{L}{2}}{\sqrt{x^2 - \left(\frac{L}{2}\right)^2}}$$

and show the following result.

Lemma 4.7
(a) *For* $D_n \le x \le \frac{c}{\sqrt{n}}$, *where* c *is a constant,*

$$k_1(x) = \tilde{k}_1\left(x, \frac{L}{2}\right) \cdot (1 + O(D_n)) \text{ uniformly in } x\ .$$

(b) *For* $x > \frac{c}{\sqrt{n}}$, *where* c *is a constant,*

$$k_1(x) = O(D_n^{1/2}) \text{ and } \tilde{k}_1(x) = O(D_n^{1/2})\ .$$

Proof. Since the proof of (b) is easy, we will prove only (a).
Since, by the mean value theorem, there exists θ_L such that $\sin\frac{L}{2} < \theta_L < \frac{L}{2}$ and

$$\frac{\frac{L}{2}}{\sqrt{x^2 - \left(\frac{L}{2}\right)^2}} - \frac{\sin\frac{L}{2}}{\sqrt{x^2 - \sin^2\frac{L}{2}}} = \left(\frac{L}{2} - \sin\frac{L}{2}\right) \cdot \frac{x^2}{(x^2 - \theta_L^2)^{3/2}}\ ,$$

we have

$$\begin{aligned} k_1(x) &> \frac{\sin\frac{L}{2}}{\sqrt{x^2 - \sin^2\frac{L}{2}}} = \tilde{k}_1(x) - \left(\frac{L}{2} - \sin\frac{L}{2}\right) \cdot \frac{x^2}{(x^2 - \theta_L^2)^{3/2}} \\ &> \tilde{k}_1(x) - \frac{1}{6}\left(\frac{L}{2}\right)^3 \cdot \frac{x^2}{\left(x^2 - \left(\frac{L}{2}\right)^2\right)^{3/2}} \\ &= \tilde{k}_1(x) \cdot \left\{1 - \frac{\frac{1}{6}\left(\frac{L}{2}\right)^2 x^2}{x^2 - \left(\frac{L}{2}\right)^2}\right\}\ . \end{aligned}$$

On the other hand, since there exists θ_x such that $\sin x < \theta_x < x$ and

$$\frac{\frac{L}{2}}{\sqrt{x^2 - \left(\frac{L}{2}\right)^2}} - \frac{\frac{L}{2}}{\sqrt{\sin^2 x - \left(\frac{L}{2}\right)^2}} = (x - \sin x) \cdot \frac{-\frac{L}{2}\,\theta_x}{\left(\theta_x^2 - \left(\frac{L}{2}\right)^2\right)^{3/2}}\ ,$$

we have

$$k_1(x) < \frac{\frac{L}{2}}{\sqrt{\sin^2 x - \left(\frac{L}{2}\right)^2}} = \tilde{k}_1(x) + (x - \sin x) \cdot \frac{\frac{L}{2}\,\theta_x}{\left(\theta_x^2 - \left(\frac{L}{2}\right)^2\right)^{3/2}}$$