Some Remarks on Degree Sets for Graphs

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Abstract

This paper is a sequel to the authors [2] and the first author [4], [5]. A degree set of a graph G is the set of degrees of vertices of G. Let n and k be any positive integers with $1 \leq k \leq n-1$, and $DG_n(k)$ be the set of all degree sets D of graphs of order n with the cardinality $|D| = k$. We shall characterize any members in $DG_n(k)$ for $k = 2, 3$, and $n-2$ respectively.

Key words: graph, degree set, degree sequence, graphical sequence.

1 Degree Sets

In this paper the terminology and notation concerning graphs follow Chartrand and Lesniak [1]. Any graphs mean always simple ones. At first we introduce some convenient notation used in the paper. As we deal with only positive integers, any variables named small letter express always positive integers unless otherwise noted. For any non-negative integers m, n with $m \leq n$ we use the following:

$[m, n] = \{m, m+1, m+2, \cdots, n-1, n\}$ and $[n] = [1, n]$ $(1 \leq n)$.

Let any monotone non-increasing n-sequence $d = (d_1, d_2, \cdots, d_n)$. If a number x appears exactly p times in d, then these terms are shortly denoted by x^p, *e.g.*, $(4,3,3,2,1,1,1) = (4, 3^2, 2, 1^3)$. In our discussion the parity of integers plays important roles frequently. So for brevity, by $a \equiv b$ we write the relation $a = b \pmod 2$, *e.g.*, $(a, b) \equiv (0, 1)$ means that a is even and b is odd.

Let G be any graph of order n, and $d(G) = (d_1, d_2, \cdots, d_n)$ be the degree sequence of G, which is listed always monotone non-increasing. Moreover let $D(G) = \{h_1, h_2, \cdots, h_k\}$ be the degree set of G, *i.e.*, $D(G)$ is the set of mutually distinct numbers in $d(G)$. Then we have

(1.1) $n - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

(1.2) $\sum_{j=1}^{n} d_j \equiv 0$.

(1.3) $n - 1 \geq h_1 > h_2 > \cdots > h_k \geq 0$.

(1.4) $d(G) = (h_1^{p_1}, h_2^{p_2}, \cdots, h_k^{p_k})$ for some k positive integers $p_1, p_2, \cdots, p_k$ with $\sum_{j=1}^{k} p_j = n$.

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In general any n-sequence $(d_1, d_2, \cdots, d_n)$ of integers with (1.2) is said to be *proper*, and any k-set $\{h_1, h_2, \cdots, h_k\}$ of integers with (1.3) is called a (n,k)*-set*. Note that any (n,k)-set is meaningful for $1 \leq k \leq n-1$. A sequence d is said to be *graphical* if there exists a graph G whose degree sequence is d, and a (n,k)-set D is called a (n,k)*-degree set* if there exists a graph of order n whose degree set is D. A k-set $\{d_1, d_2, \cdots, d_k\}$ is a (n,k)-degree set if and only if there exists a collection $\{p_1, p_2, \cdots, p_k\}$ of k positive integers with the following properties:

(1.5) $\sum_{j=1}^{k} p_j = n$.

(1.6) $(d_1^{p_1}, d_2^{p_2}, \cdots, d_k^{p_k})$ is graphical.

Let $DG_n(k)$ and $GS_n(k)$ be the set of all (n,k)-degree sets and the set of all graphical n-sequences with (n,k)-degree set.

As well known, $DG_n(1)$ consists of n singletons $\{a\}$ for $a \in [0, n-1]$ when $n \equiv 0$, and of $(n+1)/2$ singletons $\{a\}$ for even $a \in [0, n-1]$ when $n \equiv 1$. On the other hand $DG_n(n-1)$ consists of exactly two $(n, n-1)$-sets $[n-1]$ and $[0, n-2]$ (*e.g.*, see [4]). The purpose of this paper is to characterize any members in $DG_n(k)$ for $k = 2, 3$, and $n-2$. The following theorem on degree sets for graphs is due to Kapoor et al. [3, p.190].

Theorem. *Any (n,k)-set $D = \{d_1, d_2, \cdots, d_k\}$ with $d_k > 0$ is in $DG_n(k)$ for $n = d_1 + 1$.*

This is proved by the construction of graph G with $D(G) = D$. But in this paper the criterion of degree sets is due to the construction of graphical sequences.

Lemma 1. *Let $D = \{d_1, d_2, \cdots, d_k\}$ be any (n,k)-set.*

(1) *if $d_1 = n-1$ and $d_k = 0$ then $D \notin DG_n(k)$.*

(2) *if $n \equiv 1$ and $d_j \equiv 1$ for all $j \in [k]$, then $D \notin DG_n(k)$.*

(3) *if $D \in DG_n(k)$ then $c(D) = \{n-1-d_k, n-1-d_{k-1}, \cdots, n-1-d_2, n-1-d_1\} \in DG_n(k)$.*

Proof. (1) is obvious from the basic fact that if a graph of order n has a vertex v with $deg(v) = n-1$, then degree of every vertex is positive. If D satisfies the conditions in (2), then any n-sequence $(d_1^{p_1}, d_2^{p_2}, \cdots, d_k^{p_k})$ is not proper for any collection $\{p_1, p_2, \cdots, p_k\}$ with (1.5). This implies (2). (3) follows immediately from the fact that if D is a degree set of a graph G, then $c(D)$ is the degree set of the complementary graph of G. □

Now let $F_n(k)$ be the set of all (n,k)-sets $D = \{d_1, d_2, \cdots, d_k\}$ with the following properties:

(1.7) if $d_1 = n-1$ then $d_k > 0$.

(1.8) if $n \equiv 1$ then D contains at least one even number.

Then $DG_n(k) \subseteq F_n(k)$ by Lemma 1. As the above mentioned, we have $DG_n(k) = F_n(k)$ for the cases $k = 1$ and $k = n-1$. It seems to be $DG_n(k) = F_n(k)$ for any n and k with $1 \leq k \leq n-1$ even though we do not yet have the complete proof. But in the next section it is proved that $DG_n(2)$ is a proper subset of $F_n(2)$ for even $n \geq 6$.

Lemma 2. *If any $\{d_1, d_2, \cdots, d_k\} \in F_n(k)$ with $d_k > 0$ is in $DG_n(k)$, then $DG_n(k) = F_n(k)$.*

Proof. Note under the notation in Lemma 1(3) that $c(D) \in F_n(k)$ if $D \in F_n(k)$. Let $D = \{d_1, d_2, \cdots, d_k\} \in F_n(k)$ with $d_k = 0$. Then $d_1 < n-1$ by (1.7) and $c(D) = \{n-1, n-1-d_{k-1}, \cdots, n-1-d_1\} \in DG_n(k)$ by the assumption. So $D \in DG_n(k)$ from Lemma 1(3). □

2 $DG_n(2)$

In this section we shall characterize any members in $DG_n(2)$. Let us begin with some lemmas. In what follows, let $\{a, b\} \in F_n(2)$, $a > b$, and for any $p \in [n-1]$ let us define an n-sequence $d(a, b; p)$ by

$$d(a, b; p) = (a^p, b^{n-p}).$$

The condition for $d(a, b; p)$ to be proper is expressed in the form:
(P_2) $\quad pa + (n-p)b \equiv 0.$
Moreover we shall consider the condition (H) for $d(a, b; p)$:
(H) $\quad b(n-p) - p(a-p+1) \geq 0.$
Then the next is a key lemma in order to characterize any members in $DG_n(2)$, which is derived from Hässelbarth Criterion of graphical sequences in Sierksma and Hoogeveen [6] (*e.g.*, see the authors [2]).

Lemma 3. *The necessary and sufficient condition for $d(a, b; p)$ to be in $GS_n(2)$ is given as follows:*

(1) *when the case $a < p$. $d(a, b; p) \in GS_n(2)$ if and only if (P_2) holds.*

(2) *when the case $b < p \leq a$, $d(a, b; p) \in GS_n(2)$ if and only if (P_2) and (H) hold.*

(3) *when the case $p \leq b$. $d(a, b; p) \in GS_n(2)$ if and only if (P_2) holds.* □

Let $2 \leq b$. Then $\{a, b\}$ belongs to the case (3) in Lemma 3 for $p \leq 2$. Further according to the parity of (n, a, b) we can choose $p = 1$ or 2 for which (P_2) holds. So if $2 \leq b$, it follows from Lemma 3(3) that $d(a, b; p) \in GS_n(2)$ for $p = 1$ or 2. More precisely we have

Lemma 4. *Let $b \geq 2$ for $d(a, b; p)$. Then*

(1) *$d(a, b; 2) \in GS_n(2)$ for $(n, a, b) \equiv (0, 0, 1), (0, 1, 0)$ or $(1, 1, 0)$.*

(2) *$d(a, b; 1) \in GS_n(2)$ for the otherwise (n, a, b).* □

Remark 1. Let $b \geq 2$. Then from Lemma 3(3) we also have:

(1) *$d(a, b; b-1) \in GS_n(2)$ for $(n, a, b) \equiv (0, 0, 1)$.*

(2) *$d(a, b; b) \in GS_n(2)$ for the otherwise n, a, b.*

Here the assumption $b \geq 2$ is necessary for the choice of $p = b - 1 \geq 1$. □

Lemma 5. *Let $b = 1$ for $d(a, b; p)$. Then*

(1) *$d(n-1, 1; 1) \in GS_n(2)$.*

(2) *$d(a, 1; n-2) \in GS_n(2)$ if $a \leq n-3$ and $na \equiv 0$.*

(3) *when $n = 4$, $d(2, 1; 2) \in GS_4(2)$.*

Proof. (1) is obvious from the fact that $d(n-1, 1; 1)$ is the degree sequence of the complete bipartite graph $K_{n-1,1}$. (2) follows from Lemma 3(1) for $p = n-2$. When $n = 4$, $d(2, 1; 2) = (2, 2, 1, 1)$ is the degree sequence of the path P_4 of order 4. □

Lemma 6. *If $n \equiv 0$ and $n \geq 6$, then $d(n-2, 1; p) \notin GS_n(2)$ for any $p \in [n-1]$.*

Proof. We suppose that $d = d(n-2,1;p)$ is graphical for some $p \in [n-1]$. Then from (P_2), $p \equiv 0$. Hence the cases (1) and (3) in Lemma 3 do not happen. Therefore d satisfies (H) for some even $p \in [2, n-2]$. Let $f(p)$ be the left hand side of (H) for d. Then $f(p) = (n-p) - p(n-p-1) = (p - \frac{n}{2})^2 + n - \frac{n^2}{4}$. The maximum value of $f(p)$ for $p \in [2, n-2]$ is $f(2) = f(n-2) = 4 - n$. If $n > 4$ then $f(p) < 0$ for any $p \in [2, n-2]$, which is a contradiction. This completes the proof. □

Theorem 1.

(1) $DG_n(2) = F_n(2)$ *if* $n \equiv 1$.

(2) $DG_4(2) = F_4(2)$.

(3) $DG_n(2) = F_n(2) \setminus \{n-2, 1\}$ *if* $n \equiv 0$ *and* $n \geq 6$.

Proof. Let $D = \{a, b\} \in F_n(2)$ with $b > 0$. If $2 \leq b < a \leq n-1$ or $na \equiv 0, b = 1, a \neq n-2$ then $D \in DG_n(2)$ by Lemmas 4-5. If $na \equiv 1$ then $\{a, 1\} \notin F_n(2)$ by (1.8). Hence (1) follows from Lemma 2. Moreover (2) and (3) are from Lemmas 5-6. □

3 $DG_n(3)$

In this section we shall characterize any members in $DG_n(3)$. In what follows, let $\{a, b, c\} \in F_n(3), a > b > c$ and let us define an n-sequence $d(a, b, c; p, q)$ by

$$d(a, b, c; p, q) = (a^p, b^q, c^{n-t}),$$

where $p, q \in [n-2]$, $t = p + q$ and $t \leq n-1$. The condition for $d(a, b, c; p, q)$ to be proper is expressed in the form:

(P_3) $\quad pa + qb + (n-t)c \equiv 0.$

Moreover consider the conditions for $d(a, b, c; p, q)$ as follows:

(H_1) $\quad qb + (n-t)c - p(a-p+1) \geq 0.$

(H_2) $\quad (n-t)c - p(a-t+1) \geq 0.$

(H_3) $\quad (n-t)c - p(a-t+1) - q(b-t+1) \geq 0.$

Under these notations we have the following conditions for $d(a, b, c; p, q)$ to be in $GS_n(3)$, which is also derived from Hässelbarth Criterion (*e.g.*, see [2]).

Lemma 7. *The necessary and sufficient condition for* $d = d(a, b, c; p, q)$ *to be in* $GS_n(2)$ *is given as follows:*

(1) *when the case* $a < p$, $d \in GS_n(3)$ *if and only if* (P_3) *holds.*

(2) *when the case* $b < p \leq a$, $d \in GS_n(3)$ *if and only if* (P_3) *and* (H_1) *hold.*

(3) *when the case* $c < p \leq b < a < t$, $d \in GS_n(3)$ *if and only if* (P_3) *holds.*

(4) *when the case* $c < p \leq b < t \leq a$, $d \in GS_n(3)$ *if and only if* (P_3) *and* (H_2) *hold.*

(5) *when the case* $p \leq c < b < t$, $d \in GS_n(3)$ *if and only if* (P_3) *holds.*

(6) *when the case* $p \leq c < t \leq b$, $d \in GS_n(3)$ *if and only if* (P_3) *and* (H_3) *hold.*

(7) *when the case* $t \leq c$, $d \in GS_n(3)$ *if and only if* (P_3) *holds.* □

Using the above lemma, we can get actually $d(a,b,c;p,q) \in GS_n(3)$ by the suitable choice of p,t according to the parity of (n,a,b,c).

Lemma 8. *Let $n \equiv 0$ and $c > 0$. Then we have*

(1) $d(a,b,c;1,n-2) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,0,0),(0,1,0),(1,0,1)$ *or* $(1,1,1)$.

(2) $d(a,b,c;1,n-3) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,0,1)$ *or* $(1,1,0)$.

(3) $d(a,b,c;2,n-3) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,1,1)$ *or* $(1,0,0)$.

Proof. In the case (3) let $c = 1, p = 2$ and $q = n-3$. If $(a,b) \equiv (0,1)$, then $3 \leq b < a \leq n-2, t = n-1$ and (P_3) holds. Since $c < p < b < a < t$ it follows from Lemma 7(3) that $d(a,b,1;2,n-3) \in GS_n(3)$. The other cases follow from Lemma 7(5). □

By the same consideration the next follows from Lemma 7(5).

Lemma 9. *Let $n \equiv 1$ and $c > 0$. Then we have*

(1) $d(a,b,c;1,n-2) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,0,0),(0,1,1),(1,0,1)$ *or* $(1,1,0)$.

(2) $d(a,b,c;1,n-3) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,0,1)$.

(3) $d(a,b,c;2,n-3) \in GS_n(3)$ *if* $(a,b,c) \equiv (0,1,0)$ *or* $(1,0,0)$. □

Theorem 2. $DG_n(3) = F_n(3)$.

Proof. Let $D = \{a,b,c\} \in F_n(3)$ with $c > 0$. Then $D \in DG_n(3)$ by Lemmas 8-9. Hence the assertion follows from Lemma 2. □

4 $DG_n(n-2)$

Finally we shall show that $DG_n(n-2) = F_n(n-2)$ for any $n \geq 3$. Since it is known already for the case of $n \in [3,5]$, we consider the case $n \geq 6$. Any member in $F_n(n-2)$ is classified into the three classes as follows:

(4.1) $D_n(n-1;k) = [n-1] \setminus \{k\}$, *where* $k \in [n-2]$

(4.2) $D_n(n-2;k) = [0,n-2] \setminus \{k\}$, *where* $k \in [0,n-3]$

(4.3) $D_n(n-3) = [0,n-3]$.

At first we shall prove that $D_n(n-2;0) = [n-2] \in DG_n(n-2)$. For any $t \in [n-1]$ we define an n-sequence$d_3(t)$ by

$d_3(t) = (n-2, n-3, \cdots, t+1, t^3, t-1, \cdots, 2, 1)$.

Further for $n = 4m-1$ or $n = 4m$ we define an n-sequence $d_2(m)$ by

$d_2(m) = (n-2, n-3, \cdots, 2m+1, (2m)^2, (2m-1)^2, 2m-2, \cdots, 2, 1)$.

Lemma 10. *Let m be any positive integer.*

(1) *if* $n = 4m+1$ *then* $d_3(t) \in GS_n(n-2)$ *for any* $t \in [m, 3m]$.

(2) *if* $n = 4m+2$ *then* $d_3(t) \in GS_n(n-2)$ *for any* $t \in [m, 3m+1]$.

(3) *if* $n = 4m-1$ *or* $n = 4m$ *then* $d_3(t) \notin GS_n(n-2)$ *for any* $t \in [n-1]$.

(4) *if* $n = 4m-1$ *or* $n = 4m$ *then* $d_2(m) \in GS_n(n-2)$. □

The assertions (1)-(3) in the above are proved in the first author [5, Theorem 2.11]. (4) is shown by Hasselbarth Criterion. The above lemma proves that $[n-2] \in DG_n(n-2)$.

Lemma 11. *Let* $D = \{d_1, d_2, \cdots, d_k\} \in F_n(k)$ *with* $d_1 \leq n-2$ *and put* $p(D) = \{n, d_1+1, d_2+1, \cdots, d_k+1\}$. *If* $D \in DG_n(k)$, *then* $p(D) \in DG_{n+1}(k+1)$.

Proof. Let G be a graph of order n whose degree set is D. Then $p(D)$ is the degree set of the graph $G + K_1$, the join of G and the complete graph K_1 of order 1. This completes the proof. □

Let $n = 6$. From Lemma 1.5 in [5] we see that $D_6(5;1), D_6(5;3), D_6(4;0), D_6(4;2)$, and $D_6(3)$ are in $DG_6(4)$. On the other hand it is seen easily that the following 6-sequences are graphical: $\{5,4^2,3^2,1\}, \{5,3^2,2^2,1\}$,and $\{4,3^2,2^2,0\}, \{4,2^2,1^2,0\}$. So $D_6(5;2), D_6(5;4), D_6(4;1)$ and $D_6(4;3)$ are in $DG_6(4)$. Hence we have $DG_6(4) = F_6(4)$.

Theorem 3. $DG_n(n-2) = F_n(n-2)$ *for any* $n > 2$.

Proof. We prove by the induction on n. As the assertion is true for $n \leq 6$, let $n > 6$. We use the notations in (4.1)-(4.3) and Lemma 11. Note that $D_n(n-1;k+1) = p(D_{n-1}(n-3;k)$ for any $k \in [0, n-4]$ and $D_n(n-1;n-2) = p(D_{n-1}(n-4))$. So $D_n(n-1;k) \in DG_n(n-2)$ for any $k \in [n-2]$ by the inductive hypothesis and Lemma 11. Further $D_n(n-2;0) \in DG_n(n-2)$ from Lemma 10. Therefore we see that if D in $F_n(n-2)$ dose not contain zero, then $D \in DG_n(n-2)$. Hence the theorem follows from Lemma 2. □

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