# QUASI POLYTOPES AND FINITE TOPOLOGICAL SPACES

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# QUASI POLYTOPES AND FINITE TOPOLOGICAL SPACES

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## § 1. Introduction

We have investigated finite topological spaces and those simplicial structures in [1] and [2]. In [2] we have adopted the concept, called eigen values, which was important for the characterization of finite $T_0$-spaces. Moreover, in [1] we have showed that there is an equivalent correspondence that associates with each finite $T_0$-space a space called a partially polytopes.

In this note we shall consider applications of the conception. We shall state that for a space called a quasi polytope the eigen values also may be defined, and we shall consider to extend each quasi polytope to a partially polytope inclued it.

## § 2. Quasi polytopes.

In an Euclidean space $E^N$, a set of properly joined open simplexes is said to be a *quasi simplicial complex* here (see [1]).

The geometric carrier of a star-finite quasi simplicial complex $K$ is called a *quasi polytope* and is denoted by the symbol $|K|$. But, in this note quasi complexes and quasi polytopes shall always be assumed to be finite quasi complexes and finite quase polytopes, respectively.

A topological space $X$ that is homeomorphic to a quasi polytope $|K|$ is called a *triangulated space* and the quasi complex $K$ is a *triangulation* of the space $X$. Next, let $K$ be a quasi simplicial complex, then a set

$$Cl(K) = \{s \mid \exists\sigma \in K: s < \sigma\}$$

is a simplicial complex, which is said to be *induced* by $K$.

DEFINITION 1. Let $f$: $K \to L$ be a mapping of a quasi simplicial complex $K$ to a quasi simplicial complex $L$. Then $f$ is *simplicial* if there is simplical mapping $f_0$: $Cl(K) \to Cl(L)$ such that $f = f_0 | K$.

DEFINITION 2. Two quasi simplicial complexes $K$ and $L$ are said to be *isomorphic* each other if there is a bijective simplicial mapping $\varphi$ of one onto the other such that the inverse mapping $\varphi^{-1}$ also is simplicial, and in such a case we denote by $K \approx L$.

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Evidently we have

$$K \approx L \Rightarrow |K| \simeq |L| .$$

A triangulation $K$ of a quasi polytope $X$ is said to be *minimal* if the cardinality of $K$ is minimum in the cardinalities of all triangulations of $X$.

From now on, we shall consider only quasi polytopes satisfying the following property: *if $K$ and $L$ are two minimal triangulations of such a quasi polytope $X$, then $K \approx L$.*

Now, let $X$ be a quasi polytope, $K$ be a minimal trangulation of $X$, and $K^0$ be the set of vertices belonging to $Cl(K)$. Set

$$K^0 = \{v_1, v_2, \cdots, v_k\} ,$$

and

$$K = \{\sigma_1, \sigma_2, \cdots, \sigma_l\} ,$$

and for $K$ we define a $(k, l)$ matrix $A=[a_{ij}]$ as follows:

$$a_{ij} = 1 \quad \text{if} \quad v_i < \sigma_j ,$$
$$= 0 \text{ otherwse} .$$

Such a matrix is said to be a *p-matrix* of $K$.

DEFINITION 3. Let $X$ be a quasi polytope, $K$ be a minimal triangulation of $X$, and $A$ be a $p$-matrix of $K$. Then the characteristic polynomial $f(x)$ of $AA'$, that is,

$$f(x) = |xE - AA'|$$

is *the polynomial of the space $X$*. And the eigen values of $AA'$ is said to be *the eigen values of the space $X$*.

In a matrix $AA'=[c_{ij}]$, $c_{ij}$ is the number of open simplexes belonging to $K$ which have vertices $v_i$ and $v_j$ as their faces. Because of

$$c_{ij} = \sum_{r=1}^{k} a_{ir} a_{jr} ,$$

and

$$a_{ir} a_{jr} = 1 \Leftrightarrow (a_{ir} = 1 \text{ and } a_{jr} = 1) \Leftrightarrow (v_i < \sigma_r \text{ and } v_j < \sigma_r) .$$

The following theorem results easily from the matrix theory.

THEOREM 1. *Let $X$ be a quasi polytope and $K$ be a minimal triangulation of $X$. Then the eigen values of $X$ have the following properties:*

(1) *the number of the eigen values is equal to the number of vertices of open simplexes belonging to $K$;*

(2) *each eigen value is a non-negative real number;*

(3) *if there are rational roots, they are non-negative integers.*

PROOF. For (1), let $A$ be a $p$-matrix of $K$, and suppose that the number of open simplexes belonging to $K$ is $n$. Then $AA'$ is a $(n, n)$ square matrix. whence (1) implies.

For (2), $AA'$ is a non-negative Hermition matrix. Hence each eigen value is a non-negative real number.

For (3), the polynomial of $X$ is

$$f(x)=|xE-AA'|=x^n-\cdots+(-1)^n|AA'|$$

and if $f(x)=0$ has rational roots, then they are non-negative integers, since $|AA'|$ is a non-negative integer.

Here, we shall show the eigen values or the characteristic polynomials of simple spaces.

(1) The eigen values of the $n$-closed ball are $n+1$ integers:

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}.$$

In fact, a minimal triangulation of the $n$-closed ball is the closure $K_1$ of a $n$-simplex. Let $A_1$ be a $p$-matrix of $K_1$, and set $A_1A_1'=[c_{ij}]$. Each $c_{ij}$ is the number of open simplex belongiog to $K$, then

$$c_{ii}={}_nC_0+{}_nC_1+\cdots+{}_nC_n=2^n,$$

and for $i\neq j$

$$c_{ij}={}_{(n-1)}C_0+{}_{(n-1)}C_1+\cdots+{}_{(n-1)}C_{(n-1)}=2^{n-1},$$

where ${}_mC_r$ is the total number of combinations of $m$ elements taken $r$ elements at a time. Hence

$$A_1A_1'=\begin{bmatrix}2^n & & & \\ & 2^n & & 2^{n-1} \\ & & \ddots & \\ & 2^{n-1} & & 2^n\end{bmatrix},$$

that is, the diagonal elements are all $2^n$, and the other elements are all $2^{n-1}$. So that the eigen values of the space are

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}.$$

(2) The eigen values of $n$-open ball are $n+1$ integers:

$$0, 0, \cdots, n+1.$$

In fact, a minimal triangulation of a $n$-open ball is a quasi simplical complex $K_2$ consisting of one $n$-open simplex. Let $A_2$ be a $p$-matrix, then elements of $A_2A_2'$ are all 1. Hence the eigen values are $n+1$ elements, $O, O, \cdots, O, n+1$.

(3) The eigen values of the $(n-1)$-sphere are $n+1$ integers which are the difference between (1) and (2).

$$2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}-(n+1).$$

In fact, a minimal triangulation of a $(n-1)$-sphere is clearly $K_3=K_1-K_2$. Now, let $A_3$ be a $p$-matrix of $K_3$ and setting $A_3A_3'=[c_{ij}]$,

$$c_{ij}=2^n-1,$$

$$c_{ij}=2^{n-1}-1 \quad (\text{for } i\neq j),$$

so that $A_3A'_3 = A_1A'_1 - A_2A'_2$. Hence, the eigen values are $n+1$ elements, $2^{n-1}, 2^{n-1}, \cdots, 2^{n-1}, (n+2)2^{n-1}-(n+1)$.

(4) The eigen values of the Torus are 7 elements,

$$10, 10, \cdots, 10, 31 .$$

(5) The eigen values of the Möbius band are 6 elements,

$$5, 5, \cdots, 5, 23 .$$

(6) The eigen values of the projective plane are 6 elements,

$$8, 8, \cdots, 8, 26 .$$

## § 3. Regularization of quasi polytopes.

A finite quasi complex $K$ is said to be a *n-partially simplicial complex* if $K$ satisfies the following conditions:

(1) if $\sigma, \tau \in K$, $dim(\sigma \cap \tau) = k$, then a set $\{\eta \in K \mid \eta < \sigma \cap \tau\}$ has $k+1$ elements,

(2) let $K^0$ be the set of vertices of open simplexes belonging to $K$, then $K^0$ has $n$ elements.

The geometric carrier of a partially simplicial complex is called a *partially polytope.* There exists an equivalent correspondence that assigns to each partially polytope a finite $T_0$-space.

Here, for each quasi polytope we shall consider a minimal dimensional partially polytope which will be called a regularization.

Now, let $X$ be a quasi polytope, $K$ be a minimal triangulation of $X$, and $K^0$ be a set of vertices belonging to $Cl(K)$. Set

$$K = \{\sigma_1, \sigma_2, \cdots, \sigma_l\} ,$$

and

$$K^0 = \{v_1, v_2, \cdots, v_k\} .$$

For $v_p \in K^0$, let

$$V_p = \{\sigma_i \in K \mid \sigma_i > v_p\} , \tag{3.1}$$

and put

$$T_1 = \{V_1, V_2, \cdots, V_k\} .$$

Then we define an ordering on $T_1$ as follows: $V_i < V_j$ if and only if (1) or (2) implies,

$$\begin{aligned} &(1) \quad V_i \neq V_j \text{ and } V_i \subset V_j ; \\ &(2) \quad V_i = V_j \text{ and } i < j . \end{aligned} \tag{3.2}$$

Next, for $\sigma_j \in K$ let

$$W_j = \bigcap \{V_r \in T_1 \mid V_r \ni \sigma_j\} , \tag{3.3}$$

where, if there is more than two elements of (3.1) which are equal to $W_j$, then $W_j$ is defined to be the smallest element of them with respected to the ordering (3.2). Here, putting

$$T = T_1 \cup \{W_1, W_2, \cdots, W_l\},$$

we define an ordering on $T$ as follows: for $U_1, U_2 \in T$ such that $U_1 \neq U_2$, $U_1 < U_2$ if and only if (1) or (2) or (3) below, implies

(1) $U_1, U_2 \in T_1$ and $U_1 < U_2$ with respected to (3.2);

(2) $U_1, U_2 \notin T_1$ and $U_1 \subset U_2$;

(3) one is in $T_1$, say $U_1 \in T_1$ and the other is not, say $U_2 \notin T_1$, and $U_1 \subset U_2$ under the comments of (3.3).

Then $(T, \leq)$ is a partially ordered set, so that a finite $T_0$-space is defined on $X$. If $L(T)$ is the simplical presentation of $T$ (see [1]), then $\tilde{X} = |L(T)|$ is said to be a *regularization* of $X$.

Moreover, if to each $U_p \in T$ we assign an simplex

$$\tilde{\sigma} = \langle \{U \in T \mid U > U_p\} \rangle$$

and consider the collection $\tilde{K}$ of such simplexes, then we have

$$\tilde{K} \approx L(T)$$

(see [1]). Especially for $\sigma_i \in K$ let

$$\tilde{\sigma}_i = \langle \{U \in T \mid U > W_i\} \rangle .$$

Given any quasi simplicial complex $M$, we may define an ordering $\lhd$ on $M$ by

$$\sigma \lhd \tau \Leftrightarrow \sigma > \tau ,$$

then $(M, \lhd)$ is a partially ordered set. So the finite $T_0$-space is defined on $M$ and is denoted by $F(M)$. Then we define $\phi\colon |M| \to F(M)$ as follows: if $x \in |M|$, then there is a unique simplex $\sigma \in M$ such that $x \in \sigma$. So put

$$\phi(x) = \sigma .$$

Then $\phi$ is a weak homotopy equivalence (see [1]).

Now a mapping $f\colon Cl(K) \to Cl(\tilde{K})$ is defined as follows: if $\tau \in Cl(K)$, $\tau = \langle v_1, v_2 \cdots v_m \rangle$, then let $f(\tau)$ be an open simplex constructed by vertices

$$\{V_1, V_2, \cdots, V_m\} \cup \{U \in T \mid \exists \sigma \in K\colon \sigma < \tau ,\ U < \tilde{\sigma}\} .$$

Then we have

$$\begin{aligned} &(a)\ \ f(v_i) = V_i\ (i = 1,2, \cdots, k);\\ &(b)\ \ f(\sigma_j) = \tilde{\sigma}_j\ (j = 1,2, \cdots, l);\\ &(c)\ \ f \text{ is injective};\\ &(d)\ \ \sigma < \tau \Leftrightarrow f(\sigma) < f(\tau) . \end{aligned} \qquad (3.4)$$

In fact, (*a*) and (*b*) follow immediately from the definition.

For (*c*), if $\sigma, \tau \in Cl(K)$, $\sigma \neq \tau$, then we may assume that there is $v_i \in K^0$ such that $v_i < \tau$ and $v_i \not< \sigma$. Clearly $U_i < f(\sigma)$, and while, if $\rho \in K$, $\rho < \sigma$, then $v_i \not< \rho$, so that $U_i \not< \tilde{\rho}$ by the definition of $\tilde{\rho}$ $(\rho \in K)$. Hence $U_i \not< f(\sigma)$, thus $f(\tau) \neq f(\sigma)$.

For $(d)$, $\sigma<\tau \Rightarrow f(\sigma)<f(\tau)$ follows from the definition of $f$. The converse follows from the proof of $(c)$.

Next, define $h\colon K \to \tilde{K}$ by $h=f|k$, then $\sigma<\tau \Rightarrow h(\sigma)<h(\tau)$, hence $h$ is a quasi simplicial mapping.

Let $Cl(K_1)$ and $Cl(\tilde{K}_1)$ be first subdivisions of $Cl(K)$ and $Cl(\tilde{K})$, respectively. And $f_1\colon Cl(K_1) \to Cl(\tilde{K}_1)$ is defined as follows: for $\langle b(\tau_1)\, b(\tau_2)\cdots b(\tau_m)\rangle \in Cl(K_1)$ (where $b(\tau_j)$ is the barycenter of $\tau_j$), let

$$f(\langle b(\tau_1)b(\tau_2)\cdots b(\tau_m)\rangle)=\langle b(\tilde{\tau}_1)b(\tilde{\tau}_2)\cdots b(\tilde{\tau}_m)\rangle$$

where $\tilde{\tau}_i=f(\tau_i)$. Then from (3.4) $f_1$ is bijective and $f_1$, $f_1^{-1}$ are simplicial, so that

$$Cl(K_1)\approx f_1(Cl(K)).$$

Next, the first barycentric subdivision $M_1$ of a quasi simplicial complex $M$ is defined by

$$K_1=\{\langle b(\tau_1)\cdots\cdot b(\tau_j)\rangle \in Cl(K_1)\,|\,\tau_1<\cdots<\tau_j, \tau_j\in K\},$$

Now, define $h_1\colon K_1\to Cl(K_1)$ by $h_1=f_1|K_1$, then $h_1(K_1)\subset\tilde{K}_1$. Because, if $\langle b(\tau_1)\cdots b(\tau_j)\rangle\in K_1$ then $\tau_1<\cdots<\tau_j$ and $\tau_j\in K$. So from (3.4), $\tilde{\tau}_1<\cdots<\tilde{\tau}_j$ and $\tilde{\tau}_j\in\tilde{K}$, hence $\langle b(\tilde{\tau}_1)\cdots b(\tilde{\tau}_j)\rangle\in\tilde{K}_1$ thus $h_1(K_1)\subset\tilde{K}_1$. From (3.4)

$$K_1\approx h_1(K_1).$$

Therefore $h_1\colon |K_1|\to|\tilde{K}_1|$ is an embedding.

THEOREM 2. *Let $X$ be a quasi polytope, $\tilde{X}$ be a regularization of $X$, and $K$, $\tilde{K}$ be minimal triangulation $X$, $\tilde{X}$, respectively, and let $h\colon K\to\tilde{K}$ be a canonical quasi simplicial mapping defined by $h=f|k$. Then $h$ induces the simplicial mapping $h_1\colon K_1\to\tilde{K}_1$ which satisfy the following conditions*:

(1) *$h_1\colon X=|K_1|\to\tilde{X}-|\tilde{K}_1|$ is an embedding*;

(2) *$h\colon F(K)\to F(\tilde{K})$ is an embedding such that $h(F(K))$ is dense in $F(\tilde{K})$*;

(3) *let $\phi\colon |K|\to F(K)$ and $\tilde{\phi}\colon |\tilde{K}|\to F(\tilde{K})$ be two weak homotopy equivalence defined as the above, then $h\circ\phi=\tilde{\phi}\circ h$.*

PROOF. (1) has been already proved.

For (2), of $h\colon K\to\tilde{K}$,

$$\sigma_i<\sigma_j\Leftrightarrow h(\sigma_i)<h(\sigma_j).$$

Hence, of $h\colon F(K)\to F(\tilde{K})$.

$$\sigma_i\rhd\sigma_j\Leftrightarrow h(\sigma_i)\rhd h(\sigma_j).$$

Thus $h$ is a homeomorphism of $F(K)$ to $f(F(K))$. On the other hand, using the previous symbols, if $\tau\in F(\tilde{K})-h(F(K))$, then from the definition of $\tilde{K}$ there is $V_i\in T_1$ such that $\tau=\langle\{U\in T\,|\,U>V_i\}\rangle$. Now, taking $\sigma_j$ such that $\sigma_j\in V_i$,

$$\tilde{\sigma}_j=\langle\{U\in T\,|\,U>W_j\}\rangle$$

where $W_j=\bigcap\{V\in T_1|\,V>\sigma_j\}$. So $V_i\supset W_j$, that is $V_i>W_j$, hence, $\tilde{\sigma}_j>\tau$, whence

$\tilde{\sigma}_j \lhd \tau$. So, in the space $F(\tilde{K})$ the minimal basic neighborhood of $\tau$ contains $\tilde{\sigma}_j$, thus $h(F(K))$ is a dense subspace in $F(\tilde{K})$.

For (3), if $x \in |K| = |K_1|$, there is a unique simplex $\sigma_i \in K$ containing $x$. Then

$$h(\phi(x)) = h(\sigma_i) = \tilde{\sigma}_i .$$

While, there is a unique simplex $\langle b(\sigma_1) \cdots b(\sigma_i) \rangle \in K_1$ containing $x$. Then

$$h(x) \in \langle b(\tilde{\sigma}_1) \cdots b(\tilde{\sigma}_i) \rangle \subset |h(K_1)| \subset |\tilde{K}_1| .$$

So that

$$\tilde{\phi}(h(x)) = \tilde{\phi}(\tilde{\sigma}_i) = \tilde{\sigma}_i .$$

Therefore

$$h \circ \phi = \tilde{\phi} \circ h_1 .$$

## References

[1] M. Shiraki: Finite $T_0$-spaces and simplicial structures. This Journal, Vol. **2**, (1969), 17–28.
[2] M. Shiraki: On finite topological spaces II. This Journal, Vol. **2**, (1969), 1–15.