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ERROR ESTIMATES UNDER THE L^∞-NORM AND ADAPTIVE MESH REFINEMENT PROCEDURES FOR TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract

We estimate the error under the L^∞-norm for the Galerkin method to solve two point boundary value problems. And, by using those error estimates, we consider adaptive mesh refinement procedures such as the value of the error under the L^∞-norm is less than a given positive number δ.

1. Introduction

In this paper we consider the following linear two point boundary value problem:

$$\begin{aligned} Ly &\equiv -\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right)+b(x)y=f(x), \qquad x\in I=[0,\ 1], \\ y(0)&=y(1)=0 \end{aligned} \tag{1.1}$$

where for some $r\geq 1$

$$\left.\begin{aligned} &\text{(i)} && a(x)\in C^r(I), && 0<a_0\leq a(x)\leq a_1 \\ &\text{(ii)} && b(x)\in C^{r-1}(I), && 0\leq b(x) \end{aligned}\right\} x\in I.$$

For the Galerkin approximation Babuška and Rheinboldt have presented *a posteriori* error estimates under the energy norm and adaptive mesh refinement procedures which are based on those estimates ([1]-[4]). In this paper we estimate the error under the L^∞-norm and consider adaptive mesh refinement procedures such as the value of the error under the L^∞-norm is less than a given positive number δ.

On the interval I we consider a partition

$$\Delta\ :\ 0=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=1,$$

and introduce the notations

$$\left.\begin{aligned} I_i&=[x_{i-1},\ x_i] \\ h_i&=x_i-x_{i-1} \end{aligned}\right\},\qquad i=1,\cdots,\ n,$$

$$h_\Delta=\max_{1\leq i\leq n} h_i.$$

If P_r denote the collection of all polynomials of degree not greater than r, then continuous piecewise polynomial space $\mathcal{M}_\Delta^r$ is defined as usual by

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$$\mathscr{M}_\Delta^r=\{v\in C^0(I)\mid v|_{I_i}\in P_r(I_i),\ i=1,\cdots,\ n;\ v(0)=v(1)=0\}.$$

Also on the interval $J\,(J\subseteq I)$ we define

$$(u,\ v)_J=\int_J uv\,dx$$

and

$$\|u\|_{E(J)}=\left[\int_J(au'^2+bu^2)dx\right]^{1/2},\qquad u\in H^1(J).$$

Let $y_{\Delta,r}\in\mathscr{M}_\Delta^r$ be the Galerkin approximation to the solution y_0 of (1.1) determined by the relation

$$(ay'_{\Delta,r},\ v')_I+(by_{\Delta,r},\ v)_I=(f,\ v)_I,$$

for all $v\in\mathscr{M}_\Delta^r$.

Then it is known that the following error estimate holds at the knots ([5]):

THEOREM 1. *Let* $C_{\Delta,r}=\max_{0\leq i\leq n}|(y_0-y_{\Delta,r})(x_i)|$. *If* $y_0\in H^{k+1}(I)$ *with* $1\leq k\leq r$, *then there is a constant* C *such that*

$$C_{\Delta,r}\leq C\|y_0{}^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r},$$

where the constant C *is independent of* h_Δ *but dependent on* a, b, k *and* r.

In the case $r\geq 2$, it is known from Theorem 1 that the Galerkin approximation $y_{\Delta,r}$ superconverges to the exact solution y_0 at the knots.

From now on we assume that

(i) $\qquad r\geq 2,$

(ii) $\qquad y_0\in C^{k+1}(I),\qquad 1\leq k\leq r.$

2. Error estimates

In this section, for each subinterval I_i, we estimate the error under the L^∞-norm. Those error estimates play important parts in adaptive mesh refinement procedures which we will describe in next section.

First we obtain the following error estimate:

THEOREM 2. *For each subinterval* I_i *of a given partition* Δ *there is a constant* C *such that*

$$\|y_0^{(s)}-y_{\Delta,r}^{(s)}\|_{L^\infty(I_i)}\leq C(\|y_0^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r}+\|y_0^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1})/h_i^s,\qquad 0\leq s\leq k+1,$$

where the constant C *is independent of* Δ *but dependent on* a, b, k, r *and* s.

Proof. Let $\tilde{y}$ be the Lagrange interpolation polynomial of degree k to the solution y_0 of (1.1) on each subinterval I_i. Then

$$\|y_0^{(s)}-\tilde{y}^{(s)}\|_{L^\infty(I_i)}\leq C'\|y_0^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1-s},\qquad 0\leq s\leq k+1. \tag{2.1}$$

Also it follows from [5] that

$$\|y_0-y_{\Delta,r}\|_{L^\infty(I_i)}\leq C''(\|y_0^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r}+\|y_0{}^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1}),$$

and

$$\|\tilde{y}-y_{\Delta,r}\|_{L^\infty(I_i)}\leq C'''(\|y_0^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r}+\|y_0^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1}). \tag{2.2}$$

Applying Markoff's inequality to the above (2.2) gives

$$\| \tilde{y}^{(s)}-y^{(s)}_{\Delta,r}\|_{L^\infty(I_i)}\leq C''''(\| y_0^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r}+\| y_0^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1})/h_i^s, \qquad 0\leq s\leq k+1.$$

From this and (2.1), it follows that

$$\begin{aligned}\| y_0^{(s)}-y^{(s)}_{\Delta,r}\|_{L^\infty(I_i)}&\leq \| y_0^{(s)}-\tilde{y}^{(s)}\|_{L^\infty(I_i)}+\| \tilde{y}^{(s)}-y^{(s)}_{\Delta,r}\|_{L^\infty(I_i)}\\ &\leq C(\|y_0^{(k+1)}\|_{L^2(I)}h_\Delta^{k+r}+\| y_0^{(k+1)}\|_{L^\infty(I_i)}h_i^{k+1})/h_i^s,\end{aligned}$$

and the proof of Theorem 2 is complete.

Let

$$f_{\Delta,r}=-\frac{d}{dx}\left(a(x)\frac{dy_{\Delta,r}}{dx}\right)+b(x)y_{\Delta,r}, \qquad x\in \mathring{I}_i,$$

$$a_i=\min_{x\in I_i}|a(x)|.$$

Then, by using $C_{\Delta,r}$ in Theorem 1 and *a posteriori* error $f-f_{\Delta,r}$, we obtain the following result:

THEOREM 3. *For each subinterval I_i of a given partition Δ there are constants C_1 and C_2 such that*

$$\| y_0-y_{\Delta,r}\|_{E(I_i)}\leq \frac{1}{\sqrt{6a_i}}h_i\| f-f_{\Delta,r}\|_{L^2(I_i)}+\frac{C_1}{\sqrt{h_i}}C_{\Delta,r}$$

$$\| y_0-y_{\Delta,r}\|_{L^\infty(I_i)}\leq \frac{1}{2\sqrt{6a_i}}h_i^{3/2}\| f-f_{\Delta,r}\|_{L^2(I_i)}+C_2C_{\Delta,r},$$

where the constants C_1 and C_2 are independent of h_i but dependent on a and b.

Theorem 3 is proved by using the following lemma:

LEMMA. *Let e be the linear interpolant to $u\in C^1(I)$ at the endpoints* 0 *and* 1. *Then we have*

$$\| u-e\|_{L^2(I)}\leq \frac{1}{\sqrt{6}}\| u'\|_{L^2(I)},$$

$$\| u-e\|_{L^\infty(I)}\leq \frac{1}{2}\| u'\|_{L^2(I)}.$$

Proof of Theorem 3. Let $\zeta=y_0-y_{\Delta,r}$ and e be the linear interpolant to y_0 at the endpoints x_{i-1} and x_i. Then, by using Schwarz's inequality and Lemma,

$$\begin{aligned}\| \zeta\|^2_{E(I_i)}&=(a\zeta',\ \zeta')_{I_i}+(b\zeta,\ \zeta)_{I_i}\\ &=(a\zeta',\ \zeta'-e')_{I_i}+(b\zeta,\ \zeta-e)_{I_i}+(a\zeta',\ e')_{I_i}+(b\zeta,\ e)_{I_i}\\ &=(f-f_{\Delta,r},\ \zeta-e)_{I_i}+(a\zeta',\ e')_{I_i}+(b\zeta,\ e)_{I_i}\\ &\leq \| f-f_{\Delta,r}\|_{L^2(I_i)}\|\zeta-e\|_{L^2(I_i)}+\| \zeta\|_{E(I_i)}\| e\|_{E(I_i)}\\ &\leq \frac{1}{\sqrt{6a_i}}h_i\| f-f_{\Delta,r}\|_{L^2(I_i)}\| \zeta\|_{E(I_i)}+\| \zeta\|_{E(I_i)}\| e\|_{E(I_i)},\end{aligned}$$

and

$$\| \zeta\|_{E(I_i)}\leq \frac{1}{\sqrt{6a_i}}h_i\| f-f_{\Delta,r}\|_{L^2(I_i)}+\frac{C_1}{\sqrt{h_i}}C_{\Delta,r}. \tag{2.3}$$

Also (2.3) and Lemma imply that

$$\begin{aligned}\| \zeta\|_{L^\infty(I_i)}&\leq \| \zeta-e\|_{L^\infty(I_i)}+\| e\|_{L^\infty(I_i)}\\ &\leq \frac{\sqrt{h_i}}{2}\| \zeta'\|_{L^2(I_i)}+C_{\Delta,r}\\ &\leq \frac{\sqrt{h_i}}{2\sqrt{a_i}}\| \zeta\|_{E(I_i)}+C_{\Delta,r}\end{aligned}$$

$$\leq \frac{1}{2\sqrt{6}a_i} h_i^{3/2} \| f - f_{\Delta,r} \|_{L^2(I_i)} + C_2 C_{\Delta,r}.$$

Hence

$$\| \zeta \|_{L^\infty(I_i)} \leq \frac{1}{2\sqrt{6}\, a_i} h_i^{3/2} \| f - f_{\Delta,r} \|_{L^2(I_i)} + C_2 C_{\Delta,r},$$

and the proof of Theorem 3 is complete.

3. Adaptive mesh refinement procedures

Now we consider adaptive mesh refinement procedures such as the value of the error under the L^∞-norm is less than a given positive number δ.

Set

$$C(I_i, \Delta) = \frac{1}{2\sqrt{6}\, a_i} h_i^{3/2} \| f - f_{\Delta,r} \|_{L^2(I_i)}. \tag{3.1}$$

First we divide the interval I into some equal parts and compute the Galerkin approximation. Next, according as the inequality

$$C(I_i, \Delta) \leq \delta \tag{3.2}$$

holds or not for each subinterval I_i, we execute the fractionalization of the subintervals. For example we divide only the subintervals I_j such as

$$C(I_j, \Delta) = \max_{1 \leq i \leq n} C(I_i, \Delta) > \delta$$

or all the subintervals I_i such as

$$C(I_i, \Delta) > \delta$$

into two equal parts. We repeat the fractionalization till the (3.2) holds for every subintervals I_i.

In the following, we assume that (3.2) holds for every subintervals I_i. Then

$$\| y_0 - y_{\Delta,r} \|_{L^\infty(I)} \leq \delta + C_2 C_{\Delta,r} \leq (1 + C_2 C h_\Delta^{r-1}) \delta.$$

THEOREM 4. *For each subinterval I_i of a given partition Δ there is a constant C such that*

$$C(I_i, \Delta) \leq C(\| y_0^{(k+1)} \|_{L^2(I)} h_\Delta^{k+r} + \| y_0^{(k+1)} \|_{L^\infty(I_i)} h_i^{k+1}),$$

where the constant C is independent of Δ but dependent on a, b, k and r.

By using Theorem 2 this theorem is simply proved. It is known from Theorem 4 that (3.2) holds for each h_i sufficiently small.

THEOREM 5. *For a given partition Δ we assume that h_Δ is sufficiently small. Then there is a constant C such that*

$$C_{\Delta,r} \leq C \cdot \delta \cdot h_\Delta^{r-1},$$

where the constant C is independent of h_Δ but dependent on a, b and r.

Proof. Let $G(x, \xi)$ be the Green's function for (1.1). Then it follows from [5] that there is an integer j such that

$$C_{\Delta,r} = |(y_0 - y_{\Delta,r})(x_j)|$$
$$\leq \inf_{v \in \mathscr{M}_\Delta^r} \sum_{i=1}^{n} \| G(x_j, \cdot) - v \|_{E(I_i)} \| y_0 - y_{\Delta,r} \|_{E(I_i)}.$$

By using v such as Lagrange interpolation to $G(x_j, \cdot)$ on each subinterval I_i, there is a constant G_i for each subinterval I_i such that

$$\| G(x_j, \cdot)-v\|_{E(I_i)}\leq G_i h_i^r,$$

from which follows

$$C_{\Delta,r}\leq \sum_{i=1}^{n} G_i h_i^r \| y_0-y_{\Delta,r}\|_{E(I_i)}.$$

Also using Theorem 3 gives

$$\| y_0-y_{\Delta,r}\|_{E(I_i)}\leq \frac{1}{\sqrt{h_i}}(2\sqrt{a_1}\,\delta+C_1 C_{\Delta,r}).$$

Thus we multiply $G_i h_i^r$ on both sides of the above inequality and fine out the whole sum of all i such that

$$\sum_{i=1}^{n} G_i h_i^r \| y_0-y_{\Delta,r}\|_{E(I_i)}\leq (2\sqrt{a_1}\,\delta+C_1 C_{\Delta,r}) \sum_{i=1}^{n} G_i h_i^{r-1/2}.$$

Hence

$$(1-C_1 \sum_{i=1}^{n} G_i h_i^{r-1/2})C_{\Delta,r}\leq 2\sqrt{a_1}\,\delta \sum_{i=1}^{n} G_i h_i^{r-1/2}.$$

Here

$$\begin{aligned}\sum_{i=1}^{n} G_i h_i^{r-1/2} &\leq (\sum_{i=1}^{n} G_i^2)^{1/2}(\sum_{i=1}^{n} h_i^{2r-1})^{1/2}\\ &\leq C'(h_\Delta^{2r-2} \sum_{i=1}^{n} h_i)^{1/2}\\ &= C' h_\Delta^{r-1}.\end{aligned}$$

Thus for h_Δ sufficiently small

$$C_{\Delta,r}\leq C\cdot\delta\cdot h_\Delta^{r-1},$$

and the proof of Theorem 5 is complete.

Remark 1. It is known from Theorem 1 of [5] and the proof of Theorem 5 that the constants C of Theorem 1 and 5 depend on not only the values of a and b but also the value $\max_{0\leq x\leq 1} \| G^{(r+1)}(x, \cdot)\|_{L^2(I)}$ of the Green's function $G(x, \cdot)$ for (1.1).

Remark 2. It follows from the proof of Theorem 3 that there is a constant C' (which depends on a and b) such that

$$\| \zeta \|_{L^\infty(I_i)}\leq \frac{1}{2\pi a_i} h_i^{3/2} \| f-f_{\Delta,r}\|_{L^2(I_i)}+C' C_{\Delta,r}. \tag{3.3}$$

Also if we replace $\sqrt{6}$ of $C(I_i, \Delta)$ by π, then the inequality of Theorem 5 remain valid. Hence

$$\tilde{C}(I_i, \Delta)=\frac{1}{2\pi a_i} h_i^{3/2} \| f-f_{\Delta,r}\|_{L^2(I_i)}$$

is the principal term of (3.3), and can be also used in adaptive mesh refinement procedures as same as $C(I_i, \Delta)$.

4. Numerical examples

We illustrate some computational results for two sample problems. We choose $r=2$ and $\delta=10^{-4}$, and divide the interval I into sixteen equal parts at first. There are various methods of the partition such as (3.2) holds for every subintervals I_i. In this paper we use the following method:

Step 1. We divide all the subintervals I_i such as

$$C(I_i,\ \Delta)>10\delta$$

into two equal parts till

$$C(I_i,\ \Delta)\leq 10\delta$$

for every subintervals I_i.

Step 2. We replace 10δ by $\sqrt{10}\,\delta$ and divide the interval I into the same way as in Step 1.

Step 3. We replace $\sqrt{10}\,\delta$ by δ and divide the interval I into the same way as in Step 1.

We summarize the numerical results of the following problems in Table 1 and 2:

Example 1. $\quad -\varepsilon y''+y=-1 \quad (\varepsilon>0), \quad x\in I=[0,\ 1], \quad y(0)=y(1)=0.$

The function

$$y_0=\frac{e^{x/\sqrt{\varepsilon}}}{e^{1/\sqrt{\varepsilon}}+1}+\frac{e^{-x/\sqrt{\varepsilon}}}{e^{-1/\sqrt{\varepsilon}}+1}-1$$

is the unique solution and has boundary layers at the endpoints of I. The numbers of the knots which are contained in interval $(0,\ \sqrt{\varepsilon})$ are respectively 10, 9 and 11.

Example 2. $\quad -y''+y=(1-4\alpha^2)e^{2\alpha x}+(\alpha^2-1)(1+e^{\alpha})e^{\alpha x}+e^{\alpha}, \quad x\in I=[0,\ 1],$
$y(0)=y(1)=0,$

where the solution is

$$y_0=(e^{\alpha x}-1)(e^{\alpha x}-e^{\alpha}).$$

TABLE 1

ε	n	$C_{\Delta,r}$	$\max_{1\leq i\leq n} C(I_i,\Delta)$	$\max_{1\leq i\leq n} h_i$	$\min_{1\leq i\leq n} h_i$	e
10^{-4}	86	0.118(−5)	0.866(−4)	2^{-4}	2^{-10}	0.117(−4)
10^{-6}	130	0.117(−5)	0.893(−4)	2^{-5}	2^{-14}	0.122(−4)
10^{-8}	188	0.381(−6)	0.927(−4)	2^{-5}	2^{-17}	0.126(−4)

TABLE 2

α	n	$C_{\Delta,r}$	$\max_{1\leq i\leq n} C(I_i,\Delta)$	$\max_{1\leq i\leq n} h_i$	$\min_{1\leq i\leq n} h_i$	e
1	29	0.132(−7)	0.949(−4)	2^{-4}	2^{-5}	0.130(−4)
2	83	0.326(−8)	0.936(−4)	2^{-4}	2^{-7}	0.128(−4)
3	195	0.168(−7)	0.986(−4)	2^{-5}	2^{-9}	0.134(−4)

In these tables we use the notation

$$e=\max_{1\leq i\leq n}\max_{0\leq j\leq 100}|(y_0-y_{\Delta,r})(x_{i-1}+j\frac{h_i}{100})|.$$

If we use the method to divide the subintervals I_i such as

$$C(I_i,\ \Delta)>\delta$$

into two equal parts till

$$C(I_i,\ \Delta)\leq\delta$$

for every subintervals I_i, then the numbers of intervals used in the partitions are, respectively, 90, 138 and 204 for Example 1 and the same numbers as Table 2 for Example 2.

And if we replace $C(I_i, \Delta)$ by $\tilde{C}(I_i, \Delta)$ in Remark 2, then we obtain the better results than the ones in Table 1 and 2.

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