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Rep. Fac. Sci., Kagoshima Univ., (Math., Phys., & Chem.),
No. 16, p. 23-26, 1983

A NOTE ON THE EIGENSPACES FOR THE ADJACENCY MATRICES OF THE STEINER SYSTEMS

By

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(Received September 7, 1983)

1. Introduction and Summary

Let v, k, λ and t be positive integers with $v \geqq k \geqq t$ and let $[t/2]$ be s. Let P be a v-set $\{a_1, \ldots, a_v\}$ and $\mathfrak{B}$ a λ_0-set $\{B_1, \ldots, B_{\lambda_0}\}$ (a collection of k-subsets of P). The elements of P will be called points and the elements of $\mathfrak{B}$ will be called blocks. $(P, \mathfrak{B})$ is defined to be a t-(v, k, λ) design if each t-subset of P is contained in exactly λ blocks. Let $P_{(h)}$ be a set of h-subsets of P and $P_h(i)(i=1, \ldots, \binom{v}{h})$ the elements of $P_{(h)}$. Let M_h $(h=0, \ldots, k)$ be the incidence matrix for $(P, \mathfrak{B})$ defined by

$$M_h(P_h(i), B_j)=\begin{cases} 1 & \text{if } P_h(i) \subseteqq B_j \\ 0 & \text{otherwise.} \end{cases}$$

In order to state Yoshizawa's Proposition [5], we need the following row vector $\mathfrak{a}_i$ $(i=1, \ldots, v)$. $\mathfrak{a}_i=(\beta_i(1), \ldots, \beta_i(\lambda_0))$, the j-th coordinate $\beta_i(j)=\begin{cases} 1 & \text{if } a_i \in B_j \\ 0 & \text{otherwise.} \end{cases}$

He proved that the real vector space spanned by $\mathfrak{a}_i - \mathfrak{a}_j$ $(1 \leqq i, j \leqq v)$ is an eigenspace for $M_l^T M_l$ $(l=0, \ldots, t\text{-}1)$.

In this note we shall prove two propositions one of which is an extension of Yoshizawa's.

PROPOSITION 1. *$V_h = W_{h,h} M_h$ (see Cameron [1]) is an eigenspace for $M_l^T M_l$ $(l=0, \ldots, t\text{-}h)$.*

PROPOSITION 2. *Let $(P, \mathfrak{B})$ be a t-$(v, k, 1)$ design (i.e., Steiner system) such that for any block B of $\mathfrak{B}$, the set of points outside B, with blocks of the form B'-$(B \cap B')$ where B' is a block with $|B \cap B'|=t-1$ is a 2-$(v$-$k, k-t+1, c)$ design for some integer c. Then $W_{2,2}M_2$ is an eigenspace for the adjacency matrices of the Steiner system $(P, \mathfrak{B})$.*

We shall use the same notations as in [1] and [3].

2. Proof of Proposition 1

For $i, j, n=0, \ldots, [v/2]$, define C_{ij}^n to be the matrix with rows indexed by the i-subsets of P, columns indexed by the j-subsets of P, and with $(P_i(a), P_j(b))$ entry equal to 1 if $|P_i(a) \cap P_j(b)|=n$, and 0 otherwise.

We proceed in several steps to investigate $(w_h M_h) M_l^T M_l$ for $w_h \in W_{h,h}$. Note that h

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$\leqq l$ by Peterson's Theorem [3].

Step 1. We show that

$$M_h M_l^T = \sum_{j=0}^{h} \lambda_{h+l-j} C_{hl}^j. \tag{1}$$

$M_h M_l^T$ will be a matrix with rows indexed by the h-subsets of P and columns indexed by the l-subsets of P. If $P_h(a)$ is an h-subset of P and $P_l(b)$ is an l-subset of P, then $(P_h(a), P_l(b))$ entry will be equal to number of blocks of $\mathfrak{B}$ which contain $P_h(a) \cup P_l(b)$. Thus, if $|P_h(a) \cap P_l(b)| = j$, the $(P_h(a), P_l(b))$ entry is λ_{h+l-j}.
Then

$$M_h M_l^T = \sum_{j=0}^{h} \lambda_{h+j-l} C_{hl}^j.$$

For m, i, define M_m^i to be the matrix with rows indexed by the m-subsets of P, columns indexed by the blocks of $\mathfrak{B}$, and with $(P_m(a), B_b)$ entry equal to 1 if $|P_m(a) \cap B_b| = i$, and 0 otherwise. Note $M_i = M_i^i$.

Step 2. We show that

$$C_{hl}^j M_l = \sum_{p=j}^{h} \binom{p}{j}\binom{k-p}{l-j} M_h^p. \tag{2}$$

$C_{hl}^j M_l$ will be a matrix with rows indexed by the h-subsets of P and columns indexed by the blocks of $\mathfrak{B}$. If $P_h(a)$ is an h-subset of P and B_b is a block, then $(P_h(a), B_b)$ entry will be equal to the number of l-subsets of P which are contained by B_b and meet $P_h(a)$ in exactly j points. Thus, if $|P_h(a) \cap B_b| = p$, then $(P_h(a), B_b)$ entry is

$$\binom{p}{j}\binom{k-j}{l-j}.$$

$$C_{hl}^j M_l = \sum_{p=j}^{h} \binom{p}{j}\binom{k-p}{l-j} M_h^p.$$

Step 3. We show the following

LEMMA 1. If $w_h \in W_{h,h}$. then $w_h M_h^p = (-1)^{h-p}\binom{h}{p} w_h M_h$.

Proof. Let the $P_h(i)$-coordinate of w_h be $\gamma\{a_1, \ldots, a_h\}$. Let the i-th coordinate of $w_h M_h^p$ be n_B and let the i-th coordinate of $w_h M_h^{p+1}$ be n'_B. We have

$$n_B = \sum_{\{a_1,\cdots,a_p\}}{}' \sum_{\{a_{p+1},\cdots,a_h\}}{}'' \gamma\{a_1,\ldots,a_p, a_{p+1},\ldots,a_h\}$$
$$= \sum_{\{a_1,\cdots,a_p\}}{}' \frac{1}{h-p} \sum_{\{a_{p+1},\cdots,a_{h-1}\}}{}''' \sum_{a_h}{}'''' \gamma\{a_1,\ldots,a_p, a_{p+1},\ldots,a_{h-1}, a_h\},$$

where the sum $\sum'$ is taken over all p-subsets of B and the sum $\sum''$ is taken over all $(h$-$p)$-subsets of Ω-B and the sum $\sum'''$ is taken over all $(h$-p-1)-subsets of Ω-B and the sum $\sum''''$ is taken over all elements of Ω-$B \cup \{a_{p+1}, \ldots, a_{h-1}\}$. Since $W_{h,h} = \{w \in W_h \mid wB_{h-1\,h}^T = 0\}$, it follows that for any $\{a_1, \ldots, a_{h-1}\} \in P_{(h-1)}$, $\sum_{a_h} \gamma\{a_1, \ldots, a_{h-1}, a_h\} = 0$, where the sum $\sum$ is taken over all elements of Ω-$\{a_1,\ldots,a_{h-1}\}$. So

$$n_B = -\frac{1}{h-p} \sum_{\{a_{p+1},\cdots,a_{h-1}\}}{}''' \sum_{\{a_1,\cdots,a_p\}}{}' \sum_{a_h}{}^{\circ} \gamma\{a_1, \ldots, a_h\},$$

where the sum $\sum^{\circ}$ is taken over all elements of B-$\{a_1, \ldots, a_p\}$. By easy calcuation we have

$$n_B = -\frac{p+1}{h-p} n'_B.$$

It follows that

$$w_h M_h^p = -\frac{p+1}{h-p} w_h M_h^{p+1} = (-1)^{h-p}\binom{h}{p} w_h M_h.$$

This completes the proof of Lemma 1.

Step 4. By (1), (2) and Lemma 1, we have

$$(w_h M_h) M_l^T M_l = \sum_{j=0}^{h} \lambda_{h+l-j} \sum_{p=j}^{h} \binom{p}{j}\binom{k-p}{l-j}(-1)^{h-p}\binom{h}{p} w_h M_h \text{ for } w_h \in W_{h,h}.$$

In order to rewrite the above expression we need the following

LEMMA 2. $$\sum_{p=j}^{h} \binom{p}{j}\binom{k-p}{l-j}(-1)^{h-p}\binom{h}{p} = (-1)^{h-j}\binom{h}{j}\binom{k-h}{l-h}.$$

In this proof we use the combinatorial identity (see Riordan [4]).

(3) $$\binom{n-p}{m-p} = \sum_{k=0} (-1)^k \binom{n-k}{m}\binom{p}{k}.$$

Since $\binom{p}{j}\binom{h}{p} = \binom{h}{j}\binom{h-j}{p-j}$, we have

$$(-1)^{h-j} \sum_{p=j}^{h} \binom{p}{j}\binom{k-p}{l-j}(-1)^{h-p}\binom{h}{p} = (-1)^{h-j}\binom{h}{j} \sum_{p=j}^{h} \binom{h-j}{p-j}\binom{k-p}{l-j}(-1)^{h-p}.$$

If we set $r = p-j$, then we have

$$(-1)^{h-j} \sum_{p=j}^{h} \binom{p}{j}\binom{h-p}{l-j}(-1)^{h-p}\binom{h}{p} = \binom{h}{j} \sum_{r=0}^{h-j} \binom{h-j}{r}\binom{k-j-r}{l-j}(-1)^r.$$

By (3), the right hand side of the above equation is equal to $\binom{h}{j}\binom{k-h}{l-h}$. We complete the proof of Lemma 2.

Now it follows that $(w_h M_h) M_l^T M_l = \sum_{j=0}^{h} \lambda_{h+l-j} (-1)^{h-j}\binom{h}{j}\binom{k-h}{l-h} w_h M_h$ for $w_h \in W_{h,h}$.

This completes the proof of Proposition 1.

Remark. Easily we can prove Proposition 1 by Peterson's Result [3]. Peterson's Result is the following.

RESULT. If $w_h \in W_{h,h}$ and $h < l$, then $w_h C_{hl}^j = (-1)^{h-j}\binom{h}{j} w_h B_{h,l}$.

But it seems to me that this method does not apply to the proof of Proposition 2.

3. Application of Proposition 1 to Steiner systems

Let $(P, \mathfrak{B})$ be a t-$(v, k, 1)$ design (Steiner system $S(t, k, v)$). Define A_i to be the adjacency matrix, corresponding to i: That is, A_i is the matrix with rows and columns indexed by blocks, and with (B_a, B_b) entry equal to 1 if $|B_a \cap B_b| = i$ and 0 otherwise. Let I be the (λ_0, λ_0) identity matrix. Then we have

LEMMA 3. $$M_l^T M_l = \sum_{h=l}^{t-1} \binom{h}{l} A_h + \binom{k}{l} I, \text{ for } l = 0, \dots, t-1.$$

Proof. See Cameron [1].

Now let us prove Proposition 2. Note that

$$M_2A_{t-1}=(\lambda_{t-1}-1)\binom{k-2}{t-3}M_2^2+\binom{k-1}{t-2}M_2^1+cM_2^0.$$

By Lemma 1, $w_2M_2^i=(-1)^{2-i}\binom{2}{i}w_2M_2$ for $w_2\in W_{2,2}$. It follows that $W_{2,2}M_2$ is an eigenspace for A_{t-1}. Proposition 1 shows that $W_{2,2}M_2$ is an eigenspace for $M_l^TM_1$ ($l=0, \ldots, t-2$). Using Lemma 3 we obtain that $W_{2,2}M_2$ is an eigenspace for A_i ($i=0,\ldots,t-1$). Thus we complete the proof of Proposition 2.

Remark. Let us give an example of the Steiner systems satisfying the assumption of Proposition 2.

The example (Cameron [2]) is the non-degenerate Steiner system $S(t, k, v)$ such that the sets, $B'-B$, where B is a fixed block and B' is any block with $|B\cap B'|=t-1$, form a Steiner system $S(t-1, k-t+1, v-k)$ on the points outside B, for all B.

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