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Rep. Fac. Sci., Kagoshima Univ., (Math., Phys. & Chem.),
No. 24, p. 33–41, 1991.

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(Received September 10, 1991)

Abstract

Let a hypersurface S in an euclidean space R^n be implicitly defined by a differentiable function f in R^n. Then the Gaussian curvature of S is expressed in terms of f itself (cf. [6, Chap. 12]). As an application of this result, in the present paper we discuss the Gaussian curvature of the indicatrix of a Lagrange space $(R^n, \mathscr{L})$.

1. Introduction

In an euclidean xy-plane R^2, let a curve C be implicitly defined by a differentiable function f in R^2 as $f(x, y) = 0$. We put $f_1 = \partial f/\partial x$, $f_2 = \partial f/\partial y$. Around a point $\mathrm{P} \in C$ such that $f_2(\mathrm{P}) \neq 0$ the curve C is graphically expressed by a differentiable function g as $y = g(x)$. Then the curvature κ of C is given by $\kappa = y''/(1 + y'^2)^{3/2}$. If we directly calculate from

$$f_2 y' = -f_1, \qquad f_2^3 y'' = -(f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2),$$

where $f_{11} = \partial^2 f/\partial x^2$, $f_{12} = f_{21} = \partial^2 f/\partial x \partial y$, $f_{22} = \partial^2 f/\partial y^2$, we have

$$\kappa = \varepsilon \begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} /(f_1^2 + f_2^2)^{3/2} \qquad (\varepsilon = \text{sign } f_2). \tag{1.1}$$

In an euclidean xyz-space R^3, let a surface S be implicitly defined by a differentiable function f in R^3 as $f(x, y, z) = 0$. We put f_i, f_{ij} similarly. Around a point $\mathrm{P} \in S$ such that $f_3(\mathrm{P}) \neq 0$ the surface S is graphically expressed by a differentiable function g as $z = g(x, y)$, and the Gaussian curvature K of S is given by $K = (rt - s^2)/(1 + p^2 + q^2)^2$, where $p = \partial g/\partial x$, $q = \partial g/\partial y$, $r = \partial^2 g/\partial x^2$, $s = \partial^2 g/\partial x \partial y$, $t = \partial^2 g/\partial y^2$. If we directly calculate from

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This research was partially supported by Grant-in-Aid for Scientific Research (No. 03640080), Ministry of Education, Science and Culture.

$$f_3p = -f_1,\ f_3q = -f_2,$$
$$f_3^3r = -f_{11}f_3^2 + 2f_{13}f_1f_3 - f_{33}f_1^2,$$
$$f_3^3s = -f_{12}f_3^2 + f_{13}f_2f_3 + f_{23}f_1f_3 - f_{33}f_1f_2,$$
$$f_3^3t = -f_{22}f_3^2 + 2f_{23}f_2f_3 - f_{33}f_2^2,$$

we have

$$K = -\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_1 \\ f_{21} & f_{22} & f_{23} & f_2 \\ f_{31} & f_{32} & f_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix} \Big/ (f_1^2 + f_2^2 + f_3^2)^2. \tag{1.2}$$

Especially, in the case where a treated function f is a quadratic polynomial of the coordinates:

$$2f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c, \tag{1.3}$$

$$2f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d, \tag{1.4}$$

the formulas (1.1) and (1.2) are reduced to

$$\kappa = \varepsilon \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \Big/ (f_1^2 + f_2^2)^{3/2} \qquad (\varepsilon = \text{sign } f_2), \tag{1.5}$$

where $f_1 = ax + hy + g$, $f_2 = hx + by + f$, and

$$K = -\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix} \Big/ (f_1^2 + f_2^2 + f_3^2)^2, \tag{1.6}$$

where $f_1 = ax + hy + gz + p$, $f_2 = hx + by + fz + q$, $f_3 = gx + fy + cz + r$, respectively. It is noted that in these formulas the determinants appeared as the numerators are well-known constants independent on rectangular coordinate systems and the values of κ and K depend only on the magnitude of the gradient of f reciprocally.

Generally, in an n-dimensional euclidean space R^n we shall consider a hypersurface S defined by a differentiable function f in R^n as

$$S = \{x \in R^n \mid f(x) = 0,\ (\nabla f)\ (x) \neq 0\}, \tag{1.7}$$

where $x = (x_1, \ldots, x_n)$ is a rectangular coordinate system of R^n, and ∇f denotes the gradient of f:

(1.8) $$\nabla f = {}^t(f_1,\dots,f_n) \qquad (f_i = \partial_i f).$$

Throughout the present paper, we put $\partial_i = \partial/\partial x_i$ and denote a vector with components $v_1,\dots,v_n$ by an $n \times 1$ matrix ${}^t(v_1,\dots,v_n)$, but we use also an abridged notation (v_i). A letter ${}^tA$ denotes the transpose of a matrix A. The inner product $\sum_i u_i v_i$ of vectors $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ is denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$, and the length $(\sum_i v_i^2)^{1/2}$ of a vector $\boldsymbol{v} = (v_i)$ by $|\boldsymbol{v}|$. The summation convention is not used.

Now, the notion of Gaussian curvature is generally defined for a hypersurface S in R^n, and in the case where S is implicitly given by (1.7) we can get the same expression as (1.1) and (1.2) (Theorem 2.1). This expression is derived, for example, from Theorem 5 of Thorpe [6, Chap. 12, p 89], but for convenience we shall give a self-contained proof in §2, based on Lemma 2.1 concerning with the determinant of a linear transformation of a hypersubspace of a vector space R^n.

The purpose of the present paper is to apply this result to Finsler geometry. We denote by $y = (y_1,\dots,y_n)$ the canonical coordinate system of the tangent space R^n_x at each point $x \in R^n$, and put $\dot{\partial}_i = \partial/\partial y_i$. Let $(R^n, \mathscr{L})$ be a Lagrange space, where $\mathscr{L}$ is a positive-valued differentiable function in the tangent bundle of R^n and satisfies the regularity condition det $(\dot{\partial}_i \dot{\partial}_j \mathscr{L}) \neq 0$ (cf. [4, p 11], [1, p 1]).

Each tangent space R^n_x is also regarded as an n-dimensional euclidean space with the rectangular coordinate system y. A hypersurface $I_x = \{y \in R^n_x \mid \mathscr{L}(x, y) = 1\}$ in R^n_x is called the *indicatrix* at x. In §3 we shall express the Gaussian curvature of I_x in terms of $\mathscr{L}$ (Theorem 3.1).

A Lagrange space $(R^n, \mathscr{L})$ becomes a Finsler space (R^n, L) if $\mathscr{L}$ is given by $\mathscr{L} = L^2$, where L is positively homogeneous of degree 1: $L(x, \lambda y) = \lambda L(x, y)$ for $\lambda > 0$. Then Theorem 3.1 is reduced to Thorem 3.2. Given a hypersurface S_x in each tangent space R^n_x a priori, by the well-known method (cf. [3, p 105]) we have a Finsler space whose indicatrix I_x is the given S_x. Thus the Gaussian curvature of S_x is expressed in terms of a Finsler geometry. This fact seems interesting from the standpoint of application.

The authors with to express here their sincere gratitude to Professor Dr. Makoto Matsumoto and Professor Dr. Yoshihiro Ichijyō for the invaluable suggestions and encouragement. The authors are also grateful to Professor Dr. Shun-ichi Hōjō for the helpful advice in the arrangements of Theorem 2.1.

2. The Gaussian curvature of a hypersurface

We shall recall here an elementary definition of the Gaussian curvature K of a surface S in an euclidean space R^3. Let S be expressed by parameters u_1, u_2 as $x = x(u_1, u_2)$, where $x = (x_1, x_2, x_3)$ is a rectangular coordinate system of R^3. At each point $\mathrm{P} \in S$, two tangent vector fields $X_\alpha = \partial x/\partial u_\alpha$ $(\alpha = 1, 2)$ constitute a basis of the tangent plane S_P, and the unit vector field $N=(X_1\times X_2)/|X_1\times X_2|$ is orthogonal to S_P. Then paying attention to the Weingarten equation

$$N_\beta = -\sum_\alpha h^\alpha_\beta X_\alpha \qquad (N_\beta = \partial N/\partial u_\beta), \tag{2.1}$$

a linear transformation T of S_P is defined by

$$T\colon S_P \longrightarrow S_P \,|\, \boldsymbol{v} = \sum_\beta v_\beta X_\beta \longrightarrow T(\boldsymbol{v}) = -\sum_\beta v_\beta N_\beta. \tag{2.2}$$

Since T is represented by the matrix (h^α_β) with respect to the basis X_1, X_2, the determinant of T gives the Gaussian curvature K of S at P.

It is noted that the vector $\sum_\beta v_\beta N_\beta$ in (2.2) is a derivative $\nabla_{\boldsymbol{v}} N$ of N with respect to $\boldsymbol{v}$. Generally, let Ω be a differentiable geometrical object defined on an open set U of an n-dimensional euclidean space R^n, such as a function and a vector field, and let $\boldsymbol{v}=(v_i)$ be a vector at a point $\mathrm{P}\in U$. The *derivative* $\nabla_{\boldsymbol{v}}\Omega$ *of* Ω *with respect to* $\boldsymbol{v}$ is defined by

$$\nabla_{\boldsymbol{v}}\Omega = (\Omega\circ c)'\,(t_0), \tag{2.3}$$

where $x=c(t)$ is any differentiable curve such that $c(t_0)=\mathrm{P}$, $c'(t_0)=\boldsymbol{v}$ The derivative $\nabla_{\boldsymbol{v}}\Omega$ is independent on the choice of a curve c, and is expressed by

$$\nabla_{\boldsymbol{v}}\Omega = \sum_i (\partial_i\Omega)v_i. \tag{2.4}$$

Now, let (S, N) be an oriented hypersurface in R^n, where N is a unit vector field orthogonal to S. Let S_P be the tangent space of a point $\mathrm{P}\in S$. The notion of a derivative of Ω with respect to $\boldsymbol{v}\in S_P$ is also defined in the case where Ω is defined only on S. Since $\nabla_{\boldsymbol{v}}N\in S_P$ for $\boldsymbol{v}\in S_P$, we have a linear transformation T of S_P defined by

$$T\colon S_P \longrightarrow S_P \,|\, \boldsymbol{v} \longrightarrow T(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}}N. \tag{2.5}$$

This is called the *Weingarten map* of (S, N) at P. The *Gaussian curvature* K of (S, N) at P is defined by the determinant of T.

Remark 2.1. In the case of $n=3$, this definition of the Gaussian curvature coincides with the elementary definition stated above, independent on the choice of N.

In the case of $n=2$, the Gaussian curvature K of a parameterized curve C is a cuvature κ of C, if we take N to be the normal vector of C. If N is replaced by $-N$, we have $K=-\kappa$. Since the Weingarten map T is represented by an $(n-1)\times(n-1)$ matrix, if n is odd then K is independent on the choice of N, whereas if n is even then K changes the sign by turning the direction of N.

In the case where a hypersurface S in R^n is given by (1.7), it is noted that the gradient ∇f of a treated function f is orthogonal to S at each point $P \in S$. We have the following expression for the Gaussian curvature K of an oriented hypersurface (S, N).

Theorem 2.1. *Let (S, N) be an oriented hypersurface in R^n, where S is given by (1.7) using a differentiable function f in R^n, and N is a unit vector field orthogonal to S given by*

$$N = \varepsilon \nabla f / |\nabla f| \qquad (\varepsilon = \pm 1). \tag{2.6}$$

Then the Gaussian curvature K of (S, N) is given by

$$K = -\tau \begin{vmatrix} f_{ij} & f_i \\ f_j & 0 \end{vmatrix} / |\nabla f|^{n+1}, \tag{2.7}$$

where $f_i = \partial_i f$, $f_{ij} = \partial_i \partial_j f$, $\nabla f = (f_i)$, and $\tau = (-\varepsilon)^{n+1}$.

Remark 2.2. In the case where n is odd, we have $\tau = 1$. In the case where n is even, we have $\tau = -\varepsilon$. If we choose an orientation N of S by

$$N = -\nabla f / |\nabla f|, \tag{2.8}$$

we have always $\tau = 1$. Even in the case where an orientation N is given a priori, we can take f to be $\tau = 1$, because f and $-f$ give the same S.

For the proof of Theorem 2.1 we shall show that the Weingarten map T of (S, N) at $P \in S$ satisfies the following formula for any $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in S_P$:

$$\boldsymbol{u} \cdot T(\boldsymbol{v}) = -(\varepsilon / |\nabla f|) \sum_{i,j} f_{ij} u_i v_j. \tag{2.9}$$

Since $\nabla f = \varepsilon |\nabla f| \, N$ from (2.6), we have for any $\boldsymbol{v} = (v_i) \in S_P$

$$\nabla_{\boldsymbol{v}} \nabla f = \varepsilon \nabla_{\boldsymbol{v}}(|\nabla f|) N + \varepsilon |\nabla f| \nabla_{\boldsymbol{v}} N.$$

Thus from (2.5) we have for any $\boldsymbol{u} = (u_i) \in S_P$

$$\boldsymbol{u} \cdot (\nabla_{\boldsymbol{v}} \nabla f) = -\varepsilon |\nabla f| \boldsymbol{u} \cdot T(\boldsymbol{v}). \tag{2.10}$$

Since the vector field $\nabla f = (f_i)$ is defined on some open set containing S, from (2.4) we have $\nabla_{\boldsymbol{v}} \nabla f = \sum_j (\partial_j \nabla f) v_j$, so we have $\boldsymbol{u} \cdot (\nabla_{\boldsymbol{v}} \nabla f) = \sum_{i,j} f_{ij} u_i v_j$. Thus (2.9) is shown from (2.10).

The proof of Theorem 2.1 is obtained from the following lemma by putting $a_{ij} = -\varepsilon f_{ij} / |\nabla f|$, $n_i = \varepsilon f_i / |\nabla f|$.

Lemma 2.1. *Let V be an n-dimensional real vector space linearly isomorphic to a vector space R^n, and T a linear transformation of an $(n-1)$-dimensional vector subspace*

W of V. We denote any $\boldsymbol{v} \in V$ by $\boldsymbol{v} = (v_i)$ using the corresponding $(v_i) \in R^n$. Let $N = (n_i)$ be a unit vector orthogonal to W. If for any $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in W$ the inner product $\boldsymbol{u} \cdot T(\boldsymbol{v})$ is expressed by a matrix $A = (a_{ij})$ as

$$\boldsymbol{u} \cdot T(\boldsymbol{v}) = {}^t\boldsymbol{u} A \boldsymbol{v} \ (= \sum_{i,j} a_{ij} u_i v_j), \tag{2.11}$$

then the determinant K of T is given by

$$K = -\begin{vmatrix} A & N \\ {}^tN & 0 \end{vmatrix} \ \left(= -\begin{vmatrix} a_{ij} & n_i \\ n_j & 0 \end{vmatrix} \right). \tag{2.12}$$

Proof. In the proof, Greek indices take the values $1, \ldots, n-1$. We choose a basis $X_1, \ldots, X_{n-1}$ of W, and represent T by an $(n-1) \times (n-1)$ matrix $B = (b_{\alpha\beta})$, where

$$T(X_\beta) = \sum_\alpha b_{\alpha\beta} X_\alpha. \tag{2.13}$$

Then the determinant K of T is obtained as $K = \det B$.

We define $n \times n$ matrices $\tilde{B}$, X, Y by

$$\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \quad X = (X_1, \ldots, X_{n-1}, N), \quad Y = (Y_1, \ldots, Y_{n-1}, N),$$

where $Y_\beta = T(X_\beta)$, and $(n+1) \times (n+1)$ matrices $\tilde{A}$, $\tilde{X}$ by

$$\tilde{A} = \begin{pmatrix} A & N \\ {}^tN & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

Paying attention to (2.13) and $X_\alpha \cdot N = 0$, $N \cdot N = 1$, we have

$${}^tX(X\tilde{B}) = {}^tXY = \begin{pmatrix} X_\alpha \cdot Y_\beta & 0 \\ 0 & 1 \end{pmatrix},$$

from which we have $(\det X)^2 K = \det(X_\alpha \cdot Y_\beta)$.

In the same way, we have

$${}^t\tilde{X}\tilde{A}\tilde{X} = \begin{pmatrix} {}^tX_\alpha A X_\beta & {}^tX_\alpha A N & 0 \\ {}^tNAX_\beta & {}^tNAN & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

from which we have $(\det X)^2 (\det \tilde{A}) = -\det({}^tX_\alpha A X_\beta)$. Since the matrix X is regular, we have $K = -\det \tilde{A}$ from (2.11). Q.E.D.

In the special case where a treated function f is a quadratic polynomial of the coordinates, we have directly from (2.7)

Theorem 2.2. *Let (S, N) be an oriented hypersurface in R^n, where S is a regular quadratic surface defined by*

$$2f(x) = \sum_{i,j} a_{ij}x_i x_j + 2\sum_i b_i x_i + c = 0 \qquad (a_{ij} = a_{ji}), \tag{2.14}$$

and N is a unit vector field orthogonal to S given by (2.6). *Then the Gaussian curvature K of (S, N) is given by*

$$K = -\tau \begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix} / (\sum_i f_i^2)^{(n+1)/2}, \tag{2.15}$$

where $f_i = \sum_j a_{ij}x_j + b_i$ and $\tau = (-\varepsilon)^{n+1}$.

Remark 2.3. The expression (2.15) of the Gaussian curvature K of a hypersurface $\lambda f = 0$ is independent on a positive constant λ.

3. The indicatrix of a Lagrange space

Let $(R^n, \mathscr{L})$ be a Lagrange space. At each point $x \in R^n$ the indicatrix I_x is a hypersurface in the tangent space R^n_x, where R^n_x is thought to be an n-dimensional euclidean space with a rectangular coordinate system $y = (y_i)$.

We define a function f by

$$f(x, y) = \mathscr{L}(x, y) - 1, \tag{3.1}$$

and put $\dot{\nabla} f = (\dot{\partial}_i f)$, $\dot{\nabla}\mathscr{L} = (\dot{\partial}_i \mathscr{L})$. Since we have $\dot{\nabla} f = \dot{\nabla}\mathscr{L} \neq 0$ from the regularity of $\mathscr{L}$, the indicatrix I_x is expressed as

$$I_x = \{y \in R^n_x \mid f(x, y) = 0,\ (\dot{\nabla} f)\ (x, y) \neq 0\}. \tag{3.2}$$

At each $y \in I_x$ the vector field $\dot{\nabla}\mathscr{L}$ is orthogonal to I_x. Suggested by Remark 2.2, we shall assume that an orientation N_x of I_x is always

$$N = -\dot{\nabla}\mathscr{L} / |\dot{\nabla}\mathscr{L}|. \tag{3.3}$$

Then we have from Theorem 2.1

Theorem 3.1. *Let $(R^n, \mathscr{L})$ be a Lagrange space. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla}\mathscr{L} = (\dot{\partial}_i \mathscr{L})$ is given by*

$$K = -\begin{vmatrix} \dot{\partial}_i \dot{\partial}_j \mathscr{L} & \dot{\partial}_i \mathscr{L} \\ \dot{\partial}_j \mathscr{L} & 0 \end{vmatrix} / (\sum_i (\dot{\partial}_i \mathscr{L})^2)^{(n+1)/2}. \tag{3.4}$$

Now, let a Lagrange space $(R^n, \mathscr{L})$ be a Finsler space (R^n, L), where $\mathscr{L} = L^2$. Putting $l_i = \dot{\partial}_i L$, $g_{ij} = (\dot{\partial}_i \dot{\partial}_j L^2)/2$, and $g = \det(g_{ij})$, we have on the indicatrix, where $L(x, y) = 1$,

$$2^{-(n+1)} \begin{vmatrix} \dot{\partial}_i \dot{\partial}_j \mathscr{L} & \dot{\partial}_i \mathscr{L} \\ \dot{\partial}_j \mathscr{L} & 0 \end{vmatrix} = \begin{vmatrix} g_{ij} & l_i \\ l_j & 0 \end{vmatrix} = -g.$$

Thus Theorem 3.1 is reduced to

Theorem 3.2. *Let (R^n, L) be a Finsler space. At each point $x \in R^n$, the Gaussian curvature K of the indicatrix I_x oriented in the direction opposite to $\dot{\nabla} L = (l_i)$ is given by*

$$K = g/(\sum_i l_i^2)^{(n+1)/2}. \tag{3.5}$$

As an example we shall treat a Randers space $(R^n, \alpha + \beta)$, where α and β are a Riemannian metric and a non-vanishing 1-form in R^n respectively. We put

$$\alpha = (\sum_{i,j} a_{ij}(x) y_i y_j)^{1/2}, \quad \beta = \sum_i b_i(x) y_i. \tag{3.6}$$

Each indicatrix I_x of a Riemannian space (R^n, α) is a quadratic hypersurface of the coordinates y_i with the center $y = 0$:

$$2f(x, y) = \sum_{i,j} a_{ij} y_i y_j - 1 = 0, \tag{3.7}$$

whereas the indicatrix I_x of a Randers space is expressed as

$$2f(x, y) = \sum_{i,j} (a_{ij} - b_i b_j) y_i y_j + 2 \sum_i b_i y_i - 1 = 0. \tag{3.8}$$

Under the necessity of using a metric with non-central indicatrix, as the simplest possible asymmetrical modification of a Riemannian metric, Randers introduced a Finsler space with a metric $L = \alpha + \beta$, which is a unique positive-valued Finsler metric such that each indicatrix is a quadratic hypersurface of the coordinates y_i (cf. [5], [2, p 34]).

Now, from (3.8) we have $\dot{\partial}_i f = \sum_j (a_{ij} - b_i b_j) y_j + b_i$, which becomes $\dot{\partial}_i f = \sum_j a_{ij} y_j + \alpha b_i$ on the indicatrix. Since from $L = \alpha + \beta$ we have $l_i = (\sum_j a_{ij} y_j + \alpha b_i)/\alpha$, the vector $\dot{\nabla} f = (\dot{\partial}_i f)$ has the same direction as $\dot{\nabla} L = (l_i)$. Thus the vector field

$$N = -\dot{\nabla} f / |\dot{\nabla} f| \tag{3.9}$$

gives the orientation assumed in (3.3). Since we have

$$\begin{vmatrix} a_{ij} - b_i b_j & b_i \\ b_j & -1 \end{vmatrix} = -\det(a_{ij}),$$

applying Theorem 2.2 to (3.8) we have

Theorem 3.3. *Let* (R^n, L) *be a Randers space, where* $L = \alpha + \beta$. *At each point* $x \in R^n$, *the Gaussian curvature* K *of the indicatrix* I_x *is given by*

$$K = \det(a_{ij})/(\sum_i f_i^2)^{(n+1)/2}, \tag{3.10}$$

where $f_i = \sum_j a_{ij} y_j + \alpha b_i$, *provided* I_x *is oriented in the direction opposite to* $\dot{V}f = (f_i)$.

Remark 3.1. Since $f_i = \alpha l_i$ in Theorem 3.3, if we compare (3.10) with (3.5), we have $g = \det(a_{ij})/\alpha^{n+1}$ on the indicatrix, from which at any $y \in R_x^n$ we have $g = (L/\alpha)^{n+1} \det(a_{ij})$.

References

[1] O. Amici, B. Casciaro and M. Hashiguchi, Finsler metrics associated with a Lagrangian, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **20** (1987), 33–41.

[2] M. Hashiguchi and Y. Ichijyō, Randers spaces with rectilinear geodesics, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **13** (1980), 33–40.

[3] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.

[4] R. Miron, A Lagrangian theory of relativity, Sem. Geometrie şi Topologie, Univ. Timişoara, Fac, Şti. Ale Naturii, 1985.

[5] G. Randers, On an asymmetrical metric in the four-space of general relativity, Phys. Rev. (2) **59** (1941), 195–199.

[6] J. Thorpe, Elementary topics in differential geometry, Springer-Verlag, New York, 1979.