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ON AN ERROR EXPRESSION IN NUMERICAL INTEGRATION FORMULAE

By

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In this report we shall derive an error expression by making use of the distribution theory on estimating an error in numerical integration, and compare the resulting error expression with one obtained by using Peano's Kernel theorem. ([1; p. 29], [2; p. 25]).

When an arbitrary integration rule $\sum_{P=1}^{N} c_P \phi(a_P)$ is described by $Q(\phi)$, and if the error $E(\phi)$ is denoted by

$$E(\phi)=\int_a^b \phi(x)\,dx - Q(\phi),$$

we have, by Peano's Kernel theorem,

$$E(\phi)=\frac{1}{(m-1)!}\int_a^b \left[\int_a^b (t-x)_+^{m-1}\,dx - \sum_{P=1}^{N} c_P\,(a_P-x)_+^{m-1}\right]\phi^{(m)}(x)\,dx,$$

where $x_+^{m-1}=\begin{cases} x^{m-1} & \text{for } x \geqq 0 \\ 0 & \text{for } x<0 \end{cases}$, and $E(\phi)=0$ for $\phi(x)=1, x, x^2, \cdots, x^{m-1}$.

Let $\mathcal{E}^{(m)}$ be a space of functions of a real variable, whose derivatives are continuous up to order m, and we denote its dual space by $\mathcal{E}'^{(m)}$. We define by $\boldsymbol{D}$ the space of infinitely pifferentiable functions on *1*-dimensional Euclidean space R, which has compact support. Then we get the following theorem.

Theorem. *Let* $\phi \in \mathcal{E}^{(m)}$, *and let* $\alpha \in \boldsymbol{D}$ *have a support in a neighborhood of* $[a, b]$. *Suppose again that* $\alpha(x)=1$ *on* $[a, b]$. *then we have*

$$E(\phi)=\frac{1}{(m-1)!}\int_{-\infty}^{b}\left[\frac{1}{m}\left((b-x)^m-(a-x)_+^m\right)-\sum_{P=1}^{N} c_P\,(a_P-x)_+^{m-1}\right](\alpha\phi)^{(m)}(x)\,dx.$$

Proof.

Let $t \in \mathcal{E}'^{(m)}$ have a support $[a, b]$, and let $t_N=\sum_{P=1}^{N} c_P\ \delta_{(a_P)}$, where $\delta_{(a_P)}$ is the Dirac measure at the point a_P.

Then we have

$$\langle t - t_N, \phi\rangle = \frac{1}{(m-1)!}\langle (t-t_N)*x_-^{m-1}, (a\phi)^{(m)}\rangle ,$$

where $x_-^{m-1} = \begin{cases} 0 & \text{for } x \geqq 0 , \\ |x|^{m-1} & \text{for } x < 0 . \end{cases}$

Let t be a distribution that

$$t(x) = \begin{cases} 1 & \text{for } x \in [a, b] , \\ 0 & \text{for } x \notin [a, b] . \end{cases}$$

Set $\psi = (a\phi)^{(m)}$, so we have, by the linearity of the distribution,

$$\langle (t-t_N)*x_-^{m-1}, \psi\rangle = \langle t*x_-^{m-1}, \psi\rangle - \langle t_N*x_-^{m-1}, \psi\rangle .$$

Furthermore, since

$$\begin{aligned}\langle t*x_-^{m-1}, \psi\rangle &= \langle t_x, \langle y_-^{m-1}, \psi(x+y)\rangle\rangle \\ &= (-1)^{m-1}\int_a^b\int_{-\infty}^0 y^{m-1}\psi(x+y)\,dx\,dy\end{aligned}$$

from the definition of convolution, we get, by integrating by parts and by the transformation of variables,

$$\begin{aligned}\langle t*x_-^{m-1}, \psi\rangle &= -\frac{(-1)^{m-1}}{m}\int_{-\infty}^0 y^m\,dy\int_a^b \frac{d\psi(x+y)}{dx}\,dx \\ &= \int_{-\infty}^b\left[\frac{1}{m}((b-x)^m - (a-x)_+^m)\right]\psi(x)\,dx .\end{aligned}$$

On the other hand, since

$$\langle x_-^{m-1}*\delta_{(a_p)}, \psi\rangle = \langle (a_P - x)_+^{m-1}, \psi\rangle ,$$

it follows that

$$\langle t_N*x_-^{m-1}, \psi\rangle = \langle \sum_{P=1}^N c_P\,(a_P - x)_+^{m-1}, \psi\rangle .$$

Thus the theorem is proved.

The following corollary shows the relation of the theorem and an error expression obtained by Peano's Kernel theorem.

Corollary. *If we add the assumption that $E(\phi)=0$ for all polynomials $\phi(x)$ of at most degree $m-1$ to the theorem, the error expression of the theorem agrees with one obtained by Peano's Kernel theorem.*

Proof.

If we remark $\alpha\phi=\phi$ on $[a, b]$, we have

$$\int_{-\infty}^{b}\left[\frac{1}{m}((b-x)^m-(a-x)_+^m)-\sum_{P=1}^{N}c_P\,(a_P-x)_+^{m-1}\right](\alpha\phi)^{(m)}(x)\,dx$$

$$=\int_{-\infty}^{a}\left[\frac{1}{m}((b-x)^m-(a-x)^m)-\sum_{P=1}^{N}c_P\,(a_P-x)^{m-1}\right](\alpha\phi)^{(m)}(x)\,dx$$

$$+\int_{a}^{b}\left[\int_{a}^{b}(t-x)_+^{m-1}\,dt-\sum_{P=1}^{N}c_P\,(a_P-x)_+^{m-1}\right]\phi^{(m)}(x)\,dx\,,$$

since $\int_{a}^{b}(t-x)_+^{m-1}\,dt=\frac{1}{m}(b-x)^m$.

By the assumption of the corollary, the first term of the right-hand side of the above expression vanishes. Thus we get the corollary.

By this theorem, it is shown that an error expression of the numerical integration is easily obtained by using some results of the theory of distribution.

References

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