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ON HYPERBOLIC AND TRIGONOMETRIC *B*-SPLINES ON EQUALLY SPACED KNOTS

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Abstract.

The object of the present paper is to show that hyperbolic and trigonometric splines admit bases of B-splines characterized by convolution processes of exponential and trigonometric functions, respectively.

1. Introduction

Among the various classes of splines, the polynomial spline has been received the greatest attention primarily because it admits a basis of B-splines which are accurately and efficiently computed. Recently it has been shwon that trigonometric and hyperbolic splines also admit bases of B-splines ([1], [3]).

The object of the present paper is to show that these B-splines on equally spaced knots are characterized by a convolution process of exponential function. Throughout this paper, we assume that m (≥ 1) is a natural number and λ is a positive parameter. Then, by use of ϕ_λ:

(1.1) $$\phi_\lambda(x) = e^{\lambda x} \ (0 \leq x \leq 1) \text{ and } 0 \ (\text{otherwise}),$$

we may define hyperbolic B-splines:

(1.2)
$$Q_{2m-1,\lambda}(x) = (\chi * \phi_\lambda * \phi_{-\lambda} *) \cdots * \phi_{(m-1)\lambda} * \phi_{-(m-1)\lambda})(x)$$
$$Q_{2m,\lambda}(x) = (\phi_{\frac{1}{2}\lambda} * \phi_{-\frac{1}{2}\lambda} * \cdots * \phi_{(m-\frac{1}{2})\lambda} * \phi_{-(m-\frac{1}{2})\lambda})(x)$$

where $\chi(x) = \lim_{\lambda \to 0} \phi_\lambda(x)$. i. e., χ is a characteristic function on $[0, 1)$, and for $f(x)$ and $g(x)$ we denote

$$\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \qquad \text{by } (f*g)(x).$$

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For the polynomial B-spline $Q_m(x)$ of degree $m-1$, it is well-known that

$$Q_m(x) = (\underbrace{\chi * \chi * \cdots * \chi}_{m})(x). \tag{1.3}$$

Therefore we have a relation between the above defined hyperbolic B-spline $Q_{m,\lambda}$ and this polynomial one Q_m:

$$\lim_{\lambda \to 0} Q_{m,\lambda}(x) = Q_m(x).$$

By the definition of $Q_{m,\lambda}$, it may be easily shown that the hyperbolic B-spline has the following properties similar to those of the polynomial one:

(i) $Q_{m,\lambda}(x) \in C^{m-2}(-\infty, \infty)$

(ii) $Q_{m,\lambda}(x) = Q_{m,\lambda}(m-x)$

(iii) the support of $Q_{m,\lambda} = [0, m]$

(iv) $Q_{m,\lambda}(x) > 0$ on $(0, m)$

(iv) $\displaystyle\sum_{i=-\infty}^{\infty} Q_{2m-1,\lambda}(x-i) = \prod_{k=1}^{m-1} \{\sinh(\tfrac{1}{2}k\lambda)/(\tfrac{1}{2}k\lambda)\}^2$

(this equality implies a partition of unity)

(vi) $\displaystyle\int_{-\infty}^{\infty} Q_{2m-1,\lambda}(x)\,dx = \prod_{k=1}^{m-1} \{\sinh(\tfrac{1}{2}k\lambda)/(\tfrac{1}{2}k\lambda)\}^2$

$\displaystyle\int_{-\infty}^{\infty} Q_{2m,\lambda}(x)\,dx = \prod_{k=1}^{m} [\sinh\{(k-\tfrac{1}{2})\lambda\}/\{(k\tfrac{1}{2})\lambda\}]^2$

(vii) on $(i, i+1)$ with $i = 0, \pm 1, \cdots$

$$(D^2-\lambda^2)(D^2-(2\lambda)^2)\cdots(D^2-(r\lambda)^2)DQ_{m,\lambda}(x) = 0$$

$$(m=2r+1,\ r=0, 1, \cdots)$$

$$(D^2-(\tfrac{1}{2}\lambda)^2)\cdots(D^2-(r-\tfrac{1}{2})^2\lambda^2)Q_{m,\lambda}(x) = 0$$

$$(m=2r,\ r=1, 2, \cdots)$$

(viii) for $s \in \text{Span}\,\{Q_{m,\lambda}(x-i)\}_{i=-\infty}^{\infty}$,

$$\sum_{i=1}^{m-1} Q_{m,\lambda}^{(k)}(m-i)s(i) = \sum_{i=1}^{m-1} Q_{m,\lambda}(m-i)s^{(k)}(i) \tag{*}$$

$$(k=1, 2, \cdots, m-2).$$

The above consistency relation $(*)$ at $(m-1)$ consective intger points is reduced to the one at $(m-2)$ integer points if $m=5, 7, \cdots$ and $2 \leq k \leq m-2$, by making an alternating sum of $(*)$ obtained by writing down $(*)$, substracting $(*)$ with i replaced by $i+1$, adding $(*)$ with i replaced by $i+2$ and so on (for this technique, see [2]).

Our next theorem gives important relations for computing the hyperbolic B-spline $Q_{m+1,\lambda}$ of degree m from the hyperbolic B-spline $Q_{m,\lambda}$ of degree $m-1$:

Theorem 1.

(1.4) (i) $$Q_{m+1,\lambda}(x) = (2/m\lambda)\left[Q_{m,\lambda}(x)\sinh\left(\frac{1}{2}\lambda x\right) + Q_{m,\lambda}(x-1)\sinh\left\{\frac{1}{2}\lambda(m+1-x)\right\}\right]$$

(ii) $$Q'_{m+1,\lambda}(x) = Q_{m,\lambda}(x)\cosh\left(\frac{1}{2}\lambda x\right) - Q_{m,\lambda}(x-1)\cosh\left\{\frac{1}{2}\lambda(m+1-x)\right\}.$$

Letting $\lambda \to 0$ in the above relations, we have the well known ones of the polynomial B-spline:

(1.5) (i) $$Q_{m+1}(x) = (1/m)\{xQ_m(x) + (m+1-x)Q_m(x-1)\}$$

(ii) $$Q'_{m+1}(x) = Q_m(x) - Q_m(x-1).$$

Here we remark that our B-spline defined by the convolution process of the exponential function satisfies the recurrence relations with simpler coefficients than the B-spline defined by the divided difference(Schumaker [3]). In addition, these B-splines are different only in their coefficients, and so our B-splines are also considered to be a basis of the following space S:

(1.6) $$S = \{s \mid s \in F_m \text{ on } (i, i+1) \text{ with } i = 0, \pm 1, \cdots \text{ and } s \in C^{m-2}(-\infty, \infty)\}$$

where

(1.7) $$F_m = \begin{cases} \text{Span } \{1, \cosh \lambda x, \sinh \lambda x, \cdots, \cosh(r\lambda x), \sinh(r\lambda x)\} \ (m=2r+1) \\ \text{Span } \{\cosh(\frac{1}{2}\lambda x), \sinh(\frac{1}{2}\lambda x), \cdots, \cosh(r-\frac{1}{2})\lambda x, \sinh(r-\frac{1}{2})\lambda x\} \ (m=2r) \end{cases}$$

Next we shall define the trigonometric B-spline $\tilde{Q}_{m,\lambda}(x)$ by replacing λ by $i\lambda$ in the definition of the hyperbolic B-spline $Q_{m,\lambda}(x)$:

(1.8) $$\tilde{Q}_{m,\lambda}(x)=Q_{m,i\lambda}(x)$$

where $i=\sqrt{-1}$.
Let us denote $(\phi_{i\chi}*\phi_{-i\lambda})(x)$ by $\psi_{i\lambda}(x)$. Then we have

(1.9) (i) $\psi_{i\lambda}(x)=\psi_{i\lambda}(2-x)$

(ii) the suppot of $\psi_{i}\lambda(x)=[0, 2]$

(iii) $$\psi_{i\lambda}(x)=\begin{cases}(1/\lambda)\sin\lambda x & (0\leq x\leq 1)\\ (1/\lambda)\sin\lambda(2-x) & (1\leq x\leq 2).\end{cases}$$

Thus it easily follows from the definition of the trigonometric B-spline that it is real valued and has the properties (i) — (vii) except (iv) of the hyperbolic B-spline, where sinh in (v) and $-$in (vi) are to be replaced by sin and $+$, respctively. For (iv),

(1.10) $$\psi_{ik\lambda}(x)>0 \quad \text{on } (0, 2) \quad \text{for } 0<\lambda<\pi/k$$

i. e.,

(iv)′ $\tilde{Q}_{m,\lambda}(x)>0$ on $(0, m)$ for $0<\lambda<2\pi/(m-1)$

$(m=2, 3, \cdots)$

(for $m=1$, $\tilde{Q}_{1,\lambda}=\chi$, and so the above inequality is trivial).

For the trigonometric B-spline $\tilde{Q}_{m,\lambda}$, we have the follwing recursion formulas with simple coefficients (c. f. [1]):

Theorem 2.

(1.11)

(i) $$\tilde{Q}_{m+1,\lambda}(x)=(2/m\lambda)\,[\tilde{Q}_{m,\lambda}(x)\sin(\frac{1}{2}\lambda x)+\tilde{Q}_{m,\lambda}(x-1)\sin\{\frac{1}{2}\lambda(m+1-x)\}]$$

(ii) $$\tilde{Q}'_{m+1,\lambda}(x)=\tilde{Q}_{m,\lambda}(x)\cos(\frac{1}{2}\lambda x)-\tilde{Q}_{m,\lambda}(x-1)\cos\{\frac{1}{2}\lambda(m+1-x)\}.$$

2. Proofs of Properties (i) — (viii) and Theorems 1, 2

Since the properties (i) — (viii) of the hyperbolic and trigonometric B-splines are easily obtained by the definition of them, here we shall only prove the proverty (v). First, we notice that

$$(2.1) \qquad \sum_{i=-\infty}^{\infty} \chi(x-i)=1.$$

By making a covolution of this equation and $\psi_{k\lambda}(x)\ (=(\phi_{k\lambda} * \phi_{-k\lambda})(x))$, $k=1, 2, \cdots, m-1$, we have the desired relation:

$$(2.2) \qquad \sum_{i=-\infty}^{\infty} Q_{2m-1,\lambda}(x-i)=(1*\psi_{\lambda}*\psi_{2\lambda}*\cdots*\psi_{(m-)\lambda})(x)$$

$$=\prod_{k=1}^{m-1}\int_0^2 \psi_{k\lambda}(x)\,dx=\prod_{k=1}^{m-1}\{\int_0^1 \phi_{k\lambda}(x)\,dx\}\{\int_0^1 \phi_{-k\lambda}(x)\,dx\}$$

$$=\prod_{k=1}^{m-1}\{\sinh(\frac{1}{2}k\lambda)/(\frac{1}{2}k\lambda)\}^2.$$

For the trigonometric B-spline $\tilde{Q}_{m,\lambda}$, similarly we have the property (v) from which follows

$$(2.3) \qquad 1\in \text{Span } \{\tilde{Q}_{2m-1,\lambda}(x-i)\}_{i=-\infty}^{\infty}$$

for $\lambda \neq 2(p/k)\pi$ with $k=1, 2, \cdots, m-1$ and $p=1, 2, \cdots$.

Now the following lemmas are required to prove Theorem 1.

Lemma 1.

$$(2.4) \qquad \prod_{K=1}^{m-1}\{(u-\frac{1}{2}i\lambda)^2+(k-\frac{1}{2})^2\lambda^2\}$$

$$=\{u/(u+im\lambda)\}\prod_{k=1}^{m}(u^2+k^2\lambda^2).$$

Proof. We only have to notice the identity:

$$(2.5) \qquad (u-\frac{1}{2}i\lambda)^2+(k-\frac{1}{2})^2u^2=(u-ik\lambda)\{u+i(k-1)\lambda\}.$$

On denoting the left hand side of the above relation in Lemma 1 by $p_m(\lambda)$, we have

$$(2.6) \qquad 1/p_m(\lambda)-1/p_m(-\lambda)=-(2m\lambda/iu)/\prod_{k=1}^{m}(u^2+k^2\lambda^2)$$

$$1/p_m(\lambda)+1/p_m(-\lambda)=2/\prod_{k=1}^{m}(u^2+k^2\lambda^2).$$

Lemma 2.

$$(2.7)\qquad (iu+\frac{1}{2}\lambda)\prod_{k=1}^{m-1}\{(u-\frac{1}{2}i\lambda)^2+k^2\lambda^2\}$$
$$=\prod_{k=1}^{m}\{k^2+(k-\frac{1}{2})^2\lambda^2\}/\{(m-\frac{1}{2})\lambda-iu\}.$$

Proof. We have to notice the identity:

$$(2.8)\qquad (u-\frac{1}{2}i\lambda)^2+k^2\lambda^2=\{u+i(k-\frac{1}{2})\lambda\}\{u-i(k+\frac{1}{2})\lambda\}.$$

Denoting the left hand side of the above equation in equation in Lemma 2 by $r_m(\lambda)$, we have

$$(2.9)\qquad 1/r_m(\lambda)-1/r_m(-\lambda)=2(m-\frac{1}{2})\lambda/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}$$
$$1/r_m(\lambda)+1/r_m(-\lambda)=-2iu/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}.$$

For any function $f(x)$ defined on $(-\infty,\infty)$, let us denote its Fourier transformation by $\hat{f}$ (if it exists), i. e.,

$$(2.10)\qquad \hat{f}(u)=\int_{-\infty}^{\infty}e^{-iux}f(x)\,dx.$$

Then, by a simple calculation we have the following two lemmas.

Lemma 3. Let $g(x)=f(x)\sinh(\frac{1}{2}\lambda x)+f(x-1)\sinh\{\frac{1}{2}\lambda(p-x)\}$. Then

$$(2.11)\qquad \hat{g}(u)=\frac{1}{2}\{e^{\frac{1}{2}\lambda(p-1)-iu}-1\}\hat{f}(u-\frac{1}{2}i\lambda)$$
$$-\frac{1}{2}\{e^{-\frac{1}{2}\lambda(p-1)-iu}-1\}\hat{f}(u+\frac{1}{2}i\lambda).$$

Lemma 4. Let $g(x)=f(x)\cosh(\frac{1}{2}\lambda x)-f(x-1)\cosh\{\frac{1}{2}\lambda(p-x)\}$.

Then

$$(2.12)\qquad \hat{g}(u)=\frac{1}{2}\{e^{\frac{1}{2}\lambda(p-1)-iu}-1\}\hat{f}(u-\frac{1}{2}i\lambda)$$
$$-\frac{1}{2}\{e^{-\frac{1}{2}\lambda(p-1)iu}-1\}\hat{f}(u+\frac{1}{2}i\lambda).$$

Now we are ready to prove Theorem 1. By an elementary calculation, (2.13) $$\hat{\chi}(u) = -(1/iu)(e^{-iu}-1)$$

$$\hat{\psi}_{\lambda}(u) = -\{1/(u^2+\lambda^2)\}(e^{\lambda-iu}-1)(e^{-\lambda-iu}-1).$$

Hence we have

(2.14) (i) $$\hat{Q}_{2m+1,\lambda}(u) = \{(-1)^{m+1}/iu\}\,\theta_m/\prod_{k=1}^{m}(u^2+k^2\lambda^2)$$

(ii) $$\hat{Q}_{2m,\lambda}(u) = (-1)^m\bar{\theta}_m/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}$$

where

(2.15) $$\theta_m = \prod_{k=-m}^{m}(e^{k\lambda-iu}-1)$$

$$\bar{\theta}_m = \prod_{k=1}^{m}(e^{(k-\frac{1}{2})\lambda-iu}-1)(e^{-(k-\frac{1}{2})\lambda-iu}-1).$$

A little additional computation yields

(2.16) $$(e^{m\lambda-iu}-1)\hat{Q}_{2m,\lambda}(u-\frac{1}{2}i\lambda) = (-1)^m\theta_m/p_m(\lambda)$$

$$(e^{-m\lambda-iu}-1)\hat{Q}_{2m,\lambda}(u+\frac{1}{2}i\lambda) = (-1)^m\theta_m/p_m(-\lambda).$$

By Lemma 3 and (2. 16), we have

(2.17) $$\hat{Q}_{2m,\lambda}(x)=\sinh(\frac{1}{2}\lambda x)+Q_{2m,\lambda}(x-1)\sinh\{\frac{1}{2}\lambda(2m+1-x)\}$$

$$=\frac{1}{2}(-1)^m\theta_m\{1/p_m(\lambda)-1/p_m(-\lambda)\}$$

$$=(-1)^{m+1}\{m\lambda\,\theta_m/iu\}/\prod_{k=1}^{m}(u^2+k^2\lambda^2)=m\lambda\,\hat{Q}_{2m+1,\lambda}(u).$$

This completes the proof of the recursion formula (i) in Theorem 1 for m odd.

By a simple calculation, from 2. 14 (i) we have

(2.18) $$\{e^{(m-\frac{1}{2})\lambda-iu}-1\}\hat{Q}_{2m-1,\lambda}(u-\frac{1}{2}i\lambda) = (-1)^m\bar{\theta}_m/r_m(\lambda).$$

$$\{e^{(m-\frac{1}{2})\lambda-iu}-1\}\hat{Q}_{2m-1,\lambda}(u+\frac{1}{2}i\lambda) = (-1)^m\bar{\theta}_m/r_m(-\lambda).$$

Hence by Lemma 3 and (2. 18) we obtain

(2.19) $$\hat{Q}_{2m-1,\lambda}(x)\sinh(\frac{1}{2}\lambda x)+\hat{Q}_{2m-1,\lambda}(x-1)\sinh\{\frac{1}{2}\lambda(2m-x)\}$$

$$=(-1)^m(m-\frac{1}{2})\lambda\bar{\theta}_m/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}$$
$$=(m-\frac{1}{2})\lambda\hat{Q}_{2m,\lambda}(u).$$

This completes the proof of (i) Theorem 1 for m even.

Next we shall prove the differentiation formula (ii) in Theorem 1. Since

$$(2.20)\qquad (D\hat{Q}_{2m+1,\lambda})(u)=iu\hat{Q}_{2m+1,\lambda}(u),$$

we get

$$(2.21)\qquad (D\hat{Q}_{2m+1,\lambda})(u)=(-1)^{m+1}\theta_m/\prod_{k=1}^{m}(u^2+k^2\lambda^2).$$

On the other hand, by Lemma 4 we have

$$(2.22)\qquad \hat{Q}_{2m,\lambda}(x)\cosh(\frac{1}{2}\lambda x)-\hat{Q}_{2m,\lambda}(x-1)\cosh\{\frac{1}{2}\lambda(2m+1-m)\}$$
$$=\frac{1}{2}(-1)^{m+1}\theta_m\{1/p_m(\lambda)+1/p_m(-\lambda)\}$$
$$=(-1)^{m+1}\theta_m/\prod_{k=1}^{m}(u^2+k^2\lambda^2)=(D\hat{Q}_{2m+1,\lambda})(u).$$

This completes the proof of (ii) in Theorem 1 for m odd.

Similarly we have

$$(2.23)\qquad (D\hat{Q}_{2m,\lambda})(u)=(-1)^m iu\bar{\theta}_m/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}.$$

On the other hand, by Lemma 4 we have

$$(2.24)\qquad \hat{Q}_{2m-1,\lambda}(x)\cosh(\frac{1}{2}\lambda x)-\hat{Q}_{2m1,\lambda}(x-1)\cosh\{\frac{1}{2}\lambda(2m-m)\}$$
$$=\frac{1}{2}(-1)^{m+1}\bar{\theta}_m\{1/r_m(\lambda)+1/r_m(-\lambda)\}$$
$$=(-1)^m iu\bar{\theta}_m/\prod_{k=1}^{m}\{u^2+(k-\frac{1}{2})^2\lambda^2\}.$$

This completes the proof of (ii) in Theorem 1 for m even.

Now we shall prove Theorem 2. The following two lemmas are required:

Lemma 5.

(2.25)
$$\hat{f}(x)\sin\left(\frac{1}{2}\lambda x\right)+\hat{f}(x-1)\sin\left\{\frac{1}{2}\lambda(p-x)\right\}$$
$$=\frac{1}{2}i\left\{e^{-\frac{1}{2}i\lambda(p-1)-iu}\right\}\hat{f}\left(u-\frac{1}{2}\lambda\right)$$
$$-\frac{1}{2}i\left\{e^{\frac{1}{2}i\lambda(p-1)-iu}-1\right\}\hat{f}\left(u+\frac{1}{2}\lambda\right).$$

Lemma 6.

(2.26)
$$\hat{f}(x)\cos\left(\frac{1}{2}\lambda x\right)-\hat{f}(x-1)\cos\left\{\frac{1}{2}\lambda(p-x)\right\}$$
$$=-\frac{1}{2}\left\{e^{\frac{1}{2}i\lambda(p-1)-iu}-1\right\}\hat{f}\left(u+\frac{1}{2}\lambda\right)$$
$$-\frac{1}{2}\left\{e^{-\frac{1}{2}i\lambda(p-1)-iu}-1\right\}\hat{f}\left(u-\frac{1}{2}\lambda\right).$$

By making use of Lemmas 1, 2, 5 and 6, similarly as in the proof of Theorem 1 we may have Theorem 2.

References

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Appendix

In the definition of the hyperbolic B-spline $Q_{m,\lambda}$, we may use an exponential (distribution) function ϕ_λ:

$$\phi_\lambda(x)=\begin{cases}\lambda e^{\lambda x}/(e^\lambda-1) & (0\leq x<1)\\ 0 & (\text{otherwise}).\end{cases}$$

Then, since $\int_{-\infty}^{\infty}\phi_{k\lambda}(x)\,dx=1$, $k=\pm1, \pm2, \cdots$, the right hand sides of the equations in (v) and (vi) are simply equal to 1. However, in this case the coefficients involved in the recursion formulae in main Theorem 1 are more complicated than before.