Limit Distributions of Random Triangles in Hyperbolic Planes

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Abstract

For random triangles in hyperbolic planes, we study limit distributions of lengths of three sides of them. First we establish some relation between limit probability distributions of random triangles in hyperbolic planes and certain expectations concerning random triangles in Euclidean planes. Using this relation, we give an explicit expression for the limit distributions by an elliptic integral.

RANDOM TRIANGLE; HYPERBOLIC PLANE; LIMIT DISTRIBUTION; ELLIPTIC INTEGRAL

1. Introduction

The first problem concerning random triangles in Euclidean planes is perhaps the problem "what is the probability that a random triangle is acute", which was proposed and solved by Woolhouse (1886). Since that time various studies have been made on these subjects. As for recent references, see Mannion (1990), Arca (1994), Eisenberg and Sullivan (1996), Baryshnikov (1996) and so on. On the other hand, it seems to me that there has been no research on random triangles in hyperbolic planes.

In hyperbolic planes, triangles happen to enjoy some "extraordinary" properties which those in Euclidean planes do not (see Fenchel (1989)). For example, it happens that they do not have circumcenters, orthocenters and excenters. Hence a series of "natural" problems arise, one example of which is "what is the probability that a random triangle has their circumcenter".

In this paper we study random triangles in hyperbolic planes and we calculate limit probability distributions of lengths of three sides of them. In section 2, we show some connection between limit probability distributions of random triangles in hyperbolic planes and certain expectations concerning those in Euclidean planes. In section 3, we give an explicit expression for the limit probability distributions by an elliptic integral.

2. Connection between random triangles in hyperbolic planes and those in Euclidean planes

Consider a random triangle ABC in a hyperbolic plane. That is, on a disk with its center at the origin O and with a radius R, we consider a triangle ABC whose three vertices A, B, and C are mutually independent and uniformly distributed on the disk. To state more precisely, we denote the angles which the line segments OA, OB and OC make with the x-axis by θ, ϕ and ψ respectively, and moreover, denote the lengths of the line segments OA, OB and OC by ξ, η and ζ respectively. Then we assume that six random variables $\theta, \phi, \psi, \xi, \eta, \zeta$ are mutually independent, and θ, ϕ, ψ have the uniform distribution on an interval $(0,2\pi)$, and ξ, η, ζ have a common probability distribution whose density is given by $\sinh\xi \; d\xi/(\cosh R\text{-}1)$.

We study the simultaneous distribution of a=BC, b= CA and c=AB. Obviously, without loss of generality, we may assume that the vertex A lies on the x-axis. Then, by the hyperbolic trigonometry, we have

$$(1)\qquad \begin{cases} \cosh a &= \cosh\eta\cosh\zeta - \sinh\eta\sinh\zeta\cos(\psi-\phi) \\ \cosh b &= \cosh\zeta\cosh\xi - \sinh\zeta\sinh\xi\cosh\psi \\ \cosh c &= \cosh\xi\cosh\eta - \sinh\xi\sinh\eta\cos\phi. \end{cases}$$

Now it is convenient to introduce the following notations:

$$x = \cosh a, y = \cosh b, z = \cosh c$$
$$u = \cosh\xi, \upsilon = \cosh\eta, w = \cosh\zeta,$$
$$\lambda = \cos(\psi - \phi), \mu = \cos\psi, \nu = \cos\phi.$$

Then (1) can be written concisely as

$$(2)\qquad \begin{cases} x &= \upsilon w - \lambda\sqrt{\upsilon^2-1}\sqrt{w^2-1} \\ y &= wu - \mu\sqrt{w^2-1}\sqrt{u^2-1} \\ z &= u\upsilon - \nu\sqrt{u^2-1}\sqrt{\upsilon^2-1} \end{cases}$$

Moreover, it can be seen that u, υ, and w have simply the uniform distribution on an interval $(1, L+1)$, where $L = \cosh R\text{-}1$.

Now we consider the characteristic function for a suitably normalized $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$,

$$f_L(t_1, t_2, t_3) = \mathrm{E}\left[\exp\left(\frac{i}{L^2}(t_1 x + t_2 y + t_3 z)\right)\right]$$

Then we can show the following lemma.

Lemma 1. The limit characteristic function

$$f(t_1, t_2, t_3) = \lim_{L\to\infty} f_L(t_1, t_2, t_3)$$

exists and it can be expressed as

$$f(t_1, t_2, t_3) = \int_0^{2\pi}\int_0^{2\pi} d\phi\, d\psi \int_0^1\int_0^1\int_0^1 du\, dv\, dw$$

(3) $$\exp\left[it_1(1-\lambda)vw + it_2(1-\mu)wu + it_3(1-\nu)uv\right]$$

Proof By the definition of characteristic function, we have

$$f_L(t_1, t_2, t_3) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\phi\, d\psi \int_1^{L+1}\int_1^{L+1}\int_1^{L+1} \frac{du}{L}\frac{dv}{L}\frac{dw}{L}$$
$$\cdot\exp\left[\frac{it_1}{L^2}(vw - \lambda\sqrt{v^2-1}\sqrt{w^2-1}) + \frac{it_2}{L^2}(wu - \mu\sqrt{w^2-1}\sqrt{u^2-1})\right.$$
$$\left. + \frac{it_3}{L^2}(uv - \nu\sqrt{u^2-1}\sqrt{v^2-1})\right].$$

Obviously it can be written as

$$f_L(t_1, t_2, t_3) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\phi\, d\psi \int_{\frac{1}{L}}^{1+\frac{1}{L}}\int_{\frac{1}{L}}^{1+\frac{1}{L}}\int_{\frac{1}{L}}^{1+\frac{1}{L}} du\, dv\, dw$$
$$\cdot\exp\left[it_1\left(vw - \lambda\sqrt{v^2 - \frac{1}{L^2}}\sqrt{w^2 - \frac{1}{L^2}}\right) + it_2\left(wu - \mu\sqrt{w^2 - \frac{1}{L^2}}\sqrt{u^2 - \frac{1}{L^2}}\right)\right.$$
$$\left. + it_3\left(uv - \nu\sqrt{u^2 - \frac{1}{L^2}}\sqrt{v^2 - \frac{1}{L^2}}\right)\right].$$

Hence follows the lemma with the aid of the bounded convergence theorem.

Let $p(x, y, z)$ denote the probability density corresponding to the characteristic function $f(t_1, t_2, t_3)$. From (3), we can derive the following expression of $p(x, y, z)$.

Lemma 2.

(4) $p(x, y, z)$

$$= \frac{1}{8\pi^2}\frac{1}{\sqrt{xyz}}\int_0^{2\pi}\int_0^{2\pi} d\phi\, d\psi \frac{1}{\sqrt{(1-\lambda)(1-\mu)(1-\nu)}}$$
$$\cdot I\left[\frac{yz}{x} < \frac{(1-\mu)(1-\nu)}{1-\lambda}, \frac{zx}{y} < \frac{(1-\nu)(1-\lambda)}{1-\mu}, \frac{xy}{z} < \frac{(1-\lambda)(1-\mu)}{1-\nu}\right]$$

Proof Consider a characteristic function corresponding to the probability density(4):

$$\begin{aligned}\tilde{f}(t_1, t_2, t_3) &= \int_0^\infty\int_0^\infty\int_0^\infty p(x,y,z)e^{i(t_1x+t_2y+t_3z)}\,dx\,dy\,dz \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d\phi\,d\psi}{\sqrt{(1-\lambda)(1-\mu)(1-\nu)}} \\ &\quad \int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{\sqrt{xyz}}\cdot e^{i(t_1x+t_2y+t_3z)} \\ &\quad \cdot I\left[\frac{yz}{x}<\frac{(1-\mu)(1-\nu)}{1-\lambda},\ \frac{zx}{y}<\frac{(1-\nu)(1-\lambda)}{1-\mu},\ \frac{xy}{z}<\frac{(1-\lambda)(1-\mu)}{1-\nu}\right]\end{aligned}$$

From the condition

$$\frac{yz}{x}<\frac{(1-\mu)(1-\nu)}{1-\lambda},\quad \frac{zx}{y}<\frac{(1-\nu)(1-\lambda)}{1-\mu},\quad \frac{xy}{z}<\frac{(1-\lambda)(1-\mu)}{1-\nu},$$

we can derive $x<1-\lambda, y<1-\mu, z<1-\nu$. In particular we have $x<2, y<2, z<2$.
Now, changing variables by $\xi=\frac{x}{1-\lambda}$ and $\eta=\frac{y}{1-\mu}$ and $\zeta=\frac{z}{1-\nu}$, we have

$$\begin{aligned}\tilde{f}(t_1, t_2, t_3) &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}d\phi\,d\psi\int_0^1\int_0^1\int_0^1\frac{d\xi\,d\eta\,d\zeta}{\sqrt{\xi\eta\zeta}} \\ &\quad \cdot\exp[it_1(1-\lambda)\xi+it_2(1-\mu)\eta+it_3(1-\nu)\zeta]\cdot I\,[\eta\zeta<\xi,\ \zeta\xi<\eta,\ \xi\eta<\zeta]\end{aligned}$$

Furthermore we change variables by $\xi=v\,w$, $\eta=wu$, $\zeta=uv$. Then it can be easily seen that the Jacobian $\frac{\partial(\xi,\eta,\zeta)}{\partial(u,v,w)}$ equals $\sqrt{\xi\eta\zeta}$ and the condition $\eta\zeta<\xi,\ \zeta\xi<\eta,\ \xi\eta<\zeta$ is quivalent to the condition $u<1,\ v<1,\ w<1$.

Consequently we obtain

$$\begin{aligned}\tilde{f}(t_1, t_2, t_3) &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}d\phi\,d\psi\int_0^1\int_0^1\int_0^1 du\,dv\,dw \\ &\quad \cdot\exp[it_1(1-\lambda)v\,w+it_2(1-\mu)wu+it_3(1-\nu)uv]\end{aligned}$$

Thus $\tilde{f}$ is identical to f and the proof is completed.

Now we go into an Euclidean plane and consider a random triangle ABC whose three vertices A, B, and C are mutually independent and are uniformly distributed on the unit circle with center at the origin O. Putting X = BC, Y = CA and Z = AB , we investicgate an expectation

$$E\left[\frac{1}{XYZ}I\,(YZ>aX,\ ZX>bY,\ XY>cZ)\right],$$

where a, b, c are constants. We denote this expectation by $T(a, b, c)$. Without loss of generality we may assume that vertex A lies at the x-axis. We denote the angles which the line segments OB and OC make with the x-axis by ϕ and ψ respectively. Then we can see easily

$$X = \sqrt{2(1-\lambda)},\ Y = \sqrt{2(1-\mu)},\ Z = \sqrt{2(1-\nu)},$$

where λ, μ, ν denote $\cos(\psi - \phi)$, $\cos\psi$, $\cos\phi$ respectively. Accordingly we have

$$T(a,b,c) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d\phi\, d\psi}{2\sqrt{2}\sqrt{(1-\lambda)(1-\mu)(1-\nu)}}$$

$$\cdot I\left[\frac{(1-\mu)(1-\nu)}{1-\lambda} > \frac{a^2}{2},\ \frac{(1-\nu)(1-\lambda)}{1-\mu} > \frac{b^2}{2},\ \frac{(1-\lambda)(1-\mu)}{1-\nu} > \frac{c^2}{2}\right]$$

Therefore, using Lemma 2, we obtain the following result.

Theorem 1

$$p(x,y,z) = \sqrt{\frac{2}{xyz}}\, T\left(\sqrt{\frac{2yz}{x}},\ \sqrt{\frac{2zx}{y}},\ \sqrt{\frac{2xy}{z}}\right).$$

3. An expression for limit probability distributions using an elliptic integral

3.1

Our task of this section is to find an explicit expression for $p(x, y, z)$. By Theorem 1, for this purpose, it suffices to find an explicit expression for $T(a, b, c)$. We consider a random triangle ABC in an euclidean plane whose three vertices A, B, and C are mutually independent and are uniformly distributed on the unit circle with center at the origin O, and put X = BC, Y = CA and Z = AB. First we calculate a conditinal expectation when $Z = z$ is given,

$$E\left[\frac{1}{XY} I\left(\frac{a}{z} < \frac{Y}{X} < \frac{z}{b},\ XY > cz\right)\right].$$

Denoting $\angle$AOB = 2ω, we may assume without loss of generality that both the angles which the line segments OA, OB make with the x-axis are equal to ω. Furthermore, letting θ denote for the angle which the line segment OC makes with the x-axis, we define a function

$$x(\theta) = \frac{1-\cos(\theta-\omega)}{|\cos\theta - \cos\omega|}.$$

Then this conditinal expectation is given by an integral

$$\int \frac{d\theta}{2\pi}\cdot\frac{1}{2|\cos\theta - \cos\omega|},$$

where the integration is taken over a domain

$$\left\{\theta \in (0, 2\pi) \colon \frac{a}{2\sin\omega} < x(\theta) < \frac{2\sin\omega}{b},\ |\cos\theta - \cos\omega| > c\sin\omega\right\}.$$

Dividing the domain of integration into two parts,

$$D(\omega) = \left\{ \theta \in (\omega, 2\pi-\omega): \frac{a}{2\sin\omega} < x(\theta) < \frac{2\sin\omega}{b}, \cos\omega - \cos\theta > c\sin\omega \right\}$$

$$\tilde{D}(\omega) = \left\{ \theta \in (-\omega, \omega): \frac{a}{2\sin\omega} < x(\theta) < \frac{2\sin\omega}{b}, \cos\theta - \cos\omega > c\sin\omega \right\},$$

we put

$$g(\omega) = \frac{1}{4\pi}\int_{D(\omega)} \frac{d\theta}{\cos\omega - \cos\theta}$$

$$\tilde{g}(\omega) = \frac{1}{4\pi}\int_{\tilde{D}(\omega)} \frac{d\theta}{\cos\omega - \cos\theta}.$$

Then we can write $\boldsymbol{T(a, b, c)}$ as

(5) $$T(a,b,c) = T_0 + \tilde{T},$$

where

(6) $$T_0 = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} d\omega \cdot \frac{1}{2\sin\omega} \cdot g(\omega)$$

(7) $$\tilde{T} = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} d\omega \cdot \frac{1}{2\sin\omega} \cdot \tilde{g}(\omega)$$

Thorougout the remainder in the paper, without loss of generality, we suppose $\boldsymbol{a < b < c}$.

3.2

First we compute $\boldsymbol{g(\omega)}$. Since $\chi(\theta)$ is an increasing function, we can define $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$ by

$$x(\alpha) = \frac{a}{2\sin\omega}, \ \alpha \in (\omega, 2\pi - \omega)$$

and

$$x(\beta) = \frac{2\sin\omega}{b}, \ \beta \in (\omega, 2\pi - \omega)$$

respectively. Furthermore, when $1 + \boldsymbol{\cos\omega} > \boldsymbol{c\sin\omega}$, we define $\gamma = \gamma(\omega)$ by

$$\cos\omega - \cos\gamma = c\sin\omega, \gamma \in (\omega, \pi)$$

Then, noting that $\chi(\theta)$ is increasing, we see

$$D(\omega) = (\alpha(\omega), \beta(\omega)) \cap (\gamma(\omega), 2\pi - \gamma(\omega)).$$

Now we consider the following conditions :

$$C_1 = \alpha(\omega) < \gamma(\omega) < \beta(\omega) < 2\pi - \gamma(\omega)$$

$$C_2 = \alpha(\omega) < \gamma(\omega) < 2\pi - \gamma(\omega) < \beta(\omega)$$

$$C_3 = \gamma(\omega) < \alpha(\omega) < \beta(\omega) < 2\pi - \gamma(\omega)$$

$$C_4 = \gamma(\omega) < \alpha(\omega) < 2\pi - \gamma(\omega) < \beta(\omega)$$

Then

(8)
$$D(\omega)=\begin{cases}(\gamma(\omega),\beta(\omega)) & \text{if the condition } C_1 \text{ holds,}\\ (\gamma(\omega),2\pi-\gamma(\omega)) & \text{if the condition } C_2 \text{ holds,}\\ (\alpha(\omega),\beta(\omega)) & \text{if the condition } C_3 \text{ holds,}\\ (\alpha(\omega),2\pi-\gamma(\omega)) & \text{if the condition } C_4 \text{ holds,}\\ \emptyset & \text{otherwise}\end{cases}$$

Next, introducing a function

$$l(x) = \log\left|\frac{\tan\frac{\omega}{2}+x}{\tan\frac{\omega}{2}-x}\right| \quad ,$$

we can easily show

(9)
$$\int_{\theta_1}^{\theta_2}\frac{d\theta}{\cos\theta-\cos\omega} = \frac{1}{\sin\omega}\left\{l\left(\tan\frac{\theta_2}{2}\right)-l\left(\tan\frac{\theta_1}{2}\right)\right\}.$$

Moreover, from the definitions of α, β and γ, it follows that

$$\tan\frac{\alpha}{2} = t\cdot\frac{4t+a\left(1+t^2\right)}{4t-a\left(1+t^2\right)},$$

$$\tan\frac{\beta}{2} = t\cdot\frac{b\left(1+t^2\right)+4t}{b\left(1+t^2\right)-4t},$$

$$\tan\frac{\gamma}{2} = \sqrt{\frac{t(t+c)}{1-ct}},$$

where t denotes $\frac{\omega}{2}$. Hence we can derive

(10)
$$l\left(\tan\frac{\alpha}{2}\right) = \log\left(\frac{4t}{a(1+t^2)}\right),$$

$$l\left(\tan\frac{\beta}{2}\right) = \log\left(\frac{b(1+t^2)}{4t}\right),$$

$$l\left(\tan\frac{\gamma}{2}\right) = \log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right).$$

Now let S_i denote for a set of ω's for which the condition C_i is satisfied (i =1,2,3,4). Then, combining (8), (9) and (10), we obtain the following result.

Lemma 3

$$g(\omega) = \frac{1}{4\pi\sin\omega}g_i(t) \; for \omega\in S_i \; (i=1,2,3,4),$$

where $t = \tan\frac{\omega}{2}$ and

(11)
$$\begin{aligned} g_1(t) &= l\left(\tan\frac{\gamma}{2}\right) - l\left(\tan\frac{\beta}{2}\right) \\ g_2(t) &= 2\cdot l\left(\tan\frac{\gamma}{2}\right) \\ g_3(t) &= l\left(\tan\frac{\alpha}{2}\right) - l\left(\tan\frac{\beta}{2}\right) \\ g_4(t) &= l\left(\tan\frac{\alpha}{2}\right) - l\left(\tan\frac{\gamma}{2}\right)\cdot \end{aligned}$$

Furthermore, from this lemma and (6), the next lemma follows.

Lemma 4.

$$T_0 = \frac{1}{8\pi^2}\sum_{i=1}^{4}\int_{S_i}\frac{t^2+1}{t^2}\,g_i(t)\,dt\,,$$

3.3

We analyse the conditions Ci (i =1,2,3,4) in detail. For this purpose it is convinient to introduce the following quantities

$$\omega_+(x) = \arcsin\sqrt{\frac{\frac{1}{4}cx^2}{x+c-\sqrt{cx(4-cx)}}},$$

$$\omega_-(x) = \arcsin\sqrt{\frac{\frac{1}{4}cx^2}{x+c+\sqrt{cx(4-cx)}}},$$

and

$$\omega_{ab} = \arcsin\left(\frac{\sqrt{ab}}{2}\right),\quad \omega_c = \arcsin\left(\frac{2c}{c^2+1}\right).$$

Furthermore, we put

$$\sigma = \frac{4c}{c^2+1},\quad \tau = \frac{ac}{a+c-\sqrt{ac(4-ac)}}$$

and

$$\sigma_- = \frac{2\left(1-\sqrt{1-c^2}\right)}{c},\quad \sigma_+ = \frac{2\left(1+\sqrt{1-c^2}\right)}{c}$$

Then, by an elementary but tedious analysis, we can show the following lemma.

Lemma 5.

$$S_1 = \begin{cases} (\omega_+(a), \omega_+(b)) & \text{if } c > 1,\ a < b < \sigma, \\ (\omega_+(a), \omega_+(b)) & \text{if } c > 1,\ a < \sigma < b < \tau, \\ (\omega_-(b), \omega_+(b)) & \text{if } c > 1,\ a < \sigma < b, b > \tau, bc > a(b+c), \\ (\omega_+(a), \omega_+(b)) & \text{if } c < 1,\ a < b < \sigma_-, \\ \varnothing & \text{otherwise} \end{cases}$$

$$S_2 = \begin{cases} (\omega_+(b), \omega_c) & \text{if } c > 1,\ a < b < \sigma, \\ \left(\omega_+(b), \dfrac{\pi}{2}\right) & \text{if } c < 1,\ a < b < \sigma_-, \\ \varnothing & \text{otherwise} \end{cases}$$

$$S_3 = \begin{cases} (\omega_{ab}, \omega_+(a)) & \text{if } c > 1,\ a < b < \sigma, \\ (\omega_{ab}, \omega_+(a)) & \text{if } c > 1,\ a < \sigma < b < \tau, \\ (\omega_{ab}, \omega_+(a)) & \text{if } c < 1,\ a < b < \sigma_-, \\ \varnothing & \text{otherwise} \end{cases}$$

$$S_4 = \varnothing \quad \text{at all times.}$$

Combining Lemma 4 and Lemma 5, we have the following lemma, where we put

$$t_+(x) = \tan\frac{\omega_+(x)}{2},\ t_-(x) = \tan\frac{\omega_-(x)}{2},\ t_{ab} = \tan\frac{\omega_{ab}}{2} \text{ and } t_c = \tan\frac{\omega_c}{2}.$$

Lemma 7.

If $c > 1$ and $a < b < \sigma$, then

$$8\pi^2 T_0 = \int_{t_{ab}}^{t_+(a)} \frac{t^2+1}{t^2} g_3(t)\,dt + \int_{t_+(a)}^{t_+(b)} \frac{t^2+1}{t^2} g_1(t)\,dt + \int_{t_+(b)}^{t_c} \frac{t^2+1}{t^2} g_2(t)\,dt. \tag{12}$$

If $a < \sigma < b < \tau$, then

$$8\pi^2 T_0 = \int_{t_{ab}}^{t_+(a)} \frac{t^2+1}{t^2} g_3(t)\,dt + \int_{t_+(a)}^{t_+(b)} \frac{t^2+1}{t^2} g_1(t)\,dt. \tag{13}$$

If $a < \sigma < b$, $b > \tau$ and $bc > a(b+c)$, then

$$8\pi^2 T_0 = \int_{t_-(b)}^{t_+(b)} \frac{t^2+1}{t^2} g_1(t)\,dt. \tag{14}$$

If $c < 1$ and $a < b < \sigma_-$, then T_0 can be given by (12) again.

3.4

Now we calculate indefinite integrals

$$\int \frac{t^2+1}{t^2} l\left(\tan\frac{\alpha}{2}\right) dt, \int \frac{t^2+1}{t^2} l\left(\tan\frac{\beta}{2}\right) dt, \int \frac{t^2+1}{t^2} l\left(\tan\frac{\gamma}{2}\right) dt .$$

For this purpose we need the following two lemmas, the first of which can be easily established.

Lemma 7.

$$\int \frac{t^2+1}{t^2} \log\frac{1+t^2}{t} dt = \frac{t^2-1}{t}\left(\log\frac{1+t^2}{t}-1\right)+4\arctan\ t .$$

In order to state the next lemma, we introduce the following elliptic integral,

$$e(u)=e(u;k)$$

$$(15)\qquad = \int_0^u \frac{\{(1-u^2)^2-c^2\}\{c^2-2c^2(1-u^2)-(1-u^2)^2\}}{(1-u^2)\{c^2+(1-u^2)^2\}} \cdot \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} .$$

where k is a constant.

Lemma 8.

$$\int \frac{t^2+1}{t^2} \log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right) dt$$

$$=\frac{t^2-1}{t}\log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right) + \frac{2}{c\sqrt{c^2+1}} e(\sqrt{1-ct};k),$$

where $k = \dfrac{1}{\sqrt{c^2+1}}$

Proof. Integrating by parts, we have

$$\int \frac{t^2+1}{t^2} \log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right) dt$$

$$=\frac{t^2-1}{t}\log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right) - J,$$

where $$J=\iint \frac{t^2-1}{t}\cdot\frac{d}{dt}\left\{\log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right)\right\} dt .$$

Since

$$\frac{d}{dt}\left\{\log\left(\frac{\sqrt{t+c}+\sqrt{t-ct^2}}{\sqrt{t+c}-\sqrt{t-ct^2}}\right)\right\}=\frac{1-2ct-t^2}{(1+t^2)\sqrt{(t+c)(t-ct^2)}},$$

we have

$$J=\int\frac{(t^2-1)(1-2ct-t^2)}{t(t^2+1)}\cdot\frac{dt}{\sqrt{(t+c)(t-ct^2)}}.$$

By change of variables $t = x^2/c$, it can be written as

$$J=\frac{2}{c}\int\frac{(x^4-c^2)(c^2-2c^2x^2-x^4)}{t(t^2+1)}\cdot\frac{dx}{\sqrt{(x^2+c^2)(1-x^2)}}.$$

Furthermore, changing variables x by $x = \sqrt{1-u^2}$, we can complete the proof.

It is convenient to rewrite the results stated in Lemma 11 and Lemma 12 in a more concise form.

Lemma 9.

$$\int\frac{t^2+1}{t^2}l\left(\tan\frac{\alpha}{2}\right)dt = \frac{t^2-1}{t}l\left(\tan\frac{\alpha}{2}\right)+\frac{t^2-1}{t}-4\arctan t$$

$$\int\frac{t^2+1}{t^2}l\left(\tan\frac{\beta}{2}\right)dt = \frac{t^2-1}{t}l\left(\tan\frac{\beta}{2}\right)-\frac{t^2-1}{t}+4\arctan t$$

$$\int\frac{t^2+1}{t^2}l\left(\tan\frac{\gamma}{2}\right)dt = \frac{t^2-1}{t}l\left(\tan\frac{\gamma}{2}\right)+\frac{2}{c\sqrt{c^2+1}}e(\sqrt{1-ct};k),$$

where $k = \dfrac{2}{\sqrt{c^2+1}}$.

3.5

Before calculating T_o, we remark the following.

Lemma 10.

If $x < \sigma$, then

$$\alpha(\omega_+(x))=\gamma(\omega_+(x)),\ \beta(\omega_+(x))=2\pi-\gamma(\omega_+(x)).$$

If $x < \sigma$, then

$$\alpha(\omega_+(x))=2\pi-\gamma(\omega_+(x)),\ \beta(\omega_+(x))=\gamma(\omega_+(x)).$$

Now we can express T_o as follows.

Lemma 11.

If $c > 1$ and $a < b < \sigma$, then

$$8\pi^2 T_0 = -2\cot\omega_+(a) - 2\cot\omega_+(b) + 4\sqrt{\frac{4-ab}{ab}} - 2\omega_+(a) - 2\omega_+(b) + 4\omega_{ab}$$
$$- \frac{2}{c\sqrt{c^2+1}}\left\{ e\left(\sqrt{1-c\tan\omega_+(a)}\right) + e\left(\sqrt{1-c\tan\omega_+(b)}\right)\right\}.$$

If $c > 1$ and $a < \sigma < b < \tau$, then

$$8\pi^2 T_0 = -2\cot\omega_+(a) - 2\cot\omega_+(b) + 4\sqrt{\frac{4-ab}{ab}} - 2\omega_+(a) - 2\omega_+(b) + 4\omega_{ab}$$
$$+ \frac{2}{c\sqrt{c^2+1}}\{e(\sqrt{1-c\tan\omega_+(a)}) + e(\sqrt{1-c\tan\omega_+(b)})\}.$$

If $c > 1$ and $a < \sigma < b$, $b > \tau$ and $bc > a(b+c)$, then

$$8\pi^2 T_0 = -2\cot\omega_+(b) + 2\cot\omega_-(b) - 2\omega_+(b) + 2\omega_-(b)$$
$$+ \frac{2}{c\sqrt{c^2+1}}\{e(\sqrt{1-c\tan\omega_+(b)}) - e(\sqrt{1-c\tan\omega_-(b)})\}.$$

If $c < 1$ and $a < b < \sigma_-$, then

$$8\pi^2 T_0 = -2\cot\omega_+(a) - 2\cot\omega_+(b) + 4\sqrt{\frac{4-ab}{ab}} - 2\omega_+(a) - 2\omega_+(b) + 4\omega_{ab}$$
$$+ \frac{2}{c\sqrt{c^2+1}}\{-e(\sqrt{1-c\tan\omega_+(a)}) - e(\sqrt{1-c\tan\omega_+(b)}) + 2e(\sqrt{1-c})\}.$$

Proof. Since the proof are similar for all cases, we only prove for the case $c > 1$, $a < b < \sigma$. Using Lemma 3, Lemma 4 and Lemma 9, we have

$$8\pi^2 T_0 = \int_{t_{ab}}^{t_+(a)} \frac{t^2+1}{t^2}\left\{l\left(\tan\frac{\alpha}{2}\right) - l\left(\tan\frac{\beta}{2}\right)\right\} dt$$
$$+ \int_{t_+(a)}^{t_+(b)} \frac{t^2+1}{t^2}\left\{l\left(\tan\frac{\gamma}{2}\right) - l\left(\tan\frac{\beta}{2}\right)\right\} dt$$
$$+ \int_{t_+(b)}^{t_c} \frac{t^2+1}{t^2}\cdot l\left(\tan\frac{\gamma}{2}\right) dt$$
$$= \left[\frac{t^2-1}{t} l\left(\tan\frac{\alpha}{2}\right) + \frac{t^2-1}{t} - 4\arctan t\right]_{t_{ab}}^{t_+(a)}$$
$$- \left[\frac{t^2-1}{t} l\left(\tan\frac{\beta}{2}\right) + \frac{t^2-1}{t} - 4\arctan t\right]_{t_+(a)}^{t_+(b)}$$
$$+ \left[\frac{t^2-1}{t} l\left(\tan\frac{\gamma}{2}\right) + \frac{2}{c\sqrt{c^2+1}} e(\sqrt{1-ct})\right]_{t_+(a)}^{t_+(b)}$$
$$+ \left[2\cdot\frac{t^2-1}{t} l\left(\tan\frac{\gamma}{2}\right) + \frac{2}{c\sqrt{c^2+1}} e(\sqrt{1-ct})\right]_{t_+(b)}^{t_c}.$$

From Lemma 10 it follows that

$$l\left(\tan\frac{\alpha(\omega_+(a))}{2}\right)=l\left(\tan\frac{\gamma(\omega_+(a))}{2}\right)$$

and

$$l\left(\tan\frac{\beta(\omega_+(b))}{2}\right)=l\left(\tan\frac{2\pi-\gamma(\omega_+(b))}{2}\right)=l\left(\tan\frac{\gamma(\omega_+(b))}{2}\right).$$

Furthermore, we can readily see that

$$l\left(\tan\frac{\alpha(\omega_{ab})}{2}\right)=l\left(\tan\frac{\beta(\omega_{ab})}{2}\right)=\log\sqrt{\frac{b}{a}}$$

and since $t_c = 1/c$,

$$l\left(\tan\frac{\gamma(\omega_c)}{2}\right)=0.$$

Accordingly,

$$\begin{aligned}8\pi^2T_0 &= \frac{t_+(a)^2-1}{t_+(a)}+\frac{t_+(b)^2-1}{t_+(b)}-2\cdot\frac{t_{ab}^2-1}{t_{ab}}-2\omega_+(a)-2\omega_+(b)+4\omega_{ab}\\ &\quad-\frac{2}{c\sqrt{c^2+1}}\{e(\sqrt{1-ct_+(a)})+e(\sqrt{1-ct_+(b)})\}.\end{aligned}$$

Finally, noting that

$$\frac{t_+(x)^2-1}{t_+(x)} = -2\cot\omega_+(x),\ \text{and}\ \frac{t_{ab}^2-1}{t_{ab}}=-2\cot\omega_{ab}=-2\sqrt{\frac{4-ab}{ab}},$$

we can complete the proof.

3.6

Now we begin the calculation of $\tilde{T}$. Since $\chi(\theta)$ is a decreasing function, we can define $\tilde{\alpha}=\tilde{\alpha}(\omega)$ and $\tilde{\beta}=\tilde{\beta}(\omega)$ by

$$x(\tilde{\alpha}) = \frac{a}{2\sin\omega},\ \tilde{\alpha}\in(-\omega,\omega)$$

and

$$x(\tilde{\beta}) = \frac{2\sin\omega}{b},\ \tilde{\beta}\in(-\omega,\omega)$$

respectively. Furthermore, when $1-\cos\omega > c\sin\omega$, we define $\tilde{\gamma}=\tilde{\gamma}(\omega)$ by

$$\cos\tilde{\gamma}-\cos\omega = c\sin\omega,\ \tilde{\gamma}\in(0,\omega)$$

Then, noting that $\chi(\theta)$ is decreasing, we see

$$\tilde{D}(\omega) = \left(\tilde{\beta}(\omega),\tilde{\alpha}(\omega)\right)\cap\left(-\tilde{\gamma}(\omega),\tilde{\gamma}(\omega)\right).$$

Now we consider the following conditions :

$$\tilde{C}_1 \ : \ \tilde{\beta}(\omega) < -\tilde{\gamma}(\omega) < \tilde{\alpha}(\omega) < \tilde{\gamma}(\omega)$$
$$\tilde{C}_2 \ : \ \tilde{\beta}(\omega) < -\tilde{\gamma}(\omega) < \tilde{\gamma}(\omega) < \tilde{\alpha}(\omega)$$
$$\tilde{C}_3 \ : \ -\tilde{\gamma}(\omega) < \tilde{\beta}(\omega) < \tilde{\alpha}(\omega) < \tilde{\gamma}(\omega)$$
$$\tilde{C}_4 \ : \ -\tilde{\gamma}(\omega) < \tilde{\beta}(\omega) < \tilde{\gamma}(\omega) < \tilde{\alpha}(\omega)$$

Then

$$\tilde{D}(\omega) = \begin{cases} (-\tilde{\gamma}(\omega), \tilde{\alpha}(\omega)) & \text{if the condition } \tilde{C}_1 \text{ holds,} \\ (-\tilde{\gamma}(\omega), \tilde{\gamma}(\omega)) & \text{if the condition } \tilde{C}_2 \text{ holds,} \\ (\tilde{\beta}(\omega), \tilde{\alpha}(\omega)) & \text{if the condition } \tilde{C}_3 \text{ holds,} \\ (\tilde{\beta}(\omega), \tilde{\gamma}(\omega)) & \text{if the condition } \tilde{C}_4 \text{ holds,} \\ \varnothing & \text{otherwise} \end{cases}$$

On the other hand, from the definitions of $\tilde{\alpha}$ $\tilde{\beta}$ and $\tilde{\gamma}$, it follows that

$$\tan\frac{\tilde{\alpha}}{2} = t\cdot\frac{4t - a\left(1+t^2\right)}{4t + a\left(1+t^2\right)},$$
$$\tan\frac{\tilde{\beta}}{2} = t\cdot\frac{b\left(1+t^2\right) - 4t}{b\left(1+t^2\right) + 4t},$$
$$\tan\frac{\tilde{\gamma}}{2} = \sqrt{\frac{t(t-c)}{1+ct}},$$

where t denotes $\tan\dfrac{\omega}{2}$. Hence we can derive

$$l\left(\tan\frac{\tilde{\alpha}}{2}\right) = \log\left(\frac{4t}{a(1+t^2)}\right),$$
$$l\left(\tan\frac{\tilde{\beta}}{2}\right) = \log\left(\frac{b(1+t^2)}{4t}\right),$$
$$l\left(\tan\frac{\tilde{\gamma}}{2}\right) = \log\left(\frac{\sqrt{t+ct^2}+\sqrt{t-c}}{\sqrt{t+ct^2}-\sqrt{t-c}}\right).$$

Now let $\tilde{S}_i$ denote for a set of ω 's for which the condition $\tilde{C}_i$ is satisfied (i = 1, 2, 3, 4). Then, corresponding to Lemma 5, we can show the following result.

Lemma 12.

$$\tilde{S}_2 = \begin{cases} \left(\omega_c, \dfrac{\pi}{2}\right) & \text{if } a < b < \sigma_- \\ \varnothing & \text{otherwise} \end{cases}$$

On the other hand, $\tilde{S}_1, \tilde{S}_3$ and $\tilde{S}_4$ are θ at all times.

Accordingly, in contrast to to Lemma 3 and Lemma 4, we have the next simple result.

Lemma 13.

$$8\pi^2 \tilde{T} = \int_{t_c}^{1} \frac{t^2+1}{t^2}\, \tilde{g}_2(t)\, dt,$$

where

$$\tilde{g}_2(t) = 2 \cdot l\left(\tan\frac{\tilde{\gamma}}{2}\right).$$

For an integral which appears in the above lemma, we can establish the next formula.

Lemma 14.

$$\int \frac{t^2+1}{t^2}\, l\left(\tan\frac{\tilde{\gamma}}{2}\right) dt = \frac{t^2-1}{t}\, l\left(\tan\frac{\tilde{\gamma}}{2}\right) - \frac{2}{c\sqrt{c^2+1}}\, e\left(\sqrt{1-\frac{c}{t}}\,; k\right),$$

where $k = \dfrac{2}{c\sqrt{c^2+1}}$.

Proof. The proof of this lemma goes along almost the same line as that of Lemma 9. The only differrence appears in the last step of the proof of Lemma 9, where we change variables x by $x = \sqrt{1-u^2}$. However in the present proof we need change variables by $x = c/\sqrt{1-u^2}$.

Then, Lemma 13 and Lemma 14 together gives the following lemma.

Lemma 15.

$$8\pi^2 \tilde{T} = -\frac{4}{c\sqrt{c^2+1}}\, e\left(\sqrt{1-c}\right).$$

3.7

Now, combining Lemma 11 and Lemma 15, we can state our main result.

Theorem 2.

If $\boldsymbol{c} > 1$ and $\boldsymbol{a} < \boldsymbol{b} < \sigma$, then