

Note on the Enumeration of Embeddings of Real Projective Spaces

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Note on the Enumeration of Embeddings of Real Projective Spaces

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§ 1. Introduction

Recently, Y. Nomura [12] has studied the enumeration problem of liftings of a given map to a fibration and its application to the enumeration problem of immersions of certain manifolds. In this note, using his results we enumerate the non-zero cross sections of certain vector bundles, and then study the embedding problem of the real projective spaces in the euclidean spaces.

Let ξ be an orientable n -plane bundle over a CW -complex X of dimension less than $n+2$, and let $w_2(\xi)$ be the second Stiefel-Whitney class of ξ . Consider the homomorphisms

$$(1.1) \quad \begin{aligned} \Theta_{\xi}^i: H^{i-1}(X; Z) &\longrightarrow H^{i+1}(X; Z_2), \\ \Gamma_{\xi}^i: H^i(X; Z_2) &\longrightarrow H^{i+2}(X; Z_2), \end{aligned}$$

of the cohomology groups, defined by

$$\begin{aligned} \Theta_{\xi}^i(a) &= Sq^2 \rho_2 a + \rho_2 a \cdot w_2(\xi), \\ \Gamma_{\xi}^i(b) &= Sq^2 b + b \cdot w_2(\xi), \end{aligned}$$

where ρ_2 is the mod 2 reduction. Then we prove the following theorem in §§ 2-4, using Nomura's theorem [12, § 2] and the Postnikov factorization of the universal orientable $(n-1)$ -sphere bundle $BSO(n-1) \rightarrow BSO(n)$.

THEOREM A. *Let $n \geq 6$ and let ξ be an orientable n -plane bundle over a CW -complex X of dimension less than $n+2$ with a non-zero cross section. Then, the set cross (ξ) of (free) homotopy classes of non-zero cross sections of ξ is given by*

$$\text{cross}(\xi) = \begin{cases} \text{Ker } \Theta_{\xi}^n \times \text{Coker } \Theta_{\xi}^{n-1}, & \text{if } \Gamma_{\xi}^{n-1} \text{ is epimorphic,} \\ \text{Ker } \Theta_{\xi}^n \times \text{Coker } \Theta_{\xi}^{n-1} \times \text{Coker } \Gamma_{\xi}^{n-1}, & \text{if } \Theta_{\xi}^{n-1} \text{ is monomorphic,} \end{cases}$$

where $\Theta_{\xi}^i, \Gamma_{\xi}^i$ are the homomorphisms of (1.1).

This is a generalization of a part of the theorem of I. M. James [8, Th. 5.1]

for the case $\dim X \leq n$.

Applying the above theorem, we prove the following theorem in §§ 5-7, using the results of A. Haefliger [6].

THEOREM B. *Let n be an even integer and let $n \geq 10, n \neq 2^r$. Then, there exists only one isotopy class of embeddings of the real n -dimensional projective space RP^n in the real $(2n-2)$ -space R^{2n-2} .*

§ 2. Nomura's theorem

Let $h: A \rightarrow D$ be a principal fibration with fiber F , and let $p: E \rightarrow A$ and $q: T \rightarrow E$ be the principal fibrations with the classifying maps $\theta: A \rightarrow B$ and $\rho: E \rightarrow C$, respectively. For a given CW-complex X and a map $u: X \rightarrow D$, we assume that there are liftings v and w in the following commutative diagram:

$$(2.1) \quad \begin{array}{ccccc} & & T & & \\ & & \downarrow q & & \\ & & E & \xrightarrow{\rho} & C \\ & w \nearrow & \downarrow p & & \\ X & \xrightarrow{v} & A & \xrightarrow{\theta} & B \\ & \searrow u & \downarrow h & & \\ & & D & & \end{array}$$

and also we assume that w has a lifting to T .

In this section, we consider the set $[X, T; u]$ of homotopy classes of liftings $X \rightarrow T$ of u , under the following stability condition (i)-(iii) for the sequence $\{h, p, q\}$ of fibrations:

- (i) the spaces B and C are homotopy associative H -spaces,
- (ii) there exists a map $d: F \times D \rightarrow B$ such that

$$\theta m \simeq d(id_F \times h) + \theta \pi_2 \quad \text{and} \quad di_2 \simeq 0,$$

- (iii) there exists a map $c: \Omega B \times D \rightarrow C$ such that

$$\rho \mu \simeq c(id_{\Omega B} \times hp) + \rho \pi_2 \quad \text{and} \quad ci_2 \simeq 0,$$

where $m: F \times A \rightarrow A$ and $\mu: \Omega B \times E \rightarrow E$ are the actions of fibers in the principal fibrations $h: A \rightarrow D$ and $p: E \rightarrow A$, respectively, π_2 and i_2 denote the projection and the injection to the second factors, and $+$ denotes the multiplication of an H -space.

The maps d and c define the maps $d': \Omega F \times D \rightarrow \Omega B$ and $c': \Omega^2 B \times D \rightarrow \Omega C$ by $d'(\lambda, x)(t) = d(\lambda(t), x)$ and $c'(v, y)(t) = c(v(t), y)$. These maps induce the maps between homotopy sets;

$$(2.2) \quad \begin{aligned} \Theta_u: [X, F] &\longrightarrow [X, B], & \Theta'_u: [X, \Omega F] &\longrightarrow [X, \Omega B], \\ \Gamma_u: [X, \Omega B] &\longrightarrow [X, C], & \Gamma'_u: [X, \Omega^2 B] &\longrightarrow [X, \Omega C], \end{aligned}$$

by setting

$$\begin{aligned} \Theta_u(a) &= d_*(a, u), & \Theta'_u(a') &= d'_*(a', u), \\ \Gamma_u(b) &= c_*(b, u), & \Gamma'_u(b') &= c'_*(b', u), \end{aligned}$$

where $u \in [X, D]$ is a given map, and $d_*: [X, F] \times [X, D] \rightarrow [X, B]$ is the induced map of d and so on. Then it is easy to see that the maps of (2.2) are homomorphisms of groups, by the existence of a lifting of u and the above stability condition (i)–(iii). Further, we define

$$(2.3) \quad \varphi: \text{Ker } \Theta_u \longrightarrow \text{Coker } \Gamma_u$$

as follows: For a fixed lifting $v: X \rightarrow A$ of u , the correspondence $[X, F] \ni \sigma \rightarrow m_*(\sigma, v) \in [X, A; u]$ is, as is well-known, a bijection. We see easily that $\sigma \in \text{Ker } \Theta_u$ if and only if $m_*(\sigma, v)$ has a lifting to E . Let $w_\sigma: X \rightarrow E$ be a lifting of $m_*(\sigma, v)$ and define

$$\varphi(\sigma) = \rho_*(w_\sigma) \text{ mod Im } \Gamma_u.$$

It is easily shown that φ is well-defined.

The following theorem is proved by Y. Nomura [12, Cor. 2.5–6].

THEOREM. *Under the above assumptions and notations, we obtain, as a set,*

$$[X, T; u] = \begin{cases} \text{Ker } \varphi \times (\text{Ker } \Gamma_u / \text{Im } \Theta'_u) & \text{if } \Gamma'_u \text{ is an epimorphism,} \\ \text{Ker } \varphi \times (\text{Ker } \Gamma_u / \text{Im } \Theta'_u) \times \text{Coker } \Gamma'_u & \text{if } \Theta'_u \text{ is a monomorphism.} \end{cases}$$

§ 3. The Postnikov factorization of the universal orientable S^{n-1} -bundle

Let $n \geq 6$. The Postnikov factorization for the fourth stage of the universal orientable S^{n-1} -bundle $BSO(n-1) \xrightarrow{p} BSO(n)$, induced by the inclusion $SO(n-1) \subset SO(n)$, is given as follows:

$$(3.1) \quad \begin{array}{rcccl} & & E_3 & & \\ & & \downarrow p_3 & & \\ & a_3 \nearrow & E_2 & \xrightarrow{p} & K(Z_2, n+2) \\ & a_2 \nearrow & \downarrow p_2 & & \\ BSO(n-1) & a_1 \rightarrow & E_1 & \xrightarrow{\theta} & K(Z_2, n+1) \\ & p \searrow & \downarrow p_1 & & \\ & & BSO(n) & \xrightarrow{x_n} & K(Z, n) \end{array}$$

where $\chi_n \in H^n(BSO(n); Z)$ represents the Euler class, $p_1: E_1 \rightarrow BSO(n)$ is the principal fibration with the classifying map χ_n , and θ and ρ are the second and the third k -invariants, and $p_2: E_2 \rightarrow E_1$ and $p_3: E_3 \rightarrow E_2$ are the principal fibrations with the classifying maps θ and ρ , respectively. Furthermore $q_3: BSO(n-1) \rightarrow E_3$ is an $(n+2)$ -equivalence, i.e., $q_{3*}: \pi_i(BSO(n-1)) \rightarrow \pi_i(E_3)$ is isomorphic for $i < n+2$ and epimorphic for $i = n+2$.

Let $m_1: K(Z, n-1) \times E_1 \rightarrow E_1$ be the action of fiber in $p_1: E_1 \rightarrow BSO(n)$ and consider the map $v_1 = m_1(id \times q_1): K(Z, n-1) \times BSO(n-1) \rightarrow E_1$. Then, by the results of E. Thomas [14, p. 21], the second k -invariant $\theta \in H^{n+1}(E_1; Z_2)$ is characterized by the equality

$$(3.2) \quad v_1^* \theta = Sq^2 \rho_2 \epsilon_1 \times 1 + \rho_2 \epsilon_1 \times p^* w_2,$$

where $v_1^*: H^{n+1}(E_1; Z_2) \rightarrow H^{n+1}(K(Z, n-1) \times BSO(n-1); Z_2)$ and $\epsilon_1 \in H^{n-1}(K(Z, n-1); Z)$ is the fundamental class and w_2 is the second universal Stiefel-Whitney class.

Now, consider the homomorphism

$$m_1^* - \pi_2^*: H^r(E_1; Z_2) \longrightarrow H^r(K(Z, n-1) \times E_1; Z_2),$$

where π_2 is the projection to the second factor. Since $(id \times q_1)^* \pi_2^*(\theta) = 1 \times q_1^*(\theta) = 0$, we have $(id \times q_1)^*(m_1^* - \pi_2^*)(\theta) = (id \times q_1)^* m_1^*(\theta) = v_1^*(\theta)$. On the other hand, $(id \times q_1)^*: \sum_{i=0}^2 H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(E_1; Z_2) \rightarrow \sum_{i=0}^2 H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(BSO(n-1); Z_2)$ is monomorphic, because $q_1^*: H^r(E_1; Z_2) \rightarrow H^r(BSO(n-1); Z_2)$ is so for $r \leq 2$. Therefore, (3.2) shows that

$$(3.3) \quad (m_1^* - \pi_2^*)(\theta) = Sq^2 \rho_2 \epsilon_1 \times 1 + \rho_2 \epsilon_1 \times p^* w_2.$$

Similarly, let $m_2: K(Z_2, n) \times E_2 \rightarrow E_2$ be the action of fiber in $p_2: E_2 \rightarrow E_1$, and consider the map $v_2 = m_2(id \times q_2): K(Z_2, n) \times BSO(n-1) \rightarrow E_2$. Then the third k -invariant $\rho \in H^{n+2}(E_2; Z_2)$ is characterized by

$$v_2^* \rho = Sq^2 \epsilon_2 \times 1 + \epsilon_2 \times p^* w_2,$$

where $\epsilon_2 \in H^n(K(Z_2, n); Z_2)$ is the fundamental class (cf. [15, Th. 3.5]). Therefore we have

$$(3.4) \quad (m_2^* - \pi_2^*)(\rho) = Sq^2 \epsilon_2 \times 1 + \epsilon_2 \times p_2^* p_1^* w_2,$$

by the same argument as above.

§ 4. Proof of Theorem A

Continuing the previous section, we choose the maps

$$d: (K(Z, n-1) \times BSO(n), BSO(n)) \longrightarrow (K(Z_2, n+1), *),$$

$$c: (K(Z_2, n) \times BSO(n), BSO(n)) \longrightarrow (K(Z_2, n+2), *)$$

such that they represent the elements $d = Sq^2 \rho_{2^{\iota_1}} \times 1 + \rho_{2^{\iota_1}} \times w_2$ and $c = Sq^2 \iota_2 \times 1 + \iota_2 \times w_2$, respectively. Then from the equalities (3.3) and (3.4), it is easy to see that the sequence $\{p_1, p_2, p_3\}$ of principal fibrations in the diagram (3.1) satisfies the stability condition (i)–(iii) in § 2. Therefore, for a given map $\xi: X \rightarrow BSO(n)$ which has a lifting $X \rightarrow E_3$, we can define the homomorphisms

$$\Theta'_\xi: H^{i-1}(X; Z) \longrightarrow H^{i+1}(X; Z_2) \quad \text{for } i = n, n-1,$$

$$\Gamma'_\xi: H^i(X; Z_2) \longrightarrow H^{i+2}(X; Z_2) \quad \text{for } i = n, n-1,$$

corresponding to $\Theta_u, \Theta'_u, \Gamma_u$ and Γ'_u of (2.2) and these are the homomorphisms of (1.1) by definition.

We now prove Theorem A in § 1.

Let ξ be an orientable n -plane bundle over a CW -complex X of dimension less than $n+2$ and suppose that ξ has a non-zero cross section. Then the set $\text{cross}(\xi)$ of homotopy classes of non-zero cross sections of ξ is

$$\text{cross}(\xi) = [X, BSO(n-1); \xi]$$

by [9, Lemma 2.2], where $\xi: X \rightarrow BSO(n)$ denotes the classifying map of ξ . Since $\dim X < n+2$ and $q_3: BSO(n-1) \rightarrow E_3$ is an $(n+2)$ -equivalence, we obtain

$$[X, BSO(n-1); \xi] = [X, E_3; \xi]$$

by [9, Th. 3.2]. Now we can apply the theorem in § 2. Since $\dim X < n+2$, we have $H^{n+2}(X; Z_2) = 0$ and so $\text{Ker } \Gamma'_\xi = H^n(X; Z_2)$ and $\text{Ker}(\varphi: \text{Ker } \Theta'_\xi \rightarrow \text{Coker } \Gamma'_\xi) = \text{Ker } \Theta'_\xi$. This completes the proof.

EXAMPLE. Let ξ be a $(2n-1)$ -plane bundle over the real $2n$ -dimensional complex projective space CP^n with a non-zero cross section. Then the set $\text{cross}(\xi)$ is equal to Z , the set of integers. In fact, $\Theta'_\xi: H^{2n-3}(CP^n; Z) \rightarrow H^{2n-1}(CP^n; Z_2)$ is obviously monomorphic and $\text{Coker } \Theta'_\xi = 0$. Also $\text{Ker}(\Theta'_\xi: H^{2n-2}(CP^n; Z) \rightarrow H^{2n}(CP^n; Z_2))$ is equal to Z and $\text{Coker}(\Gamma'_\xi: H^{2n-2}(CP^n; Z_2) \rightarrow H^{2n}(CP^n; Z_2))$ is Z_2 or 0 .

§ 5. Enumeration of embeddings

Let M be an n -dimensional differentiable closed manifold, M^* be its reduced symmetric product obtained from $M \times M - \Delta$ (Δ is the diagonal of M) by identifying (x, y) with (y, x) and let η be the real line bundle over M^* associated with the

double covering $M \times M - \Delta \rightarrow M^*$. Then the set $[M \subset R^{2n-2}]$ of isotopy classes of embeddings of M into R^{2n-2} for $n \geq 8$ is equal to the set of homotopy classes of cross sections of the associated S^{2n-3} -bundle $(M \times M - \Delta) \times_{Z_2} S^{2n-3} \rightarrow M^*$ and so equal to $\text{cross}((2n-2)\eta)$, by the theorem of A. Haefliger [6, § 1].

Since M^* is an open $2n$ -manifold, there is a proper Morse function on M^* with no critical points of index $2n$ by [13, Lemma 1.1] and so M^* has the homotopy type of a CW -complex of dimension less than $2n$ by [11, Th.3.5]. Therefore we obtain the following proposition from Theorem A.

PROPOSITION. *Let $n \geq 8$ and let M be an n -dimensional differentiable closed manifold which is embedded in R^{2n-2} . Then the set $[M \subset R^{2n-2}]$ of isotopy classes of embeddings of M into R^{2n-2} is given by*

$$[M \subset R^{2n-2}] = \begin{cases} \text{Ker } \Theta^{2n-2} \times \text{Coker } \Theta^{2n-3}, & \text{if } \Gamma \text{ is epimorphic,} \\ \text{Ker } \Theta^{2n-2} \times \text{Coker } \Theta^{2n-3} \times \text{Coker } \Gamma, & \text{if } \Theta^{2n-3} \text{ is monomorphic,} \end{cases}$$

where the homomorphisms

$$\Theta^i: H^{i-1}(M^*; Z) \longrightarrow H^{i+1}(M^*; Z_2) \quad \text{for } i=2n-2, 2n-3,$$

$$\Gamma: H^{2n-3}(M^*; Z_2) \longrightarrow H^{2n-1}(M^*; Z_2),$$

are defined by

$$\Theta^i(a) = Sq^2 \rho_2 a + (n-1) \rho_2 a \cdot v^2,$$

$$\Gamma(b) = Sq^2 b + (n-1) b \cdot v^2,$$

and $v \in H^1(M^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $M \times M - \Delta \rightarrow M^*$.

COROLLARY. *In addition to the conditions of the above proposition, we assume that $H_1(M; Z_2) = 0$. Then we have*

$$[M \subset R^{2n-2}] = H^{2n-3}(M^*; Z) \times \text{Coker } \Theta^{2n-3}.$$

PROOF. Since $H_1(M; Z_2) = 0$, we have $H_1(M \times M, \Delta; Z_2) = 0$ by the exact sequence of the pair $(M \times M, \Delta)$ and so $H^{2n-1}(M \times M - \Delta; Z_2) = H_1(M \times M, \Delta; Z_2) = 0$ by the Poincaré duality. Therefore, the Thom-Gysin exact sequence of the double covering $M \times M - \Delta \rightarrow M^*$:

$$\dots \rightarrow H^{2n-1}(M \times M - \Delta; Z_2) \rightarrow H^{2n-1}(M^*; Z_2) \rightarrow H^{2n}(M^*; Z_2) \quad (=0)$$

shows that $H^{2n-1}(M^*; Z_2) = 0$ and we have the desired result by the above pro-

position.

§ 6. Remarks on the cohomology of $(RP^n)^*$

Let $G_{n+1,2}$ be the Grassmann manifold of 2-planes in R^{n+1} . By [2, Th. 11], the mod 2 cohomology of $G_{n+1,2}$ is given by

$$H^*(G_{n+1,2}; Z_2) = Z_2[x, y]/(a_n, a_{n+1}),$$

where $\deg x = 1, \deg y = 2$ and $a_r = \sum_i \binom{r-i}{i} x^{r-2i} y^i$ ($r = n, n+1$).

S. Feder [4], [5] and D. Handel [7] investigated the mod 2 cohomology of the reduced symmetric product $(RP^n)^*$ of the n -dimensional real projective space RP^n and they showed that

(6.1) $H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as basis of $H^*(G_{n+1,2}; Z_2)$ -module, where $v \in H^1((RP^n)^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \rightarrow (RP^n)^*$ and there are the relations

$$v^2 = vx, Sq^1 y = xy, \text{ and } x^{2^{r+1}-1} = 0 \text{ for } n = 2^r + s, 0 \leq s < 2^r.$$

We study $H^*((RP^n)^*; Z)$ for even n . According to [7, (3.4)], there exists a fibration

$$V_{n+1,2} \longrightarrow SZ_{n+1,2} \longrightarrow BG,$$

such that $V_{n+1,2}$ is the Stiefel manifold of 2-frames in R^{n+1} , $SZ_{n+1,2}$ is a $(2n-1)$ -dimensional closed manifold having the homotopy type of $(RP^n)^*$ and BG is the classifying space of a group G of order 8 (as a matter of fact, G is the dihedral group D_4). Let p be an odd prime. The E_2 -term of the mod p cohomology spectral sequence of the above fibration is given by

$$E_2^{s,t} = H^s(BG; \underline{H}^t(V_{n+1,2}; Z_p)),$$

which is the cohomology with local coefficients $\{H^t(V_{n+1,2}; Z_p)\}$. Since $H^*(V_{n+1,2}; Z_p) = H^*(S^{2n-1}; Z_p)$ for even n by [1, (10.5)], we have

$$E_2^{s,t} = \begin{cases} H^s(BG; \underline{H}^0(V_{n+1,2}; Z_p)) & \text{for } t=0 \\ H^s(BG; \underline{H}^{2n-1}(V_{n+1,2}; Z_p)) & \text{for } t=2n-1 \\ 0 & \text{for } t \neq 0, 2n-1. \end{cases}$$

Since the action of $\pi_1(BG)$ on $H^0(V_{n+1,2}; Z_p)$ is trivial and $H^i(BG; Z_p) = 0$ for $i > 0$ by [3, Chap. 12, Cor. 2.7], we have

$$E_2^{s,0} = H^s(BG; Z_p) = \begin{cases} Z_p & s=0 \\ 0 & s \neq 0. \end{cases}$$

These imply that $H^s((RP^n)^*; Z_p) = 0$ for $0 < s < 2n - 1$ and so

(6.2) *the orders of elements of $H^s((RP^n)^*; Z)$ for $0 < s < 2n - 1$ are powers of 2.*

Using the above facts, we determine the groups $H^{2n-3}((RP^n)^*; Z)$ and $\rho_2 H^{2n-4}((RP^n)^*; Z)$. Let $n = 2^r + s$, $0 < s < 2^r$ and s be even. By (6.1) and the Poincaré duality for the manifold $SZ_{n+1,2}$,

(6.3) *the mod 2 cohomology groups $H^t((RP^n)^*; Z_2)$ for $2n - 4 \leq t \leq 2n - 1$ are given as follows:*

t	$H^t((RP^n)^*; Z_2)$	basis
$2n - 1$	Z_2	$vx^{2^{r+1}-2}y^s$
$2n - 2$	$Z_2 + Z_2$	$vx^{2^{r+1}-3}y^s, x^{2^{r+1}-2}y^s$
$2n - 3$	$Z_2 + Z_2 + Z_2$	$vx^{2^{r+1}-4}y^s, x^{2^{r+1}-3}y^s, vx^{2^{r+1}-2}y^{s-1}$
$2n - 4$	$Z_2 + Z_2 + Z_2 + Z_2$	$vx^{2^{r+1}-5}y^s, x^{2^{r+1}-4}y^s, vx^{2^{r+1}-3}y^{s-1}, x^{2^{r+1}-2}y^{s-1}$

Consider the exact sequence associated with $0 \rightarrow Z \xrightarrow{\times 2} Z \xrightarrow{\beta_2} Z_2 \rightarrow 0$:

$$\begin{aligned} \dots \rightarrow H^{2n-4}((RP^n)^*; Z) \xrightarrow{\rho_2} H^{2n-4}((RP^n)^*; Z_2) \xrightarrow{\beta_2} H^{2n-3}((RP^n)^*; Z) \xrightarrow{\times 2} \\ H^{2n-3}((RP^n)^*; Z) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*; Z_2) \xrightarrow{\beta_2} H^{2n-2}((RP^n)^*; Z) \rightarrow \dots, \end{aligned}$$

where β_2 is the Bockstein homomorphism. By simple calculations, we have the following relations for the elements of $H^{2n-3}((RP^n)^*; Z_2)$ by (6.1):

$$\begin{aligned} Sq^1(vx^{2^{r+1}-4}y^s) &= vx^{2^{r+1}-3}y^s, & Sq^1(x^{2^{r+1}-3}y^s) &= x^{2^{r+1}-2}y^s, \\ vx^{2^{r+1}-2}y^{s-1} &= Sq^1(vx^{2^{r+1}-3}y^{s-1}) = \rho_2 \beta_2(vx^{2^{r+1}-3}y^{s-1}). \end{aligned}$$

These imply that $\rho_2 H^{2n-3}((RP^n)^*; Z)$ is Z_2 generated by $vx^{2^{r+1}-2}y^{s-1}$. Hence we have

(6.4) $H^{2n-3}((RP^n)^*; Z) = Z_2$ generated by $\beta_2(vx^{2^{r+1}-3}y^{s-1})$

by (6.2) and the above exact sequence.

This shows that $\rho_2: H^{2n-3}((RP^n)^*; Z) \rightarrow H^{2n-3}((RP^n)^*; Z_2)$ is a monomorphism. Furthermore $\rho_2 H^{2n-4}((RP^n)^*; Z) = \text{Ker } \beta_2 = \text{Ker}(Sq^1: H^{2n-4}((RP^n)^*; Z) \rightarrow H^{2n-3}((RP^n)^*; Z_2))$, because $Sq^1 = \rho_2 \beta_2$. On the other hand, we have the relations:

$$\begin{aligned} Sq^1(vx^{2^{r+1}-5}y^s) &= 0, & Sq^1(x^{2^{r+1}-4}y^s) &= 0, \\ Sq^1(vx^{2^{r+1}-3}y^{s-1}) &= vx^{2^{r+1}-2}y^{s-1}, & Sq^1(x^{2^{r+1}-2}y^{s-1}) &= 0. \end{aligned}$$

Therefore, by (6.3), we have

$$(6.5) \quad \rho_2 H^{2n-4}((RP^n)^*; Z) = Z_2 + Z_2 + Z_2 \text{ generated by } \{vx^{2r+1-5}y^s, x^{2r+1-4}y^s, x^{2r+1-2}y^{s-1}\}.$$

§ 7. Proof of Theorem B

We now prove Theorem B in § 1.

The existence of embeddings of RP^n in R^{2n-2} is shown in [7, Th. 4.1] and [10, Th. 7.2.2]. To prove that any two embeddings of RP^n in R^{2n-2} are isotopic, we apply the proposition in § 5 for $M=RP^n$, where the homomorphisms

$$\begin{aligned} \Theta^i: H^{i-1}((RP^n)^*; Z) &\longrightarrow H^{i+1}((RP^n)^*; Z_2) \quad \text{for } i=2n-2, 2n-3, \\ \Gamma: H^{2n-3}((RP^n)^*; Z_2) &\longrightarrow H^{2n-1}((RP^n)^*; Z_2) \end{aligned}$$

are defined by $\Theta^i(a) = Sq^2 \rho_2 a + \rho_2 av^2$ and $\Gamma(b) = Sq^2 b + bv^2$. We see that Θ^{2n-2} is a monomorphism by (6.4) and the following relations:

$$\begin{aligned} \Theta^{2n-2}(\beta_2(vx^{2r+1-3}y^{s-1})) &= Sq^2(vx^{2r+1-2}y^{s-1}) + vx^{2r+1-2}y^{s-1}v^2 \\ &= vx^{2r+1-2}y^s \neq 0 \text{ (by (6.3)).} \end{aligned}$$

Also, the equation $\Gamma(vx^{2r+1-2}y^{s-1}) = vx^{2r+1-2}y^s$ and (6.3) imply that Γ is an epimorphism. Consider the homomorphism $\Theta': \rho_2 H^{2n-4}((RP^n)^*; Z) \rightarrow H^{2n-2}((RP^n)^*; Z_2)$ defined by $\Theta'(a) = Sq^2 a + av^2$. Then we have the relations

$$\begin{aligned} \Theta'(x^{2r+1-2}y^{s-1}) &= x^{2r+1-2}y^s, \\ \Theta'(x^{2r+1-4}y^s) &= vx^{2r+1-3}y^s + \binom{s}{2}x^{2r+1-2}y^s. \end{aligned}$$

These and (6.3), (6.5) show that Θ' is an epimorphism, and so is $\Theta^{2n-3} = \Theta' \rho_2$. This completes the proof of Theorem B.

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