Smooth invariant classes for singular integrals: Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

| 著者 | 川崎隆和 |
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Smooth invariant classes for singular integrals

Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

By Takahide KUROKAWA

Abstract. It is well known that the $L^p$-spaces are invariant for singular integrals. In this paper we establish invariance of certain classes which consist of smooth functions.

1. Introduction and preliminaries

Let $R^n$ be the $n$-dimensional Euclidean space. Elements of $R^n$ are denoted by $x = (x_1, \ldots, x_n)$. For a domain $\Omega \subset R^n$, we denote by $C^\infty(\Omega)$ the set of all infinitely differentiable functions on $\Omega$. A function $k(x)$ is called a smooth Calderon-Zygmund kernel if $k(x)$ satisfies the following three conditions:

1. $k(x) \in C^\infty(R^n - \{0\})$, 
2. $k(x)$ is homogeneous of degree $-n$, 
3. $\int_\Sigma k(x)dS(x) = 0$

where $\Sigma$ is the unit sphere $\{|x| = 1\}$ and $dS$ is the surface element of $\Sigma$ (cf. [Sa: Chap.6]). For a smooth Calderon-Zygmund kernel $k(x)$ we consider the singular integral

$$Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x)$$

where

$$K_\epsilon f(x) = \int_{|x-y| \geq \epsilon} k(x-y)f(y)dy.$$ 

For $1 < p < \infty$ we let

$$L^p(R^n) = \{f : ||f||_p = (\int |f(x)|^pdx)^{1/p} < \infty\}.$$ 

The $L^p$-theory of singular integrals ([Sa: Chap.6], [St: Chap.II] and [SW: Chap.VI]) shows that the $L^p$-spaces $(1 < p < \infty)$ are invariant for singular integrals. Namely, for $f \in L^p$, $Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x)$ exists for almost every $x \in R^n$ and $Kf \in L^p$.

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For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $D_j$ denotes the differentiation with respect to $x_j$ $(j = 1, \cdots, n)$. The Lizorkin space $\Phi$ is defined by

$$\Phi = \{ \varphi \in \mathcal{S} : \int \varphi(x) x^\alpha dx = 0 \quad \text{for any multi-index } \alpha \}$$

where $\mathcal{S}$ is the Schwartz space (see [Li: §2 in Chap.II] and [SKM: §25]). The discussion in [Ku1: §2] shows that the Lizorkin space $\Phi$ is also invariant for singular integrals. Further, in [Ku2] we proved that the class $C^{\infty, +}(R^n)$ is invariant for singular integrals where

$$C^{\infty, +}(R^n) = \cup_{r>0} C^{\infty, r}(R^n)$$

with

$$C^{\infty, r}(R^n) = \{ f \in C^\infty(R^n) : \sup_{x \in R^n} (1 + |x|)^r |D^\alpha f(x)| < \infty \text{ for any } \alpha \}.$$ 

In this article we investigate invariance of the following class $C^{\infty, r}(R^n)$: For positive number $r$ we let

$$C^{\infty, r}(R^n) = \{ f \in C^\infty(R^n) : \sup_{x \in R^n} (1 + |x|)^{r+|\alpha|} |D^\alpha f(x)| < \infty \text{ for any } \alpha \}.$$ 

We introduce a topology on $C^{\infty, r}$ that makes the space a Fréchet space. Toward this end we introduce a countable family of seminorms $\{p_{t, r}\}_{t=0, 1, 2, \ldots}$ defined by

$$p_{t, r}(f) = \sum_{|\alpha| = t} \sup_{x \in R^n} (1 + |x|)^{r+t} |D^\alpha f(x)|.$$ 

We prove that $Kf$ is a continuous linear operator on $C^{\infty, r}(R^n)$ for $0 < r < n$ (Theorem 2.4). We use the symbol $C$ for a generic positive constant whose value may be different at each occurrence.

2. Invariance of the space $C^{\infty, r}$ $(0 < r < n)$

We prepare three lemmas.

**Lemma 2.1.** Let $q + s + n < 0$ and $s + n > 0$. Then

$$I_{q, s}(x) = \int_{|x-y| \geq \max(|x|/2, 1)} |x-y|^q (1 + |y|)^s dy \leq C(1 + |x|)^{q+s+n}. $$
Proof. First, let \( |x| \leq 2 \). Since \( |x| \leq 2 \) implies \( (1 + |x - y|)/3 \leq 1 + |y| \leq 3(1 + |x - y|) \), we see that

\[
I_{q,s}(x) \leq \max(3^s, 3^{-s}) \int_{|x - y| \geq 1} |x - y|^q (1 + |x - y|)^s dy = C_{q,s} < \infty
\]

by the condition \( q + s + n < 0 \).

Next, let \( |x| > 2 \). We devide \( I_{q,s}(x) \) as follows:

\[
I_{q,s}(x) = I^1_{q,s}(x) + I^2_{q,s}(x) + I^3_{q,s}(x)
\]

where

\[
I^1_{q,s}(x) = \int_{|y| < |x|/2} |x - y|^q (1 + |y|)^s dy,
\]

\[
I^2_{q,s}(x) = \int_{|y| \geq |x|/2, |x - y| > |y|} |x - y|^q (1 + |y|)^s dy
\]

and

\[
I^3_{q,s}(x) = \int_{|x - y| \geq |x|/2, |x - y| \leq |y|} |x - y|^q (1 + |y|)^s dy.
\]

For \( I^1_{q,s}(x) \), since \( |y| < |x|/2 \) implies \( (1 + |x|)/4 < |x - y| \), we have

\[
I^1_{q,s}(x) \leq 4^{-q}(1 + |x|)^q \int_{|y| < |x|/2} (1 + |y|)^s dy \leq C(1 + |x|)^{q + s + n}
\]

by the conditions \( q < 0 \) and \( s + n > 0 \). For \( I^2_{q,s}(x) \), since \( 1 \leq |x|/2 \leq |y| \) and \( |x - y| > |y| \) imply \( |x - y| > (1 + |y|)/2 \), we obtain

\[
I^2_{q,s}(x) \leq 2^{-q} \int_{|y| \geq |x|/2} (1 + |y|)^{q + s} dy \leq C(1 + |x|)^{q + s + n}
\]

by the conditions \( q < 0 \) and \( q + s + n < 0 \). For \( I^3_{q,s}(x) \), since \( 1 \leq |x|/2 \leq |x - y| \) and \( |x - y| \leq |y| \) imply \( 1 + |x - y| \leq 1 + |y| \leq 3(1 + |x - y|) \) and \( |x - y| < 1 + |x - y| \leq 2|x - y| \), we get

\[
I^3_{q,s}(x) \leq 2^{-q} \max(1, 3^s) \int_{|x - y| \geq |x|/2} (1 + |x - y|)^{q + s} dy \leq C(1 + |x|)^{q + s + n}
\]

by the conditions \( q < 0 \) and \( q + s + n < 0 \). The estimates (2.1), (2.2), (2.3) and (2.4) give the lemma.

Lemma 2.2. If \( f \in C^\infty_c(R^n) \) \( (r > 0) \), then \( Kf \in C^\infty(R^n) \) and \( D^\alpha(Kf)(x) = K(D^\alpha f)(x) \) for any \( \alpha \).
Proof. First, we prove that $K_{\epsilon} f \in C^\infty(R^n)$ and $D^\alpha(K_{\epsilon} f)(x) = K_{\epsilon}(D^\alpha f)(x)$. For $T > 0$, let $B_T = \{ x : |x| < T \}$. It suffices to show that $K_{\epsilon} f \in C^\infty(B_T)$ and $D^\alpha(K_{\epsilon} f)(x) = K_{\epsilon}(D^\alpha f)(x)$ on $B_T$. Since $1 + |y| \leq (1 + T)(1 + |x - y|)$ for $x \in B_T$, we have

$$|k(y)D^\alpha f(x - y)| \leq \frac{C}{|y|^n(1 + |y|)^{r + |\alpha|}} , \quad x \in B_T$$

by the condition $f \in C^{\infty,r}(R^n)$ and (1.2). Therefore we can apply the differentiation under the integral sign, and hence

$$D^\alpha(K_{\epsilon} f)(x) = \int_{|y| \geq \epsilon} k(y)D^\alpha f(x - y)dy , \quad x \in B_T.$$ 

This implies the necessary conclusions. Next we prove that $D^\alpha K_{\epsilon} f(x)$ converges uniformly on $R^n$ as $\epsilon$ tends to 0 for any $\alpha$. Let $0 < \epsilon < \eta$. By (1.3) we have

$$|D^\alpha K_{\epsilon} f(x) - D^\alpha K_{\eta} f(x)| = |K_{\epsilon} D^\alpha f(x) - K_{\eta} D^\alpha f(x)|$$

$$= \left| \int_{x - y < \eta} k(x - y)D^\alpha f(y)dy \right|$$

$$= \left| \int_{x - y < \eta} k(x - y)(D^\alpha f(y) - D^\alpha f(x))dy \right|.$$

By the mean value theorem of calculus we see that

$$|D^\alpha f(y) - D^\alpha f(x)| = \left| \sum_{j=1}^n D^{\alpha + \epsilon_j} f(y + \theta(y - x))(y_j - x_j) \right|$$

$$\leq C|x - y| \sum_{j=1}^n \frac{1}{(1 + |y + \theta(y - x)|)^r}$$

$$\leq C|x - y|$$

where $0 < \theta < 1$. Therefore by (1.2) we get

$$|D^\alpha K_{\epsilon} f(x) - D^\alpha K_{\eta} f(x)| \leq C \int_{|x - y| < \eta} |x - y|^{1 - n}dy = C(\eta - \epsilon).$$

Hence $D^\alpha K_{\epsilon} f(x)$ converges uniformly on $R^n$ as $\epsilon$ tends to 0 for any $\alpha$. This implies that $K f(x) \in C^\infty(R^n)$ and $D^\alpha(K f)(x) = K(D^\alpha f)(x)$ for any $\alpha$. We complete the proof of the lemma.
The following lemma follows from Gauss’s divergence theorem.

**Lemma 2.3.** Let $D$ be a bounded domain with $C^\infty$-boundary $\partial D$. Let $n(x) = (n_1(x), \ldots, n_n(x))$ denote the outer unit normal to the boundary $\partial D$ at the point $x \in \partial D$. We assume that $g$ and $h$ have continuous partial derivatives on a neighborhood of the closure of $D$. Then

$$\int_D g(x) D_j h(x) dx = \int_{\partial D} g(x) h(x) n_j(x) dS(x) - \int_D D_j g(x) h(x) dx$$

where $dS$ represents the surface element of $\partial D$.

Now we prove our main result.

**Theorem 2.4.** Let $0 < r < n$. If $f \in C^{\infty,r}(\mathbb{R}^n)$, then

$$p_{\ell, r}(Kf) \leq C \left\{ \begin{array}{l}
(\sum_{k=0}^{\ell-1} p_{k, r}(f) + p_{\ell+1, r}(f)), \quad \ell \geq 1 \\
(p_{0, r}(f) + p_{1, r}(f)), \quad \ell = 0,
\end{array} \right.$$ 

and hence $Kf$ is a continuous linear operator on $C^{\infty,r}(\mathbb{R}^n)$.

**Proof.** Let $f \in C^{\infty,r}(\mathbb{R}^n)$. It follows from Lemma 2.2 that $Kf \in C^{\infty}(\mathbb{R}^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any $\alpha$. Let $|\alpha| = \ell$. We have

$$D^\alpha Kf(x) = KD^\alpha f(x)$$

$$= \lim_{\epsilon \to 0} \int_{|x-y| \leq \max(|x|/2,1)} k(x-y)D^\alpha f(y)dy$$

$$+ \int_{|x-y| > \max(|x|/2,1)} k(x-y)D^\alpha f(y)dy$$

$$= K_1(D^\alpha f)(x) + K_2(D^\alpha f)(x).$$

By (1.2) and (1.3) we obtain

$$|K_1(D^\alpha f)(x)| = \left| \lim_{\epsilon \to 0} \int_{|x-y| \leq \max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x))dy \right|$$

$$= \left| \int_{|x-y| \leq \max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x))dy \right|$$

$$\leq C \int_{|x-y| \leq \max(|x|/2,1)} \frac{|D^\alpha f(y) - D^\alpha f(x)|}{|x-y|^n} dy.$$ 

Since $f \in C^{\infty,r}(\mathbb{R}^n)$, by the mean value theorem of calculus we obtain

$$|D^\alpha f(y) - D^\alpha f(x)| = \left| \sum_{j=1}^{n} D^{\alpha+\epsilon_j} f(x + \theta(y-x))(y_j - x_j) \right|$$

$$\leq C \frac{|x-y|}{(1 + |x + \theta(y-x)|)^{r+\ell+1}} p_{\ell+1, r}(f)$$
where \(0 < \theta < 1\). Further, since \(|x - y| \leq \max(|x|/2, 1)\) implies \(1 + |x + \theta(y - x)| \geq (1 + |x|)/2\), we have

\[
(2.5) \quad |K_1(D^\alpha f)(x)| \leq C \frac{p_{\ell+1,r}(f)}{(1 + |x|)^{r+\ell + 1}} \int_{|x-y| \leq \max(|x|/2, 1)} |x-y|^{1-n}dy
\]

\[
= C \frac{p_{\ell+1,r}(f)}{(1 + |x|)^{r+\ell + 1}} \max(|x|/2, 1)
\]

\[
\leq C \frac{P_{\ell+1,r}(f)}{(1 + |x|)^{r+\ell}}.
\]

The multi-index \(e_j\) denotes the ordered \(n\)-tuple that has 1 in the \(j\)th spot and 0 everywhere else \((j = 1, \cdots, n)\). In case \(\ell \geq 1\), we let \(\alpha = e_{j_1} + \cdots + e_{j_\ell}\). By Lemma 2.3 we have

\[
K_2(D^\alpha f)(x) = \lim_{M \to \infty} \int_{M > |x-y| > \max(|x|/2, 1)} k(x-y)D^\alpha f(y)dy
\]

\[
= \lim_{M \to \infty} \int_{|y| < |x-y| = |x|=M} k(x-y)D^{e_{j_1}} f(y)n_{j_1}(y)dS(y)
\]

\[
+ \sum_{k=2}^{\ell} \lim_{M \to \infty} \int_{|y| < |x-y| = |x|=M} D^{e_{j_1} + \cdots + e_{j_{k-1}}} k(x-y)D^{e_{j_k}} f(y)n_{j_k}(y)dS(y)
\]

\[
+ \int_{|y| < |x-y| = \max(|x|/2, 1)} k(x-y)D^{e_{j_1}} f(y)n_{j_1}(y)dS(y)
\]

\[
+ \sum_{k=2}^{\ell} \int_{|y| < |x-y| = \max(|x|/2, 1)} D^{e_{j_1} + \cdots + e_{j_{k-1}}} k(x-y)D^{e_{j_k}} f(y)n_{j_k}(y)dS(y)
\]

\[
+ \lim_{M \to \infty} \int_{M > |x-y| > \max(|x|/2, 1)} D^\alpha k(x-y)f(y)dy
\]

\[
= \lim_{M \to \infty} I_1^{1,M} + \sum_{k=2}^{\ell} \lim_{M \to \infty} I_1^{k,M}(x) + I_3^1(x) + \sum_{k=2}^{\ell} I_3^k(x) + I_3^0(x).
\]

In case \(\ell = 0\), we have

\[
K_2(D^\alpha f)(x) = K_2 f(x) = I_3(x).
\]

By the condition \(f \in C^{\infty,r}(R^n)\) and (1.2) we have

\[
|I_1^{k,M}(x)| \leq C p_{\ell-k,r}(f) \int_{|y| < |x-y| = M} |x-y|^{-n-k+1}(1 + |y|)^{-r-\ell+k}dS(y),
\]

\[
|I_3^k(x)| \leq C p_{\ell-k,r}(f) \int_{|y| < |x-y| = \max(|x|/2, 1)} |x-y|^{-n-k+1}(1 + |y|)^{-r-\ell+k}dS(y).
\]
for \( k = 1, 2, \ldots, \ell, \) and
\[
|I_3(x)| = \left| \int_{\{|y|:|x-y|>|x|/2,1]\}} D^n k(x-y) f(y) dy \right|
\leq C p_{0,r}(f) \int_{\{|y|:|x-y|>|x|/2,1]\}} |x-y|^{-n-\ell}(1+|y|)^{-r} dy.
\]

We may assume that \( M \geq 2|x| \). Therefore, since \( |x-y| \geq 2|x| \) implies \( (1+|x-y|)/2 \leq 1 + |y| \leq 3(1+|x-y|)/2 \), we obtain
\[
|I_{1}^{k,M}(x)| \leq C p_{\ell-k,r}(f) M^{-n-k+1}(1+M)^{-r-\ell+k} \int_{\{|y|:|x-y|=M\}} dS(y)
= C p_{\ell-k,r} M^{-k}(1+M)^{-r-\ell+k} \to 0 \quad (M \to \infty).
\]

Since \( |x-y| = \max(|x|/2,1) \) implies \( (1+|x|)/2 \leq 1 + |y| \leq 3(1+|x|)/2 \), we get
\[
|I_{2}^{k}(x)| \leq C p_{\ell-k,r}(f)(\max(|x|/2,1))^{-n-k+1}(1+|x|)^{-r-\ell+k}(\max(|x|/2,1))^{n-1}
\leq C p_{\ell-k,r}(1+|x|)^{-r-\ell}.
\]

Furthermore, since \( 0 < r < n \), Lemma 2.1 gives
\[
|I_3(x)| \leq C p_{0,r}(f)(1+|x|)^{-r-\ell}.
\]

Thus
\[
|K_2(D^n f)(x)| \leq C (1+|x|)^{-r-\ell} \sum_{k=1}^{\ell} p_{\ell-k,r}(f) = C (1+|x|)^{-r-\ell} \sum_{k=0}^{\ell-1} p_{k,r}(f).
\]

The estimates (2.5) and (2.6) give the theorem.

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Takahide KUROKAWA
Department of Mathematics
and Computer Science
Faculty of Science
Kagoshima University
Kagoshima, 890-0065
Japan
E-mail: kurokawa@sci.kagoshima-u.ac.jp