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Smooth invariant classes for singular integrals

Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

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Abstract. It is well known that the L^p-spaces are invariant for singular integrals. In this paper we establish invariance of certain classes which consist of smooth functions.

1. Introduction and preliminaries

Let R^n be the n-dimensional Euclidean space. Elements of R^n are denoted by $x = (x_1, \cdots, x_n)$. For a domain $\Omega \subset R^n$, we denote by $C^\infty(\Omega)$ the set of all infinitely differentiable functions on Ω. A function $k(x)$ is called a smooth Calderon-Zygmund kernel if $k(x)$ satisfies the following three conditions:

(1.1) $k(x) \in C^\infty(R^n - \{0\})$,

(1.2) $k(x)$ is homogeneous of degree $-n$,

(1.3) $\int_\Sigma k(x)dS(x) = 0$

where Σ is the unit sphere $\{|x| = 1\}$ and dS is the surface element of Σ (cf. [Sa: Chap.6]). For a smooth Calderon-Zygmund kernel $k(x)$ we consider the singular integral

$$Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x)$$

where

$$K_\epsilon f(x) = \int_{|x-y| \geq \epsilon} k(x-y)f(y)dy.$$

For $1 < p < \infty$ we let

$$L^p(R^n) = \{f : ||f||_p = (\int |f(x)|^p dx)^{1/p} < \infty\}.$$

The L^p-theory of singular integrals ([Sa: Chap.6], [St: Chap.II] and [SW: Chap.VI]) shows that the L^p-spaces $(1 < p < \infty)$ are invariant for singular integrals. Namely, for $f \in L^p$, $Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x)$ exists for almost every $x \in R^n$ and $Kf \in L^p$.

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For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$, we denote $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}, x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$, where D_j denotes the differentiation with respect to x_j $(j=1,\cdots,n)$. The Lizorkin space Φ is defined by

$$\Phi=\{\varphi\in\mathcal{S}:\int\varphi(x)x^\alpha dx=0 \quad \text{for any multi-index } \alpha\}$$

where $\mathcal{S}$ is the Schwartz space (see [Li: §2 in Chap.II] and [SKM: §25]). The discussion in [Ku1: §2] shows that the Lizorkin space Φ is also invariant for singular integrals. Further, in [Ku2] we proved that the class $\mathcal{C}^{\infty,+}(R^n)$ is invariant for singular integrals where

$$\mathcal{C}^{\infty,+}(R^n)=\cup_{r>0}\mathcal{C}^{\infty,r}(R^n)$$

with

$$\mathcal{C}^{\infty,r}(R^n)=\{f\in C^\infty(R^n):\sup_{x\in R^n}(1+|x|)^r|D^\alpha f(x)|<\infty \text{ for any } \alpha\}.$$

In this article we investigate invariance of the following class $C^{\infty,r}(R^n)$: For positive number r we let

$$C^{\infty,r}(R^n)=\{f\in C^\infty(R^n):\sup_{x\in R^n}(1+|x|)^{r+|\alpha|}|D^\alpha f(x)|<\infty \text{ for any } \alpha\}.$$

We introduce a topology on $C^{\infty,r}$ that makes the space a Fréchet space. Toward this end we introduce a countable family of seminorms $\{p_{\ell,r}\}_{\ell=0,1,2,\cdots}$ defined by

$$p_{\ell,r}(f)=\sum_{|\alpha|=\ell}\sup_{x\in R^n}(1+|x|)^{r+\ell}|D^\alpha f(x)|.$$

We prove that Kf is a continuous linear operator on $C^{\infty,r}(R^n)$ for $0<r<n$ (Theorem 2.4). We use the symbol C for a generic positive constant whose value may be different at each occurrence.

2. Invariance of the space $C^{\infty,r}$ $(0<r<n)$

We prepare three lemmas.

LEMMA 2.1. *Let $q+s+n<0$ and $s+n>0$. Then*

$$I_{q,s}(x)=\int_{|x-y|\geq\max(|x|/2,1)}|x-y|^q(1+|y|)^s dy\leq C(1+|x|)^{q+s+n}.$$

PROOF. First, let $|x| \leq 2$. Since $|x| \leq 2$ implies $(1+|x-y|)/3 \leq 1+|y| \leq 3(1+|x-y|)$, we see that

$$I_{q,s}(x) \leq \max(3^s, 3^{-s}) \int_{|x-y|\geq 1} |x-y|^q (1+|x-y|)^s dy = C_{q,s} < \infty \tag{2.1}$$

by the condition $q+s+n<0$.

Next, let $|x|>2$. We devide $I_{q,s}(x)$ as follows:

$$I_{q,s}(x) = I^1_{q,s}(x) + I^2_{q,s}(x) + I^3_{q,s}(x)$$

where

$$I^1_{q,s}(x) = \int_{|y|<|x|/2} |x-y|^q (1+|y|)^s dy,$$

$$I^2_{q,s}(x) = \int_{|y|\geq|x|/2,|x-y|>|y|} |x-y|^q (1+|y|)^s dy$$

and

$$I^3_{q,s}(x) = \int_{|x-y|\geq|x|/2,|x-y|\leq|y|} |x-y|^q (1+|y|)^s dy.$$

For $I^1_{q,s}(x)$, since $|y|<|x|/2$ implies $(1+|x|)/4 < |x-y|$, we have

$$I^1_{q,s}(x) \leq 4^{-q}(1+|x|)^q \int_{|y|<|x|/2} (1+|y|)^s dy \leq C(1+|x|)^{q+s+n} \tag{2.2}$$

by the conditions $q<0$ and $s+n>0$. For $I^2_{q,s}(x)$, since $1 \leq |x|/2 \leq |y|$ and $|x-y|>|y|$ imply $|x-y|>(1+|y|)/2$, we obtain

$$I^2_{q,s}(x) \leq 2^{-q} \int_{|y|\geq|x|/2} (1+|y|)^{q+s} dy \leq C(1+|x|)^{q+s+n} \tag{2.3}$$

by the conditions $q<0$ and $q+s+n<0$. For $I^3_{q,s}(x)$, since $1 \leq |x|/2 \leq |x-y|$ and $|x-y| \leq |y|$ imply $1+|x-y| \leq 1+|y| \leq 3(1+|x-y|)$ and $|x-y| < 1+|x-y| \leq 2|x-y|$, we get

$$I^3_{q,s}(x) \leq 2^{-q}\max(1,3^s) \int_{|x-y|\geq|x|/2} (1+|x-y|)^{q+s} dy \leq C(1+|x|)^{q+s+n} \tag{2.4}$$

by the conditions $q<0$ and $q+s+n<0$. The estimates (2.1), (2.2), (2.3) and (2.4) give the lemma.

LEMMA 2.2. *If $f \in C^{\infty,r}(R^n)$ $(r>0)$, then $Kf \in C^{\infty}(R^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any α.*

PROOF. First, we prove that $K_\epsilon f \in C^\infty(R^n)$ and $D^\alpha(K_\epsilon f)(x) = K_\epsilon(D^\alpha f)(x)$. For $T > 0$, let $B_T = \{x : |x| < T\}$. It suffices to show that $K_\epsilon f \in C^\infty(B_T)$ and $D^\alpha(K_\epsilon f)(x) = K_\epsilon(D^\alpha f)(x)$ on B_T. Since $1+|y| \leq (1+T)(1+|x-y|)$ for $x \in B_T$, we have

$$|k(y)D^\alpha f(x-y)| \leq \frac{C}{|y|^n(1+|y|)^{r+|\alpha|}}, \quad x \in B_T$$

by the condition $f \in C^{\infty,r}(R^n)$ and (1.2). Therefore we can apply the differentiation under the integral sign, and hence

$$D^\alpha(K_\epsilon f)(x) = \int_{|y|\geq\epsilon} k(y)D^\alpha f(x-y)dy, \quad x \in B_T.$$

This implies the neccessary conclusions. Next we prove that $D^\alpha K_\epsilon f(x)$ converges uniformly on R^n as ϵ tends to 0 for any α. Let $0 < \epsilon < \eta$. By (1.3) we have

$$\begin{aligned} |D^\alpha K_\epsilon f(x) - D^\alpha K_\eta f(x)| &= |K_\epsilon D^\alpha f(x) - K_\eta D^\alpha f(x)| \\ &= |\int_{\epsilon\leq|x-y|<\eta} k(x-y)D^\alpha f(y)dy| \\ &= |\int_{\epsilon\leq|x-y|<\eta} k(x-y)(D^\alpha f(y) - D^\alpha f(x))dy|. \end{aligned}$$

By the mean value theorem of calculus we see that

$$\begin{aligned} |D^\alpha f(y) - D^\alpha f(x)| &= |\sum_{j=1}^n D^{\alpha+e_j} f(y+\theta(y-x))(y_j - x_j)| \\ &\leq C|x-y| \sum_{j=1}^n \frac{1}{(1+|y+\theta(y-x)|)^r} \\ &\leq C|x-y| \end{aligned}$$

where $0 < \theta < 1$. Therefore by (1.2) we get

$$|D^\alpha K_\epsilon f(x) - D^\alpha K_\eta f(x)| \leq C \int_{\epsilon\leq|x-y|<\eta} |x-y|^{1-n}dy = C(\eta - \epsilon).$$

Hence $D^\alpha K_\epsilon f(x)$ converges uniformly on R^n as ϵ tends to 0 for any α. This implies that $Kf(x) \in C^\infty(R^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any α. We complete the proof of the lemma.

The following lemma follows from Gauss's divergence theorem.

LEMMA 2.3. *Let D be a bounded domain with C^∞-boundary ∂D. Let $\mathbf{n}(x) = (\mathbf{n}_1(x), \cdots, \mathbf{n}_n(x))$ denote the outer unit normal to the boundary ∂D at the point $x \in \partial D$. We assume that g and h have continuous partial derivatives on a neighborhood of the closure of D. Then*

$$\int_D g(x)D_jh(x)dx = \int_{\partial D} g(x)h(x)\mathbf{n}_j(x)dS(x) - \int_D D_jg(x)h(x)dx$$

where dS represents the surface element of ∂D.

Now we prove our main result.

THEOREM 2.4. *Let $0 < r < n$. If $f \in C^{\infty,r}(R^n)$, then*

$$p_{\ell,r}(Kf) \leq C \begin{cases} (\sum_{k=0}^{\ell-1} p_{k,r}(f) + p_{\ell+1,r}(f)), & \ell \geq 1 \\ (p_{0,r}(f) + p_{1,r}(f)), & \ell = 0, \end{cases}$$

and hence Kf is a continuous linear operator on $C^{\infty,r}(R^n)$.

PROOF. Let $f \in C^{\infty,r}(R^n)$. It follows from Lemma 2.2 that $Kf \in C^\infty(R^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any α. Let $|\alpha| = \ell$. We have

$$\begin{aligned} D^\alpha Kf(x) &= KD^\alpha f(x) \\ &= \lim_{\epsilon\to 0}\int_{\epsilon\leq|x-y|\leq\max((|x|/2,1)} k(x-y)D^\alpha f(y)dy \\ &\qquad + \int_{|x-y|>\max(|x|/2,1)} k(x-y)D^\alpha f(y)dy \\ &= K_1(D^\alpha f)(x) + K_2(D^\alpha f)(x). \end{aligned}$$

By (1.2) and (1.3) we obtain

$$\begin{aligned} |K_1(D^\alpha f)(x)| &= |\lim_{\epsilon\to 0}\int_{\epsilon\leq|x-y|\leq\max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x))dy| \\ &= |\int_{|x-y|\leq\max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x))dy| \\ &\leq C\int_{|x-y|\leq\max(|x|/2,1)} \frac{|D^\alpha f(y) - D^\alpha f(x)|}{|x-y|^n}dy. \end{aligned}$$

Since $f \in C^{\infty,r}(R^n)$, by the mean value theorem of calculus we obtain

$$\begin{aligned} |D^\alpha f(y) - D^\alpha f(x)| &= |\sum_{j=1}^n D^{\alpha+e_j}f(x+\theta(y-x))(y_j - x_j)| \\ &\leq C\frac{|x-y|}{(1+|x+\theta(y-x)|)^{r+\ell+1}}p_{\ell+1,r}(f) \end{aligned}$$

where $0<\theta<1$. Further, since $|x-y| \leq \max(|x|/2, 1)$ implies $1+|x+\theta(y-x)| \geq (1+|x|)/2$, we have

$$(2.5)\qquad \begin{aligned} |K_1(D^\alpha f)(x)| &\leq C\frac{p_{\ell+1,r}(f)}{(1+|x|)^{r+\ell+1}}\int_{|x-y|\leq \max(|x|/2,1)}|x-y|^{1-n}dy \\ &= C\frac{p_{\ell+1,r}(f)}{(1+|x|)^{r+\ell+1}}\max(|x|/2,1) \\ &\leq C\frac{P_{\ell+1,r}(f)}{(1+|x|)^{r+\ell}}. \end{aligned}$$

The multi-index e_j denotes the ordered n-tuple that has 1 in the jth spot and 0 everywhere else ($j=1,\cdots,n$). In case $\ell \geq 1$, we let $\alpha = e_{j_1}+\cdots+e_{j_\ell}$. By Lemma 2.3 we have

$$\begin{aligned} K_2(D^\alpha f)(x) &= \lim_{M\to\infty}\int_{M>|x-y|>\max(|x|/2,1)} k(x-y)D^\alpha f(y)dy \\ &= \lim_{M\to\infty}\int_{\{y:|x-y|=M\}} k(x-y)D^{\alpha-e_{j_1}}f(y)\mathbf{n}_{j_1}(y)dS(y) \\ &\quad +\sum_{k=2}^{\ell}\lim_{M\to\infty}\int_{\{y:|x-y|=M\}} D^{e_{j_1}+\cdots+e_{j_{k-1}}}k(x-y)D^{\alpha-e_{j_1}-\cdots-e_{j_k}}f(y)\mathbf{n}_{j_k}(y)dS(y) \\ &\quad +\int_{\{y:|x-y|=\max(|x|/2,1)\}} k(x-y)D^{\alpha-e_{j_1}}f(y)\mathbf{n}_{j_1}(y)dS(y) \\ &\quad +\sum_{k=2}^{\ell}\int_{\{y:|x-y|=\max(|x|/2,1)\}} D^{e_{j_1}+\cdots+e_{j_{k-1}}}k(x-y)D^{\alpha-e_{j_1}-\cdots-e_{j_k}}f(y)\mathbf{n}_{j_k}(y)dS(y) \\ &\quad +\lim_{M\to\infty}\int_{M>|x-y|>\max(|x|/2,1)} D^\alpha k(x-y)f(y)dy \\ &= \lim_{M\to\infty} I_1^{1,M}+\sum_{k=2}^{\ell}\lim_{M\to\infty} I_1^{k,M}(x)+I_2^1(x)+\sum_{k=2}^{\ell} I_2^k(x)+I_3(x). \end{aligned}$$

In case $\ell=0$, we have

$$K_2(D^\alpha f)(x) = K_2 f(x) = I_3(x).$$

By the condition $f \in C^{\infty,r}(R^n)$ and (1.2) we have

$$|I_1^{k,M}(x)| \leq Cp_{\ell-k,r}(f)\int_{\{y:|x-y|=M\}}|x-y|^{-n-k+1}(1+|y|)^{-r-\ell+k}dS(y),$$

$$|I_2^k(x)| \leq Cp_{\ell-k,r}(f)\int_{\{y:|x-y|=\max(|x|/2,1)\}}|x-y|^{-n-k+1}(1+|y|)^{-r-\ell+k}dS(y)$$

for $k = 1, 2, \cdots, \ell$, and

$$\begin{aligned} |I_3(x)| &= \left|\int_{\{y:|x-y|>\max(|x|/2,1)\}} D^\alpha k(x-y)f(y)dy\right| \\ &\le Cp_{0,r}(f)\int_{\{y:|x-y|>\max(|x|/2,1)\}} |x-y|^{-n-\ell}(1+|y|)^{-r}dy. \end{aligned}$$

We may assume that $M \ge 2|x|$. Therefore, since $|x-y| \ge 2|x|$ implies $(1+|x-y|)/2 \le 1+|y| \le 3(1+|x-y|)/2$, we obtain

$$\begin{aligned} |I_1^{k,M}(x)| &\le Cp_{\ell-k,r}(f)M^{-n-k+1}(1+M)^{-r-\ell+k}\int_{\{y:|x-y|=M\}} dS(y) \\ &= Cp_{\ell-k,r}M^{-k}(1+M)^{-r-\ell+k} \to 0 \quad (M \to \infty). \end{aligned}$$

Since $|x-y| = \max(|x|/2,1)$ implies $(1+|x|)/2 \le 1+|y| \le 3(1+|x|)/2$, we get

$$\begin{aligned} |I_2^k(x)| &\le Cp_{\ell-k,r}(f)(\max(|x|/2,1))^{-n-k+1}(1+|x|)^{-r-\ell+k}(\max(|x|/2,1))^{n-1} \\ &\le Cp_{\ell-k,r}(1+|x|)^{-r-\ell}. \end{aligned}$$

Furthermore, since $0 < r < n$, Lemma 2.1 gives

$$|I_3(x)| \le Cp_{0,r}(f)(1+|x|)^{-r-\ell}.$$

Thus

$$(2.6) \qquad |K_2(D^\alpha f)(x)| \le C(1+|x|)^{-r-\ell}\sum_{k=1}^{\ell} p_{\ell-k,r}(f) = C(1+|x|)^{-r-\ell}\sum_{k=0}^{\ell-1} p_{k,r}(f).$$

The estimates (2.5) and (2.6) give the theorem.

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