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Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

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Smooth invariant classes for singular integrals

Dedicated to Professor Shoji Tsuboi on the occasion of his 60th birthday

By Takahide KUROKAWA

Abstract. It is well known that the \( L^p \)-spaces are invariant for singular integrals. In this paper we establish invariance of certain classes which consist of smooth functions.

1. Introduction and preliminaries

Let \( R^n \) be the \( n \)-dimensional Euclidean space. Elements of \( R^n \) are denoted by \( x = (x_1, \ldots, x_n) \). For a domain \( \Omega \subset R^n \), we denote by \( C^\infty(\Omega) \) the set of all infinitely differentiable functions on \( \Omega \). A function \( k(x) \) is called a smooth Calderon-Zygmund kernel if \( k(x) \) satisfies the following three conditions:

(1.1) \( k(x) \in C^\infty(R^n - \{0\}) \),

(1.2) \( k(x) \) is homogeneous of degree \(-n\),

(1.3) \( \int_\Sigma k(x)dS(x) = 0 \)

where \( \Sigma \) is the unit sphere \( \{|x|=1\} \) and \( dS \) is the surface element of \( \Sigma \) (cf. [Sa: Chap.6]). For a smooth Calderon-Zygmund kernel \( k(x) \) we consider the singular integral

\[
Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x)
\]

where

\[
K_\epsilon f(x) = \int_{|x-y| \geq \epsilon} k(x-y)f(y)dy.
\]

For \( 1 < p < \infty \) we let

\[
L^p(R^n) = \{ f : \|f\|_p = \left( \int |f(x)|^pdx \right)^{1/p} < \infty \}.
\]

The \( L^p \)-theory of singular integrals ([Sa: Chap.6], [St: Chap.II] and [SW: Chap.VI]) shows that the \( L^p \)-spaces \( (1 < p < \infty) \) are invariant for singular integrals. Namely, for \( f \in L^p \), \( Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x) \) exists for almost every \( x \in R^n \) and \( Kf \in L^p \).
For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $D_j$ denotes the differentiation with respect to $x_j$ ($j = 1, \cdots, n$). The Lizorkin space $\Phi$ is defined by

$$\Phi = \{ \varphi \in S : \int \varphi(x)x^{\alpha}dx = 0 \text{ for any multi-index } \alpha \}$$

where $S$ is the Schwartz space (see [Li: §2 in Chap.II] and [SKM: §25]). The discussion in [Ku1: §2] shows that the Lizorkin space $\Phi$ is also invariant for singular integrals. Further, in [Ku2] we proved that the class $C^{\infty,+}(\mathbb{R}^n)$ is invariant for singular integrals where

$$C^{\infty,+}(\mathbb{R}^n) = \bigcup_{r>0} C^{\infty,r}(\mathbb{R}^n)$$

with

$$C^{\infty,r}(\mathbb{R}^n) = \{ f \in C^{\infty}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1 + |x|)^r |D^\alpha f(x)| < \infty \text{ for any } \alpha \}.$$ 

In this article we investigate invariance of the following class $C^{\infty,r}(\mathbb{R}^n)$: For positive number $r$, we let

$$C^{\infty,r}(\mathbb{R}^n) = \{ f \in C^{\infty}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1 + |x|)^{r+|\alpha|} |D^\alpha f(x)| < \infty \text{ for any } \alpha \}.$$ 

We introduce a topology on $C^{\infty,r}$ that makes the space a Fréchet space. Toward this end we introduce a countable family of seminorms $\{p_{\ell,r}\}_{\ell=0,1,2,\cdots}$ defined by

$$p_{\ell,r}(f) = \sum_{|\alpha| = \ell} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{r+\ell} |D^\alpha f(x)|.$$ 

We prove that $Kf$ is a continuous linear operator on $C^{\infty,r}(\mathbb{R}^n)$ for $0 < r < n$ (Theorem 2.4). We use the symbol $C$ for a generic positive constant whose value may be different at each occurrence.

### 2. Invariance of the space $C^{\infty,r}$ ($0 < r < n$)

We prepare three lemmas.

**Lemma 2.1.** Let $q + s + n < 0$ and $s + n > 0$. Then

$$I_{q,s}(x) = \int_{|x-y| \geq \max(1,|x|/2,1)} |x-y|^q (1 + |y|)^s dy \leq C(1 + |x|)^{q+s+n}.$$
PROOF. First, let $|x| \leq 2$. Since $|x| \leq 2$ implies $(1 + |x - y|)/3 \leq 1 + |y| \leq 3(1 + |x - y|)$, we see that

(2.1) \quad I_{q,s}(x) \leq \max(3^s, 3^{-s}) \int_{|x - y| \geq 1} |x - y|^q (1 + |x - y|)^s dy = C_{q,s} < \infty

by the condition $q + s + n < 0$.

Next, let $|x| > 2$. We divide $I_{q,s}(x)$ as follows:

$$I_{q,s}(x) = I^1_{q,s}(x) + I^2_{q,s}(x) + I^3_{q,s}(x)$$

where

$$I^1_{q,s}(x) = \int_{|y| < |x|/2} |x - y|^q (1 + |y|)^s dy,$$

$$I^2_{q,s}(x) = \int_{|y| \geq |x|/2, |x - y| > |y|} |x - y|^q (1 + |y|)^s dy$$

and

$$I^3_{q,s}(x) = \int_{|x - y| \geq |x|/2, |x - y| \leq |y|} |x - y|^q (1 + |y|)^s dy.$$

For $I^1_{q,s}(x)$, since $|y| < |x|/2$ implies $(1 + |x|)/4 < |x - y|$, we have

(2.2) \quad I^1_{q,s}(x) \leq 4^{-q} (1 + |x|)^q \int_{|y| < |x|/2} (1 + |y|)^s dy \leq C(1 + |x|)^{q+s+n}

by the conditions $q < 0$ and $s + n > 0$. For $I^2_{q,s}(x)$, since $1 \leq |x|/2 \leq |y|$ and $|x - y| > |y|$ imply $|x - y| > (1 + |y|)/2$, we obtain

(2.3) \quad I^2_{q,s}(x) \leq 2^{-q} \int_{|y| \geq |x|/2} (1 + |y|)^{q+s} dy \leq C(1 + |x|)^{q+s+n}

by the conditions $q < 0$ and $q + s + n < 0$. For $I^3_{q,s}(x)$, since $1 \leq |x|/2 \leq |x - y|$ and $|x - y| \leq |y|$ imply $1 + |x - y| \leq 1 + |y| \leq 3(1 + |x - y|)$ and $|x - y| < 1 + |x - y| \leq 2|x - y|$, we get

(2.4) \quad I^3_{q,s}(x) \leq 2^{-q} \max(1, 3^s) \int_{|x - y| \geq |x|/2} (1 + |x - y|)^{q+s} dy \leq C(1 + |x|)^{q+s+n}

by the conditions $q < 0$ and $q + s + n < 0$. The estimates (2.1), (2.2), (2.3) and (2.4) give the lemma.

**Lemma 2.2.** If $f \in C^{\infty,c}(R^n)$ ($r > 0$), then $Kf \in C^{\infty}(R^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any $\alpha$. 
Proof. First, we prove that $K_\epsilon f \in C^\infty(R^n)$ and $D^\alpha(K_\epsilon f)(x) = K_\epsilon(D^\alpha f)(x)$. For $T > 0$, let $B_T = \{ x : |x| < T \}$. It suffices to show that $K_\epsilon f \in C^\infty(B_T)$ and $D^\alpha(K_\epsilon f)(x) = K_\epsilon(D^\alpha f)(x)$ on $B_T$. Since $1 + |y| \leq (1 + T)(1 + |x - y|)$ for $x \in B_T$, we have

$$|k(y)D^\alpha f(x - y)| \leq \frac{C}{|y|^n(1 + |y|)^{r+|\alpha|}}, \quad x \in B_T$$

by the condition $f \in C^{\infty,r}(R^n)$ and (1.2). Therefore we can apply the differentiation under the integral sign, and hence

$$D^\alpha(K_\epsilon f)(x) = \int_{|y| \geq \epsilon} k(y)D^\alpha f(x - y)dy, \quad x \in B_T.$$ 

This implies the necessary conclusions. Next we prove that $D^\alpha K_\epsilon f(x)$ converges uniformly on $R^n$ as $\epsilon$ tends to 0 for any $\alpha$. Let $0 < \epsilon < \eta$. By (1.3) we have

$$|D^\alpha K_\epsilon f(x) - D^\alpha K_\eta f(x)| = |K_\epsilon(D^\alpha f(x) - K_\eta D^\alpha f(x))|$$

$$= \int_{|y| \leq |x - y| < \eta} k(x - y)D^\alpha f(y)dy|$$

$$= \int_{|y| \leq |x - y| < \eta} k(x - y)(D^\alpha f(y) - D^\alpha f(x))dy|.$$ 

By the mean value theorem of calculus we see that

$$|D^\alpha f(y) - D^\alpha f(x)| = \left| \sum_{j=1}^n D^{\alpha+\epsilon_j} f(y + \theta(y - x))(y_j - x_j) \right|$$

$$\leq C|x - y| \sum_{j=1}^n \frac{1}{(1 + |y + \theta(y - x)|)^r}$$

$$\leq C|x - y|$$

where $0 < \theta < 1$. Therefore by (1.2) we get

$$|D^\alpha K_\epsilon f(x) - D^\alpha K_\eta f(x)| \leq C \int_{|y| \leq |x - y| < \eta} |x - y|^{1-n}dy = C(\eta - \epsilon).$$

Hence $D^\alpha K_\epsilon f(x)$ converges uniformly on $R^n$ as $\epsilon$ tends to 0 for any $\alpha$. This implies that $Kf(x) \in C^\infty(R^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any $\alpha$. We complete the proof of the lemma.
The following lemma follows from Gauss's divergence theorem.

**Lemma 2.3.** Let $D$ be a bounded domain with $C^\infty$-boundary $\partial D$. Let $\mathbf{n}(x) = (\mathbf{n}_1(x), \ldots, \mathbf{n}_n(x))$ denote the outer unit normal to the boundary $\partial D$ at the point $x \in \partial D$. We assume that $g$ and $h$ have continuous partial derivatives on a neighborhood of the closure of $D$. Then

$$
\int_D g(x) D_j h(x) dx = \int_{\partial D} g(x) h(x) \mathbf{n}_j(x) dS(x) - \int_D D_j g(x) h(x) dx
$$

where $dS$ represents the surface element of $\partial D$.

Now we prove our main result.

**Theorem 2.4.** Let $0 < r < n$. If $f \in C^{\infty,r}(\mathbb{R}^n)$, then

$$
p_{\ell,r}(Kf) \leq C \left\{ \begin{array}{ll}
\left( \sum_{k=0}^{\ell-1} p_{k,r}(f) + p_{\ell+1,r}(f) \right), & \ell \geq 1 \\
p_{0,r}(f) + p_{1,r}(f), & \ell = 0,
\end{array} \right.
$$

and hence $Kf$ is a continuous linear operator on $C^{\infty,r}(\mathbb{R}^n)$.

**Proof.** Let $f \in C^{\infty,r}(\mathbb{R}^n)$. It follows from Lemma 2.2 that $Kf \in C^{\infty}(\mathbb{R}^n)$ and $D^\alpha(Kf)(x) = K(D^\alpha f)(x)$ for any $\alpha$. Let $|\alpha| = \ell$. We have

$$
D^\alpha Kf(x) = KD^\alpha f(x)
$$

$$
= \lim_{\epsilon \to 0} \int_{|x-y| \leq \max(|x|/2,1)} k(x-y) D^\alpha f(y) dy + \int_{|x-y| > \max(|x|/2,1)} k(x-y) D^\alpha f(y) dy
$$

$$
= K_1(D^\alpha f)(x) + K_2(D^\alpha f)(x).
$$

By (1.2) and (1.3) we obtain

$$
|K_1(D^\alpha f)(x)| = \left| \lim_{\epsilon \to 0} \int_{|x-y| \leq \max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x)) dy \right|
$$

$$
= \int_{|x-y| \leq \max(|x|/2,1)} k(x-y)(D^\alpha f(y) - D^\alpha f(x)) dy
$$

$$
\leq C \int_{|x-y| \leq \max(|x|/2,1)} \frac{|D^\alpha f(y) - D^\alpha f(x)|}{|x-y|^n} dy.
$$

Since $f \in C^{\infty,r}(\mathbb{R}^n)$, by the mean value theorem of calculus we obtain

$$
|D^\alpha f(y) - D^\alpha f(x)| = \left| \sum_{j=1}^{n} D^{\alpha+\epsilon_j} f(x + \theta(y-x))(y_j - x_j) \right|
$$

$$
\leq C \frac{|x-y|}{(1 + |x + \theta(y-x)|)^{r+1+\epsilon_1}} p_{\ell+1,r}(f)
$$
where $0 < \theta < 1$. Further, since $|x - y| \leq \max(|x|/2, 1)$ implies $1 + |x + \theta(y - x)| \geq (1 + |x|)/2$, we have

$$
(2.5) \quad |K_1(D^\alpha f)(x)| \leq C \frac{P_{t+1,r}(f)}{(1 + |x|)^{r+\ell+1}} \int_{|x-y| \leq \max(|x|/2, 1)} |x-y|^{1-n}dy
$$

$$
= C \frac{P_{t+1,r}(f)}{(1 + |x|)^{r+\ell+1}} \max(|x|/2, 1)
$$

$$
\leq C \frac{P_{t+1,r}(f)}{(1 + |x|)^{r+\ell}}.
$$

The multi-index $\varepsilon_j$ denotes the ordered $n$-tuple that has 1 in the $j$th spot and 0 everywhere else ($j = 1, \cdots, n$). In case $\ell \geq 1$, we let $\alpha = \varepsilon_{j_1} + \cdots + \varepsilon_{j_\ell}$. By Lemma 2.3 we have

$$
K_2(D^\alpha f)(x) = \lim_{M \to \infty} \int_{M > |x - y| > \max(|x|/2, 1)} k(x - y)D^\alpha f(y)dy
$$

$$
= \lim_{M \to \infty} \int_{\{y: |x - y| = M\}} k(x - y)D^{\alpha - \varepsilon_{j_1}} f(y)n_{j_1}(y)dS(y)
$$

$$
+ \sum_{k=2}^\ell \lim_{M \to \infty} \int_{\{y: |x - y| = M\}} D^{\varepsilon_{j_1} + \cdots + \varepsilon_{j_{k-1}}} k(x - y)D^{\alpha - \varepsilon_{j_1} - \cdots - \varepsilon_{j_k}} f(y)n_{j_k}(y)dS(y)
$$

$$
+ \int_{\{y: |x - y| = \max(|x|/2, 1)\}} k(x - y)D^{\alpha - \varepsilon_{j_1}} f(y)n_{j_1}(y)dS(y)
$$

$$
+ \sum_{k=2}^\ell \int_{\{y: |x - y| = \max(|x|/2, 1)\}} D^{\varepsilon_{j_1} + \cdots + \varepsilon_{j_{k-1}}} k(x - y)D^{\alpha - \varepsilon_{j_1} - \cdots - \varepsilon_{j_k}} f(y)n_{j_k}(y)dS(y)
$$

$$
+ \lim_{M \to \infty} \int_{M > |x - y| > \max(|x|/2, 1)} D^\alpha k(x - y)f(y)dy
$$

$$
= \lim_{M \to \infty} I_1^{1,M} + \sum_{k=2}^\ell \lim_{M \to \infty} I_k^{1,M}(x) + I_2^1(x) + \sum_{k=2}^\ell I_k^2(x) + I_3(x).
$$

In case $\ell = 0$, we have

$$
K_2(D^\alpha f)(x) = K_2 f(x) = I_3(x).
$$

By the condition $f \in C^{\cdot,r}(R^n)$ and (1.2) we have

$$
|I_1^{k,M}(x)| \leq C P_{t-k,r}(f) \int_{\{y: |x - y| = M\}} |x - y|^{-n-k+1}(1 + |y|)^{-r-\ell+k}dS(y),
$$

$$
|I_2^k(x)| \leq C P_{t-k,r}(f) \int_{\{y: |x - y| = \max(|x|/2, 1)\}} |x - y|^{-n-k+1}(1 + |y|)^{-r-\ell+k}dS(y)
$$
for $k = 1, 2, \cdots, \ell$, and

$$|I_3(x)| = \int_{\{y:|x-y| > \max(|x|/2,1)\}} D^\alpha k(x-y) f(y) dy \leq C \rho_0, r(f) \int_{\{y:|x-y| > \max(|x|/2,1)\}} |x-y|^{-n-\ell} (1 + |y|)^{-r} dy.$$ 

We may assume that $M \geq 2|x|$. Therefore, since $|x-y| \geq 2|x|$ implies $(1 + |x-y|)/2 \leq 1 + |y| \leq 3(1 + |x-y|)/2$, we obtain

$$|I_1^{k,M}(x)| \leq C \rho_{r, k, r}(f) M^{-n-k+1} (1 + M)^{-r-\ell+k} \int_{\{y:|x-y|=M\}} dS(y) = C \rho_{r, k, r} M^{-k} (1 + M)^{-r-\ell+k} \to 0 \quad (M \to \infty).$$

Since $|x-y| = \max(|x|/2, 1)$ implies $(1 + |x|)/2 \leq 1 + |y| \leq 3(1 + |x|)/2$, we get

$$|I_2^k(x)| \leq C \rho_{r, k, r}(f)(\max(|x|/2, 1))^{-n-k+1} (1 + |x|)^{-r-\ell+k}(\max(|x|/2, 1))^{n-1} \leq C \rho_{r, k, r}(1 + |x|)^{-r-\ell}.$$

Furthermore, since $0 < r < n$, Lemma 2.1 gives

$$|I_3(x)| \leq C \rho_{0, r}(f)(1 + |x|)^{-r-\ell}.$$

Thus

$$(2.6) \quad |K_2(D^\alpha f)(x)| \leq C (1 + |x|)^{-r-\ell} \sum_{k=1}^\ell \rho_{\ell-k, r}(f) = C (1 + |x|)^{-r-\ell} \sum_{k=0}^{\ell-1} \rho_{k, r}(f).$$

The estimates (2.5) and (2.6) give the theorem.

References


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