Chandrasekhar-Type Recursive Wiener Filter Using Covariance Information in Linear Discrete-Time Wide-Sense Stationary Stochastic Systems

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Abstract. This paper designs a Chandrasekhar-type recursive Wiener filter for the white observation noise in linear discrete-time wide-sense stationary stochastic systems. The system matrix in the state-space model of the signal, the crossvariance function of the state variable of the signal with the observed value, the observation matrix for the signal, the variance of the white observation noise and the observed value are assumed to be known. In particular, this paper extends the Chandrasekhar-type recursive Wiener filter for a scalar observation equation to the case of a vector observation equation. A characteristic of the Chandrasekhar-type filter is to calculate the filter gain directly by solving the recursive difference equations.
Keywords. Chandrasekhar-type filter, linear stochastic systems, covariance information, white observation noise, RLS Wiener filter

1. Introduction

In detection and estimation theory (Trees, 1968), the continuous-time estimation problems using the covariance information have been studied extensively. Correspondingly, the continuous-time recursive least-squares (RLS) estimation algorithms using the covariance information are derived (Nakamori, 1991, 1996a). Recently, the continuous-time Chandrasekhar-type RLS Wiener filter is proposed (Nakamori, 2000) for the white and white plus coloured observation noise. In Nakamori (1996b, 1997a, 1997b), the discrete-time RLS Wiener filter is designed for the white observation noise. In Nakamori et. al (2003), the discrete-time Chandrasekhar-type RLS Wiener filter is designed for the case of the scalar observation equation. In (Sugisaka, 2000), the Chandrasekhar-type recursive Wiener filter for the vector observation equation is proposed. However, in (Sugisaka, 2000), there seem to be errornous expressions in the derivation process of the Chandrasekhar-type filter.

In particular, this paper extends the Chandrasekhar-type RLS Wiener filter for the scalar observation equation to the case of the vector observation equation. Namely, this paper, in the case of the vector observation equation, newly designs the Chandrasekhar-type recursive Wiener filter for the white observation noise in linear discrete-time wide-sense stationary stochastic systems. The characteristic of the Chandrasekhar-type filter is to calculate the filter gain directly by solving the recursive difference equations. The filter necessitates the following information. (1) The system matrix. (2) The observation matrix for the signal from the state vector. (3) The crossvariance function of the state variable for the signal with the observed value. (4) The variance of white observation noise. (5) The observed value. The procedure to calculate the above quantities (1)-(4) from the observed value is studied in (Nakamori, 1997b). In Kalman filter, the information of the state-space model generating the signal process is necessary. For the p-dimensional discrete-time observation equation and the n-dimensinal state vector, the number of difference equations included in the current Chandrasekhar-type filter is $2n\times p+n$. Whereas the numbers of the difference equations in the RLS Wiener filter is $\frac{n(n+1)}{2}+n$. This indicates, for $p<\frac{n+1}{4}$, that the number of equations of the proposed Chandrasekhar-type recursive Wiener filter is less than that of the RLS Wiener filter.

2. Linear least-squares filtering problems

In this section, linear least-squares filtering problems using the covariance information are introduced for the white observation noise.

Let an n-dimensional discrete-time state-difference equation and a p-dimensional discrete-time observation equation be represented by

$$x(k+1)=Fx(k)+Bu(k),\ E[u(k)u(s)]=\sigma^2\delta_K(k-s),$$
$$y(k)=z(k)+v(k),\quad z(k)=Hx(k) \tag{1}$$

in linear wide-sense stationary stochastic systems. Here, $x(k)$ is a state vector, u(k) is a white noise input, $y(k)$ is an observed value and $z(k)$ is a zero-mean signal. Also, F is a system matrix, B is an $n \times r$ input matrix, H is a $p \times n$ observation matrix for the state vector $x(k)$ and $\delta_K(k-s)$ represents the Kronecker delta function.

Let $v(k)$ be the white observation noise with the variance R.

$$E[v(k)v^T(s)] = R\delta_K(k-s) \tag{2}$$

Here, it is assumed that the signal $z(\cdot)$ and the observation noise $v(\cdot)$ are uncorrelated mutually.

$$E[z(k)v^T(s)] = 0, \quad 0 \le s, t < \infty \tag{3}$$

Let us assume that the filtering estimate $\hat{x}(k,k)$ of the state variable $x(k)$ is given by

$$\hat{x}(k,k) = \sum_{i=1}^{k} h(k,i)y(i), \tag{4}$$

where $h(k,s)$ represents the $n \times p$ impulse response function. Minimizing the mean-square value of the filtering error $x(k) - \hat{x}(k,k)$

$$J = E[(x(k) - \hat{x}(k,k))^T (x(k) - \hat{x}(k,k))], \tag{5}$$

we obtain the Wiener-Hopf equation (Sage and Melsa, 1971):

$$E[x(k)y^T(s)] = \sum_{i=1}^{k} h(k,i)E[y(i)y^T(s)]. \tag{6}$$

Let $K_{xy}(k,s)$ represent the crosscovariance function of the state vector $x(k)$ with the observed value $y(s)$. If we substitute (1) into (6), and use (2) and (3), we obtain

$$h(k,s)R = K_{xy}(k,s) - \sum_{i=1}^{k} h(k,i)HK_{xy}(i,s). \tag{7}$$

(7) is the equation which the optimal impulse response function $h(k,s)$ satisfies in linear discrete-time least-squares filtering problem for the white observation noise.

3. Chandrasekhar-type recursive Wiener filtering algorithm

In [Theorem 1], the Chandrasekhar-type recursive Wiener algorithm for the linear filtering estimate is shown.

[Theorem 1]

Let $\hat{z}(k,k)$ represent the filtering estimate of the signal $z(k)$. Let the system matrix F, the observation matrix H, the crossvariance $K_{xy}(0)$ of the state variable with the observed value, the variance R of the white observation noise $v(k)$ and the observed value $y(k)$ be given in linear wide-sense stationary stochastic systems. Then the Chandrasekhar-type recursive Wiener algorithm for the filtering estimate $\hat{z}(k,k)$ consists of the equations (8)-(10) for the white observation noise.

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k) = H\hat{x}(k,k)$

$$\hat{x}(k,k) = F\hat{x}(k-1,k-1) + h(k,k)(y(k) - HF\hat{x}(k-1,k-1)) \tag{8}$$

Initial condition: $\hat{x}(1,1) = h(1,1)y(1)$

Filter gain for the filtering estimate of $x(k)$: $h(k,k)$

$$h(k,k) = (h(k-1,k-1) - Fh(k-1,1)h^T(k-1,1)F^T H^T)\,(I - HFh(k-1,1)h^T(k-1,1)F^T H^T)^{-1} \tag{9}$$

Initial condition:

$$\mathrm{h}(1,1) = \mathrm{K}_{\mathrm{xy}}(0)(\mathrm{R} + \mathrm{HK}_{\mathrm{xy}}(0))^{-1}$$

$$h(k,1) = Fh(k-1,1) - h(k,k)HFh(k-1,1) \tag{10}$$

Initial condition:

$$\mathrm{h}(1,1) = \mathrm{K}_{\mathrm{xy}}(0)(\mathrm{R} + \mathrm{HK}_{\mathrm{xy}}(0))^{-1}$$

The proof of [Theorem 1] is deferred to the Appendix. From the proof in the Appendix, it should be noted that (9) is valid for sufficiently large value of k, where the stationary property for the autocovariance function of the filtering estimate is satisfied. For the value of k relatively small, where the stationary property of the covariance function is not satisfied, (9) might be regarded as sub-optimal expression.

The conditions on the convergence of the Chandrasekhar-type recursive Wiener filter are that the system matrix F and the matrix $F - h(k,k)HF$ are stable and that the matrix $I - HFh(K-1,1)h^T(k-1,1)F^T H^T$ is positive definite.

It is notified that (8) represents the innovations state-space model for the filtering estimate $\hat{x}(k,k)$ of the state variable $x(k)$. As shown in (9), in the Chandrasekhar-type filtering algorithm, the filter gain is directly updated.

Now, let us compare the Chandrasekhar-type recursive Wiener filter in [Theorem 1] with the RLS Wiener filter (Nakamori, 1996b) in [Theorem 2] using the covariance information.

[Theorem 2]

Let F be the system matrix of order n in the state-space model for the signal $z(k)$. Let H be the observation matrix for $z(k)$. Let $K_{xy}(k,k)\,(=K_{xy}(0))$ be the crossvariance function of the state vector $x(k)$ with the observed value $y(k)$. Let $\hat{z}(k,k)$ represent the filtering estimate of $z(k)$. Then the RLS Wiener algorithm for the filtering estimate $\hat{z}(k,k)$ consists of the following equations (11)-(13) for white observation noise.

Filtering estimate of the signal $z(k)$:

$$\hat{z}(k,k) = H\hat{x}(k,k)$$
$$\hat{x}(k,k) = F\hat{x}(k-1,k-1)+$$
$$h(k,k)(y(k)-HF\hat{x}(k-1,k-1)),\ \hat{x}(0,0)=0 \tag{11}$$

Filter gain for the filtering estimate of $x(k)$: $h(k,k)$

$$h(k,k) = (K_{xy}(k,k)-FS(k-1)F^T H^T)$$
$$(R+HK_{xy}(k,k)-HFS(k-1)F^T H^T)^{-1} \tag{12}$$

Autovariance function of the filtering estimate $\hat{x}(k,k)$: a square matrix $S(k)$ of order n.

$$S(k) = FS(k-1)F^T + h(k,k)(K_{xy}^T(k,k)-HFS(k-1)F^T),$$
$$S(0)=0 \tag{13}$$

The number of the difference equations included in the current Chandrasekhar-type recursive Wiener filter is $2n\times p+n$. The number of the difference equations in the filter of [Theorem 2] is $\frac{n(n+1)}{2}+n$. This means, for $p<\frac{n+1}{4}$, that the number of equations of the proposed Chandrasekhar-type recursive filter is less than that of the RLS Wiener filter of [Theorem 2] in linear discrete-time wide-sense stationary stochastic systems.

4. A numerical simulation example

Let the observed value y(k) be given by a scalar observation equation

$$y(k) = Hx(k) + v(k),\ z(k) = Hx(k),$$
$$x(k) = [x_1(k) \quad x_2(k) \quad \cdots \quad x_n(k)]^T,\ z(k) = x_1(k). \tag{14}$$

Let us consider to estimate a vowel signal spoken by one of the authors. Its phonetic symbol is written as "/i:/". The sampling frequency of the voice signal $z(k)$ is 10.025(kHz). The autocovariance data of the signal are calculated in terms of the N(=5,000) sampled signal data. Let the stochastic process of the vowel signal be modeled in terms of the AR process of order n.

$$z(k) = -a_1 z(k-1) - a_2 z(k-2) - \cdots - a_n z(k-n) + e(k), \quad E[e(k)e(s)] = \sigma^2 \delta_K(k-s) \tag{15}$$

Let $K_z(i)$, $i = 1, \ldots, n$, represent the autocovariance data of the signal $z(k)$. The AR parameters a_i, $i = 1, \ldots, n$, are calculated by the Yule-Walker equations.

$$\begin{bmatrix} K_z(0) & K_z(1) & \cdots & \cdots & K_z(n-1) \\ K_z(1) & K_z(0) & \cdots & \cdots & K_z(n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_z(n-2) & \cdots & \cdots & K_z(0) & K_z(1) \\ K_z(n-1) & K_z(n-2) & \cdots & \cdots & K_z(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} -K_z(1) \\ -K_z(2) \\ \vdots \\ -K_z(n-1) \\ -K_z(n) \end{bmatrix} \tag{16}$$

By refering to (Nakamori, 1997a, b) the $1 \times n$ observation vector H, the crossvariance function $K_{xy}(k,k)(=K_{xy}(0))$, the system matrix F and the autovariance function $K_x(k, k)$ are obtained in terms of the autocovariance data of the signal as follows:

$$H = [1 \quad 0 \quad \cdots \quad 0], \tag{17}$$

$$K_{xy}(k,k) = [K_z(0) \quad K_z(1) \quad \cdots \quad K_z(n-1) \quad K_z(n)]^T, \tag{18}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, \tag{19}$$

$$K_x(k,k) = \begin{bmatrix} K_z(0) & K_z(1) & \cdots & \cdots & K_z(n-1) \\ K_z(1) & K_z(0) & \cdots & \cdots & K_z(n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_z(n-2) & \cdots & \cdots & K_z(0) & K_z(1) \\ K_z(n-1) & K_z(n-2) & \cdots & \cdots & K_z(0) \end{bmatrix}. \tag{20}$$

$K_x(k, k)$ is also called the Hankel matrix (Akaike, 1974). As indicated in Nakamori (1997b), a finite dimensional realization for $z(k)$ exists if and only if the rank of the Hankel matrix is n.

Let the order of the AR process be $n=10$. If we substitute the estimates of F, H and $K_{xy}(0)(=K_x(0)H^T)$ into the proposed estimation algorithm of [Theorem 1], the filtering estimate $\hat{z}(k,k)$ of $z(k)$ is calculated. **Fig.1** illustrates the signal $z(k)(=x_1(k))$, and the filtering estimate $\hat{z}(k,k)$ vs. k for the S/N ratio (SNR) 5 [dB]. Here, the variance of the signal process is 1.0873. **Fig.2** illustrates the mean-square values (MSVs) of the filtering errors by the Chandrasekhar-type recursive Wiener filter in [Theorem 1] and by the RLS Wiener filter in [Theorem 2] vs. SNR. Here, the MSV is calculated by $\dfrac{\sum_{k=1}^{300}(z(k)-\hat{z}(k,k))^2}{300}$. From **Fig.2**, it is shown that the proposed Chandrasekhar-type recursive Wiener filter has almost the same estimation accuracy with the RLS Wiener filter in [Theorem 2]. The computation time of the Chandrasekhar-type filter is just the 1/3 of that by the RLS Wiener filter.

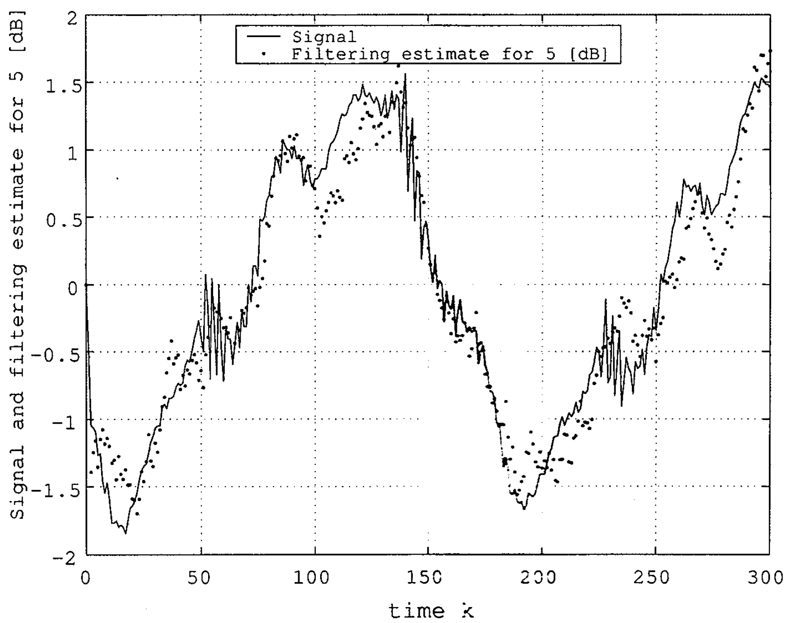

Fig.1 Signal $z(k)(=x_1(k))$ and the filtering estimate $\hat{z}(k,k)$ vs. k for the S/N ratio (SNR) $5[dB]$.

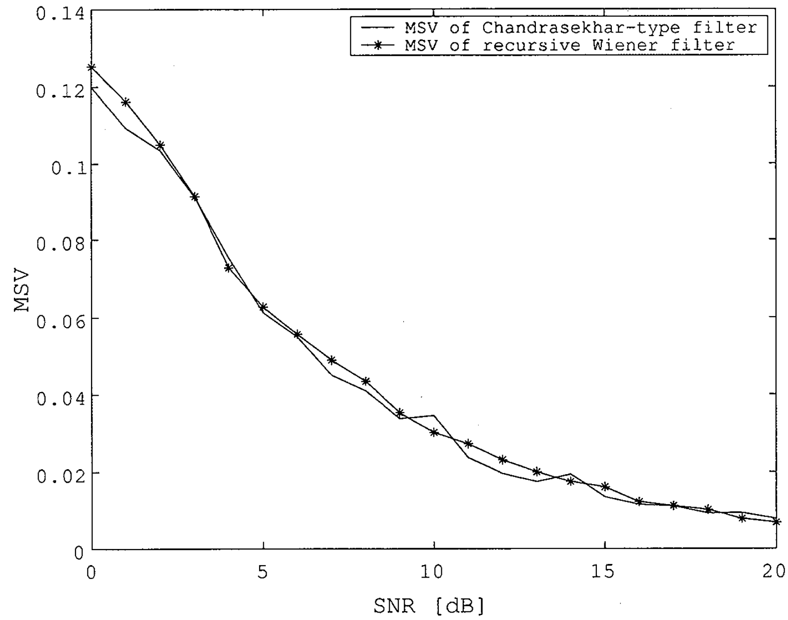

Fig.2 Mean-square values of the filtering errors by the RLS Chandrasekhar-type filter in [Theorem 1] and the RLS Wiener filter in [Theorem 2] vs. SNR.

5. Conclusions

In this paper, in the case of the vector observation equation, the Chandrasekhar-type recursive Wiener filter has been devised for white observation noise in linear discrete-time wide-sense stationary stochastic systems. The proposed Chandrasekhar-type filter has almost the same estimation accuracy with the RLS Wiener filter. Also, for $p < \frac{n+1}{4}$, the number of equations of the proposed Chandrasekhar-type recursive Wiener filter is less than that of the RLS Wiener filter of [Theorem 2].

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APPENDIX. Proof of **[Theorem 1]**

In (7), putting $k \to k-1$ and $s \to s-1$, we have

$$h(k-1,s-1)R = K_{xy}(k-1,s-1) - \sum_{i=1}^{k-1} h(k-1,i)HK_{xy}(i,s-1)$$

$$= K_{xy}(k-1,s-1) - \sum_{i=2}^{k} h(k-1,i-1)HK_{xy}(i,s). \tag{A.1}$$

Subtracting (A.1) from (7), we obtain

$$(h(k,s)-h(k-1,s-1))R = K_{xy}(k,s) - K_{xy}(k-1,s-1) - h(k,1)HK_{xy}(1,s) - \sum_{i=2}^{k}(h(k,i)-h(k-1,i-1))HK_{xy}(i,s). \tag{A.2}$$

From the wide-sense stationarity for the crosscovariance function, $K_{xy}(k,s) = K_{xy}(k-1,s-1) = K_{xy}(k-s)$ is valid. Hence, (A.2) becomes

$$(h(k,s)-h(k-1,s-1))R = -h(k,1)HK_{xy}(1,s) - \sum_{i=2}^{k}(h(k,i)-h(k-1,i-1))HK_{xy}(i,s). \tag{A.3}$$

Introducing the function $J(k,s)$ which satisfies

$$J(k-1,s-1)R = K_{xy}(1,s) - \sum_{i=2}^{k} J(k-1,i-1)HK_{xy}(i,s)$$

$$= K_{xy}(1,s) - \sum_{i=1}^{k-1} J(k-1,i)HK_{xy}(i,s-1), \tag{A.4}$$

from (A.3) and (A.4), we obtain

$$h(k,s)-h(k-1,s-1) = -h(k,1)HJ(k-1,s-1). \tag{A.5}$$

Pre-multiplying H to both sides of (A.4), we have

$$HJ(k-1,s-1)R = HK_x(1,s-1)F^T H^T - \sum_{i=2}^{k} HJ(k-1,i-1)HK_{xy}(i,s). \tag{A.6}$$

From (A.1) we have

$$HFh(k-1,s-1)R = HFK_{xy}(k-1,s-1) - \sum_{i=2}^{k} HFh(k-1,i-1)HK_{xy}(i,s). \tag{A.7}$$

Surely, $HFh(k-1,s-1)R$ is the estimation error covariance function of $Hx(\bullet)$, $HFE[x(k-1)x^T(s-1)]H^T - HFE[\hat{x}(k-1,k-1)\hat{x}^T(s-1,s-1)]H^T$. For sufficient large value of k, the second terms in (A.6) and (A.7) might approach the stationary covariance functions. Since the autocovariance function of the observed value $HK_{xy}(\bullet,\bullet)$ is the symmetric function, from (A.6) and (A.7) with the wide-sense stationary property for the covariance functions $HJ(k-1,s-1)R$ and $HFh(k-1,s-1)R$, we see that

$$(HJ(k-1,s-1)|_{s=k})^T = HFh(k-1,s-1)_{s=2} = HFh(k-1,1). \tag{A.8}$$

Hence, putting $s \to k$ in (A.5), we obtain

$$h(k,k)-h(k-1,k-1)=-h(k,1)h^T(k-1,1)F^TH^T. \tag{A.9}$$

The initial value of h(k,k) at $k=1$ is given by $h(1,1) = K_{xy}(0)(R+HK_{xy}(0))^{-1}$ from (7). For the value of k relatively small, where the stationary property of the covariance function is not satisfied, (A.9) might be regarded as sub-optimal expression. (A.9) coincides with the equation derived by Sugisaka (2000).

Now, from (7), the function $h(k,s)$ satisfies

$$h(k,s)R=K_{xy}(k,s)-\sum_{i=1}^{k}h(k,i)HK_{xy}(i,s). \tag{A.10}$$

Subtracting the equation by putting $k \to k-1$ in (A.10) from (A.10), we have

$$(h(k,s)-h(k-1,s))R=K_{xy}(k,s)-K_{xy}(k-1,s)-h(k,k)HK_{xy}(k,s)-$$
$$\sum_{i=1}^{k-1}(h(k,i)-h(k-1,i))HK_{xy}(i,s). \tag{A.11}$$

Comparing the equation obtained by substituting $K_{xy}(k,s)=FK_{xy}(k-1,s)$ into (A.11) with (A.10), we obtain

$$h(k,s)-h(k-1,s)=Fh(k-1,s)-h(k-1,s)-h(k,k)Fh(k-1,s). \tag{A.12}$$

Hence, $h(k,s)$ is updated by

$$h(k,s)=Fh(k-1,s)-h(k,k)HFh(k-1,s). \tag{A.13}$$

Putting $s=1$ in (A.13), we obtain

$$h(k,1)=Fh(k-1,1)-h(k,k)HFh(k-1,1). \tag{A.14}$$

The initial value on the difference equation (A.14) at $k=1$ is $h(1,1) = K_{xy}(0)(R+HK_{xy}(0))^{-1}$ from (7).

From (A.9) and (A.14), we obtain

$$h(k,k)=(h(k-1,k-1)-Fh(k-1,1)h^T(k-1,1)F^TH^T)$$
$$(I-HFh(k-1,1)h^T(k-1,1)F^TH^T)^{-1}. \tag{A.15}$$

The filtering estimate $\hat{x}(k,k)$ is rewritten as

$$\hat{x}(k,k)=h(k,k)y(k)+\sum_{i=1}^{k-1}h(k,i)y(i)$$
$$=F\hat{x}(k-1,k-1)+h(k,k)(y(k)-HF\hat{x}(k-1,k-1)) \tag{A.16}$$

from (4) and (A.13). The initial value on the difference equation (A.17) for $\hat{x}(k,k)$ at $k=1$ is $\hat{x}(1,1)=h(1,1)y(1)$ from (4) (Q.E.D.).

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