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QUASI-BIVARIANT CHERN CLASSES OBTAINED BY RESOLUTIONS OF SINGULARITIES

SHOJI YOKURA*

Dedicated to Professor Shoji Tsuboi on the occasion of his sixtieth birthday

ABSTRACT. The bivariant theory was introduced by W. Fulton and R. MacPherson to unify both covariant and contravariant theories. They posed the problem of unique existence of a bivariant Chern class and J.-P. Brasselet showed the existence of a bivariant Chern class in the category of analytic varieties with cellular morphisms. The uniqueness problem is still open. In this paper, without assuming "cellularness" of morphisms, using resolution of singularites, we construct a "quasi-bivariant" theory $\mathbb{F}_\infty$ of constructible functions and a "quasi-bivariant" homology theory $\mathbb{H}_\infty$ and we show that there exists a unique "quasi-Grothendieck" transformation $\gamma_\infty : \mathbb{F}_\infty \to \mathbb{H}_\infty$ satisfying that γ_∞ for morphisms to a point becomes the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$. We also show that if a bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$ exists, then $\gamma : \mathbb{F} \to \mathbb{H}$ is "uniquely embedded" into $\gamma_\infty : \mathbb{F}_\infty \to \mathbb{H}_\infty$.

§1 INTRODUCTION

W. Fulton and R. MacPherson [FM] (also see [F]) introduced the notion of bivariant theory and they conjectured the existence of a bivariant Chern class, i.e., *a Grothendieck transformation* from the bivariant theory $\mathbb{F}$ of constructible functions to the bivariant homology theory $\mathbb{H}$ satisfying that for a morphism from a nonsingular variety X to a point the value of the characteristic function $1\!\!1_X$ of X is the Poincaré dual of the total Chern class of X. If such a bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$ exists, then restricted to morphisms to a point the bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$ becomes *the Chern-Schwartz-MacPherson transformation* $c_* : F \to H_*$ (see [BS], [Ke], [Kw1], [Ma], [Sa1], [Sc1, Sc2]etc.). The conjecture was solved by J.-P. Brasselet [B] for the category of complex analytic varieties whose morphisms are assumed to be *cellular.* But the problem of whether "cellularness" of morphisms can be dropped (note that it is a "folklore" that any analytic morphism is perhaps celluar) and the problem of uniqueness have been unresolved since then.

In [Y1] we have showed that for a morphism with nonsingular target variety the bivariant Chern class is uniquely determined and furthermore we have showed that the

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bivariant Chern class of a bivariant constructible function is expressed as the Chern-Schwartz-MacPherson class of the bivariant constructible function followed by capping with the pullback of the total Segre class of the nonsingular target variety. This "twisted" Chern-Schwartz-MacPherson class is called the *Ginzburg-Chern class*, since this class was already treated (implicitly) in Ginzburg's paper [G1] (cf. [G2] also). Note that this result of [Y1] generalizes the result (due to J. Zhou [Z1, Z2]) that for a morphism with a traget variety being a nonsingular curve the bivariant Chern classes constructed by J.-P. Brasselet [B] and C.Sabbah [Sa2] are both identical. Furthermore, in [Y2, Y3, Y4, Y5] we have studied on Ginzburg-Chern classes, in particular the problem of whether Ginzburg-Chern classes can be captured as a Grothendieck transformation. For the definition of the Ginzburg-Chern class it is clearly essential and crucial that the target variety of a given morphism is nonsingular, otherwise one cannot define such a class. In this paper, for a morphism with a singular target variety, we use resolution of singularites and construct "quasi-bivariant" Chern classes, which do not necessarily behave as in the usual bivariant theory; that is why it is called "quasi-bivariant".

§2 Bivariant Theories

In this section we quickly recall some basic things of the Bivariant Theory which we need in this paper.

A bivariant theory $\mathbb{B}$ on a category $\mathcal{C}$ with values in an abelian category is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category $\mathcal{C}$ a graded abelian group

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations:
(Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.
(Pushforward operations): For morphisms $f : X \to Y$ and $g : Y \to Z$ with f proper, the pushforward operation

$$f_\bigstar : \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.
(Pullback operations): For a fiber square

$$\begin{CD}
X' @>{g'}>> X \\
@V{f'}VV @VV{f}V \\
Y' @>{g}>> Y,
\end{CD}$$

the pullback operation

$$g^{\star} : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven axioms (see [FM, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,
(B-6) pushforward and pullback commute, and
(B-7) projection formula.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{C}$. Then a *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism $X \to Y$ in the category $\mathcal{C}$, which preserves the above three basic operations:

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_{\star}\alpha) = f_{\star}\gamma(\alpha)$, and
(iii) $\gamma(g^{\star}\alpha) = g^{\star}\gamma(\alpha)$.

Fulton-MacPherson's bivariant theory $\mathbb{F}(X \xrightarrow{f} Y)$ of constructible functions consists of all the constructible functions on X which satisfy the local Euler condition with respect to f. Here a constructible function $\alpha \in F(X)$ is said to satisfy the *local Euler condition with respect to* f if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the equality $\alpha(x) = \chi(B_\epsilon \cap f^{-1}(z); \alpha)$ holds, where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$ (cf. [B], [Sa2]). The three operations on $\mathbb{F}$ are defined as follows:

(i) product operation:

$$\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z)$$

is defined by $\alpha \bullet \beta := \alpha \cdot f^*\beta$,

(ii) the pushforward operation:

$$f_{\star} : \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$$

is the usual pushforward f_*, i.e., $f_{\star}(\alpha)(y) := \int c_*(\alpha|_{f^{-1}(y)})$ and (iii)the pullback operation: for a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^\star : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$$

is the functional pullback g'^*, i.e., $g^\star(\alpha)(x') := \alpha(g'(x'))$.

Note that $\mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ consists of all locally constant functions and $\mathbb{F}(X \to pt) = F(X)$.

Let $\mathbb{H}$ be Fulton-MacPherson's bivariant homology theory, constructed from the cohomology theory. For a morphism $f : X \to Y$, choose a morphism $\phi : X \to \mathbb{R}^n$ such that $\Phi := (f, \phi) : X \to Y \times \mathbb{R}^n$ is a closed embedding. Then the i-th bivariant homology group $\mathbb{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_\phi),$$

where X_ϕ is defined to be the image of the morphism $\Phi = (f, \phi)$. The definition is independent of the choice of ϕ. Note that instead of taking the Euclidean space $\mathbb{R}^n$ we can take a manifold M so that $i : X \to M$ is a closed embedding and then consider the graph embedding $f \times i : X \to Y \times M$. See [FM, §3.1] for more details of $\mathbb{H}$. In particular, note that if Y is nonsingular, $\mathbb{H}(X \to Y)$ is isomorphic to the homology group $H_*(X)$ of the source variety X by the Alexander duality isomorphism.

§3 Bivariant Chern classes for morphisms with nonsingular target varieties

In [FM] W. Fulton and R. MacPherson conjectured and later J.-P. Brasselet [B] affirmatively solved the following

Theorem (3.1). *(J.-P. Brasselet) For the category of complex analytic varieties with celluar morphisms, there exists a Grothendieck transformation*

$$\gamma : \mathbb{F} \to \mathbb{H}$$

such that for a morphism $f : X \to pt$ from a nonsingular variety X to a point pt and the bivariant constructible function $1\!\!1_f := 1\!\!1_X$ the following normalization condition holds:

$$\gamma(1\!\!1_f) = c(TX) \cap [X].$$

In [Z 1, Z 2] J. Zhou showed that the bivariant Chern classes constructed by J.-P. Brasselet [B] and by C. Sabbah [Sa 2] are identical in the case when the target variety is a nonsingular curve. And the present author showed the following uniqueness theorem of bivariant Chern classes for morphisms whose target varieties are nonsingular and *of any dimension*:

Theorem (3.2). *([Y1, Theorem (3.7)]) If there exists a bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$, then it is unique when restricted to morphisms whose target varieties are nonsingular; explicitly, for a morphism $f : X \to Y$ with Y nonsingular and for any bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ the bivariant Chern class $\gamma(\alpha)$ is expressed by*

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha)$$

where $s(TY)$ is the Segre class of the tangent bundle. □

The "twisted" Chern-Schwartz-MacPherson homology class $f^*s(TY) \cap c_*$ is named *Ginzburg-Chern class* [G1, G2] (also see [CG]). Let us denote $f^*s(TY) \cap c_*$ simply by γ^{Gin}.

Thus, conversely, it is quite natural to ask *if the correspondence defined by Ginzburg-Chern class*

$$\gamma^{\mathrm{Gin}} : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y)$$

becomes a Grothendieck transformation.

In [Y3] we dealt with a very naïve and more tractable situation where we consider (smooth) morphisms of nonsingular varieties and in [Y4] we made our category much broader and the morphisms which we consider are any morphisms with *only being required that the target varieties are nonsingular.* Another requirement in [Y4] is that the pullback homomorphisms are considered only for *smooth* morphisms, not for *any* morphism. In [Y5] we dropped this extra requirement.

First we recall the following theorems:

Theorem (3.3). *([Y4]) For a morphism $f : X \to Y$ with Y nonsingular, we define*

$$\mathbb{GF}(X \xrightarrow{f} Y)$$

to be the set of all constructible functions $\alpha \in F(X)$ satisfying that for any fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

with Y' also nonsingular and for any constructible function $\beta' \in F(Y')$ the following equality holds:

$$\gamma^{\mathrm{Gin}}(g^{\bigstar}\alpha \bullet \beta') = \gamma^{\mathrm{Gin}}(g^{\bigstar}\alpha) \bullet \gamma^{\mathrm{Gin}}(\beta')$$

Then $\mathbb{GF}$ becomes a bivariant theory of constructible functions with the same operations as in the bivariant theory $\mathbb{F}$ of constructible functions. Furthermore we have that $\mathbb{GF}(X \to pt) = F(X)$. Here it should be noted that the bivariant pullback is considered only for morpshims of nonsingular varieties. □

Theorem (3.4). *([Y4]) For morphisms whose target varieties are nonsingular,*

$$\gamma^{\mathrm{Gin}} : \mathbb{GF} \to \mathbb{H}$$

is the unique Grothendieck transformation satisfying that γ^{Gin} for morphisms to a point is the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$, except for that $g^{\star}\gamma^{\mathrm{Gin}} = \gamma^{\mathrm{Gin}} g^{\star}$ for a smooth morphism g of nonsingular varieties. □

With the bivariant theory $\mathbb{GF}$, it is not clear at all whether $g^{\star}\gamma^{\mathrm{Gin}} = \gamma^{\mathrm{Gin}} g^{\star}$ holds for *any* morphism of nonsingular varieties. However, by taking into definition its compatibility with pullback homomorphism, we showed the following theorem.

Theorem (3.5). *([Y5]) For a morphism $f : X \to Y$ with Y nonsingular we define $\widetilde{\mathbb{GF}}(X \xrightarrow{f} Y)$ to be the set of all constructible functions $\alpha \in F(X)$ satisfying the following two conditions ($\sharp$) and ($\flat$) : for any fiber square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

with Y' nonsingular

($\sharp$) the following equality holds for any constructible function $\beta' \in F(Y')$:

$$\gamma^{\mathrm{Gin}}(g^{\star}\alpha \bullet \beta') = \gamma^{\mathrm{Gin}}(g^{\star}\alpha) \bullet \gamma^{\mathrm{Gin}}(\beta'),$$

($\flat$)

$$\gamma^{\mathrm{Gin}}(g^{\star}\alpha) = g^{\star}\gamma^{\mathrm{Gin}}(\alpha).$$

Then $\widetilde{\mathbb{GF}}$ becomes a bivariant theory with the same operations as in $\mathbb{F}$ and furthermore the Ginzburg-Chern class

$$\gamma^{\mathrm{Gin}} : \widetilde{\mathbb{GF}} \to \mathbb{H}$$

becomes the unique Grothendieck transformation satisfying that γ^{Gin} for morphisms to a point is the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$. And also $\widetilde{\mathbb{GF}}(X \to pt) = F(X)$. □

If we let us denote the set of all constructible functions $\alpha \in F(X)$ satisfying the condition ($\flat$) by $F^{\flat}(X)$ we have

$$\widetilde{\mathbb{GF}}(X \to Y) = \mathbb{GF}(X \to Y) \cap F^{\flat}(X).$$

To prove $\mathbb{GF}(X \to pt) = F(X)$ and $\widetilde{\mathbb{GF}}(X \to pt) = F(X)$, we need the cross product formula of the Chern-Schwartz-MacPherson classes due to M. Kwieciński [Kw2] (cf. [KY]): $c_*(\alpha \times \beta) = c_*(\alpha) \times c_*(\beta)$.

Remark (3.6). It is easy to see that the inclusion $\mathbb{F}(X \xrightarrow{f} Y) \subset \widetilde{\mathbb{GF}}(X \xrightarrow{f} Y)$ is equivalent to the existence of the bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$. The Brasselet theorem [B] and the uniqueness theorem of the bivariant Chern class [Y1] imply that if $f : X \to Y$ is cellular and Y is nonsingular, we have $\mathbb{F}(X \xrightarrow{f} Y) \subset \widetilde{\mathbb{GF}}(X \xrightarrow{f} Y)$.

Remark (3.7). In [Sch] Jörg Schürmann has recently generalized our construction of the bivariant theory $\widetilde{\mathbb{GF}}$ and the bivariant Chern class $\gamma : \widetilde{\mathbb{GF}} \to \mathbb{H}$.

Remark (3.8). In [E] L. Ernström considered the unique existence problem of *an operational bivariant Chern class* $\gamma^{\mathrm{op}} : \mathbb{F} \to \mathbb{A}$, where $\mathbb{A}$ is the so-called *bivariant intersection theory* or *operational bivariant theory of Chow homology groups* (see [FM, §§8-9], and [F, Chapter 17]). And his work [E] was modified and completed by [EY1] and [EY2].

§4 Quasi-bivariant Chern classes for arbitrary morphisms

Theorem (3.4) is the bivariant Chern class theorem in the case of morphisms whose target varieties are nonsingular. In this section, still using the Ginzburg-Chern classes, we will try to treat the general case when target varieties are possibly singular.

For a morphism $f : X \to Y$ with Y being singular, consider a bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$. Let $\gamma : \mathbb{F} \to \mathbb{H}$ be a bivariant Chern class. At the moment, we do not know an explicit description of the bivariant Chern class $\gamma(\alpha)$ of the bivariant constructible function α. Let $\pi : \widetilde{Y} \to Y$ be a resolution of singularities and consider the following fiber square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\pi}} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow[\pi]{} & Y. \end{array}$$

Then $\widetilde{\pi}^*\alpha = \pi^\bigstar\alpha \in \mathbb{F}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y})$. Since the target variety $\widetilde{Y}$ is nonsingular, it follows from Theorem (3.1) that we have

$$\pi^\bigstar\gamma(\alpha) = \gamma(\pi^\bigstar\alpha) = \widetilde{f}^* s(T\widetilde{Y}) \cap c_*(\widetilde{\pi}\alpha).$$

However, since the resolution of singularities is not unique, the above bivariant class $\pi^\bigstar\gamma(\alpha)$ is not uniquely determined and it does depend on the choice of the resolution $\pi : \widetilde{Y} \to Y$. Our original motivation of the present work was to try to view all the different bivariant classes $\pi^\bigstar\gamma(\alpha)$ constructed by using the resolutions of singularities as "the same" or "equivalent" in some sense.

Let Y be a possibly singular variety and let $\mathcal{R}_Y$ be the collection $\{(Y', g)\}$ of resolution of singularities $g : Y' \to Y$ of Y, where Y' is nonsingular and $g|_{Y' \setminus g^{-1}(Y_{\mathrm{sing}})} : Y' \setminus g^{-1}(Y_{\mathrm{sing}}) \to Y \setminus Y_{\mathrm{sing}}$ is an isomorphism with Y_{sing} denoting the singular set of Y. When Y is nonsingular, $\mathcal{R}_Y$ is defined to be just $\{(Y, \mathrm{id}_Y)\}$. The reason for this requirement is that otherwise $\mathcal{R}_Y$ consists of all automorphisms of Y, which is not necessary for our purpose.

For two elements $g_1 : Y_1 \to Y$, $g_2 : Y_2 \to Y$ of $\mathcal{R}_Y$ we define the order $\leq$ as follows: $g_1 \leq g_2$ if and only if there exists a morphism $g_{12} : Y_2 \to Y_1$ such that $g_2 = g_1 \circ g_{12}$.

Proposition (4.1). *For a possibly singular variety Y, the ordered set $(\mathcal{R}_Y, \leq)$ is a directed set.*

Proof. Let $g_1 : Y_1 \to Y$ and $g_2 : Y_2 \to Y$ be two resolution of singularities of Y. We want to show that there exists a resolution of singularities $g_3 : Y_3 \to Y$ such that $g_1 \leq g_3$ and $g_2 \leq g_3$. Consider the fiber product

$$\begin{array}{ccc} Y_1 \times_Y Y_2 & \xrightarrow{\widetilde{g_1}} & Y_2 \\ {\scriptstyle \widetilde{g_2}}\downarrow & & \downarrow{\scriptstyle g_2} \\ Y_1 & \xrightarrow[g_1]{} & Y. \end{array}$$

Note that the fiber product $Y_1 \times_Y Y_2$ is not necessarily nonsingular even if Y_1 and Y_2 are nonsingular. So we consider a resolution of singularities of $Y_1 \times_Y Y_2$:

$$\pi : \widetilde{Y_1 \times_Y Y_2} \to Y_1 \times_Y Y_2.$$

Let us set Y_3 to be $\widetilde{Y_1 \times_Y Y_2}$, and we set $g_3 : Y_3 \to Y$ to be the composite

$$g_3 := g_1 \circ \widetilde{g_2} \circ \pi = g_2 \circ \widetilde{g_1} \circ \pi.$$

Which means that $g_1 \leq g_3$ and $g_2 \leq g_3$. $\square$

Let $f : X \to Y$ be a morphism of two possibly singular varieties. For a resolution of singularities $g : Y' \to Y$ in $\mathcal{R}_Y$ we consider the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

and the bivariant group

$$\mathbb{F}_g(X \xrightarrow{f} Y) := \widetilde{\mathbb{GF}}(X' \xrightarrow{f'} Y').$$

Furthermore for any ordered pair $g_1 < g_2$ in $\mathcal{R}_Y$, i.e., the composite $g_2 : Y_2 \xrightarrow{g_{12}} Y_1 \xrightarrow{g_1} Y$, we consider the fiber squares:

$$\begin{array}{ccccc} X_2 & \xrightarrow{g_{12}'} & X_1 & \xrightarrow{g_1'} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y_2 & \xrightarrow[g_{12}]{} & Y_1 & \xrightarrow[g_1]{} & Y, \end{array}$$

from which we get the homomorphism

$$g_{12}^{\bigstar} : \widetilde{\mathbb{GF}}(X_1 \xrightarrow{f'} Y_1) \to \widetilde{\mathbb{GF}}(X_2 \xrightarrow{f''} Y_2),$$

which is denoted by

$$\mathbb{F}_{g_1, g_2} : \mathbb{F}_{g_1}(X \xrightarrow{f} Y) \to \mathbb{F}_{g_2}(X \xrightarrow{f} Y).$$

Then we have the following proposition:

Proposition (4.2). *For a morphism $f : X \to Y$ of possibly singular varieties, the following system*

$$\left\{ \mathbb{F}_g(X \xrightarrow{f} Y), \mathbb{F}_{g_1,g_2} \right\}$$

is an inductive system of abelian groups.

Proof. This is nothing but the functoriality of the pullback homomorphism

$$g_{12}^{\bigstar} : \widetilde{\mathbb{GF}}(X_1 \xrightarrow{f'} Y_1) \to \widetilde{\mathbb{GF}}(X_2 \xrightarrow{f''} Y_2)$$

, i.e., Axiom (B-3). □

We define

$$\mathbb{F}_\infty(X \xrightarrow{f} Y) := \varinjlim_{(Y',g)\in\mathcal{R}_Y} \mathbb{F}_g(X \xrightarrow{f} Y) = \varinjlim_{(Y',g)\in\mathcal{R}_Y} \widetilde{\mathbb{GF}}(X' \xrightarrow{f'} Y').$$

Similarly we define the corresponding homology version as follows:

$$\mathbb{H}_\infty(X \xrightarrow{f} Y) := \varinjlim_{(Y',g)\in\mathcal{R}_Y} \mathbb{H}_g(X \xrightarrow{f} Y) = \varinjlim_{(Y',g)\in\mathcal{R}_Y} \mathbb{H}(X' \xrightarrow{f'} Y').$$

For any resolution of singularities $\nu : \widetilde{Y} \to Y$ we have the fiber square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\nu}} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{\nu} & Y \end{array}$$

and $\mathbb{F}_\nu(X \xrightarrow{f} Y) = \widetilde{\mathbb{GF}}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y})$ and $\mathbb{H}_\nu(X \xrightarrow{f} Y) = \mathbb{H}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y})$. Thus, by the definition of $\mathbb{F}_\infty$ and $\mathbb{H}_\infty$, we have the canonical homomorphisms

$$\mathbb{F}^\nu : \widetilde{\mathbb{GF}}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) \to \mathbb{F}_\infty(X \xrightarrow{f} Y), \qquad \mathbb{H}^\nu : \mathbb{H}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) \to \mathbb{H}_\infty(X \xrightarrow{f} Y),$$

which are defined, respectively, by $\mathbb{F}^\nu(\alpha) := [\alpha]$ the equivalence class of α and $\mathbb{H}^\nu(\widetilde{x}) := [\widetilde{x}]$ the equivalence class of $\widetilde{x}$.

The above groups cannot in general become a bivariant theory in the sense of Fulton and MacPherson. Indeed, the three basic bivariant operations of product, pushforward and pullback are not always defined on $\mathbb{F}_\infty$ and $\mathbb{H}_\infty$. What we can say so far is that (i) when they are restricted to morphisms with nonsingular target varieties they become bivariant theories as stated in §3 and that (ii) they satisfy the seven axioms of Bivariant Theory listed in §2 whenever these three operations are defined on them. In this sense we shall call the above assignments $\mathbb{F}_\infty$ and $\mathbb{H}_\infty$ *the quasi-bivariant theory of constructible functions* and *the quasi-bivariant homology theory* respectively.

Now for each $g \in \mathcal{R}_Y$, consider the following

$$\gamma^{\mathrm{Gin}} : \widetilde{\mathbb{GF}}(X' \to Y') \to \mathbb{H}(X' \to Y'),$$

which shall be denoted by

$$\gamma_g^{\mathrm{Gin}} : \mathbb{F}_g(X \xrightarrow{f} Y) \to \mathbb{H}_g(X \xrightarrow{f} Y).$$

Then the inductive limit

$$\gamma_\infty := \varinjlim_{g \in \mathcal{R}_Y} \gamma_g^{\mathrm{Gin}}$$

of the Ginzburg-Chern classes is surely a transformation from $\mathbb{F}_\infty$ to $\mathbb{H}_\infty$. We can see that γ_∞ preserves the three operations whenever they are defined and in this sense $\gamma_\infty : \mathbb{F}_\infty \to \mathbb{H}_\infty$ shall be called a *quasi-Grothendieck transformation.* By definition it is obvious that γ_∞ for morphisms to a point is the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$. And it follows from Theorem (3.4) that it becomes the bivariant Chern class for morphisms whose target varieties are nonsingular. In this sense $\gamma_\infty : \mathbb{F}_\infty \to \mathbb{H}_\infty$ shall be called *a quasi-bivariant Chern class.*

Also it follows from the definition of γ_∞ that γ_∞ is uniquely determined so that for a morphism $f : X \to Y$ and any resolution of singularities $\nu : \widetilde{Y} \to Y$ the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathbb{GF}}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) & \xrightarrow{\gamma^{\mathrm{Gin}}} & \mathbb{H}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) \\
{\scriptstyle \mathbb{F}^\nu}\downarrow & & \downarrow{\scriptstyle \mathbb{H}^\nu} \\
\mathbb{F}_\infty(X \xrightarrow{f} Y) & \xrightarrow[\gamma_\infty]{} & \mathbb{H}_\infty(X \xrightarrow{f} Y).
\end{array}$$

Therefore we obtain the following theorem:

Theorem (4.3). *There exists a unique quasi-Grothendieck transformation*

$$\gamma_\infty : \mathbb{F}_\infty \to \mathbb{H}_\infty,$$

satisfying (i) γ_∞ for morphisms to a point is the Chern-Schwartz-MacPherson class transformation $c_* : F \to H_*$ and (ii) for a morphism $f : X \to Y$ and any desinglarization $\nu : \widetilde{Y} \to Y$ the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathbb{GF}}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) & \xrightarrow{\gamma^{\mathrm{Gin}}} & \mathbb{H}(\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}) \\
{\scriptstyle \mathbb{F}^\nu}\downarrow & & \downarrow{\scriptstyle \mathbb{H}^\nu} \\
\mathbb{F}_\infty(X \xrightarrow{f} Y) & \xrightarrow[\gamma_\infty]{} & \mathbb{H}_\infty(X \xrightarrow{f} Y).
\end{array}$$

Remark (4.4). It is not clear whether the condition (ii) in Theorem (4.3) can be dropped.

This is a "solution via Ginzburg-Chern classes" to the original existence and uniqueness problem concerning the looked-for bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$. To attack this original problem, we need another approach, which remains to be seen.

Furthermore we can show the following theorem concerning relationships between $\gamma : \mathbb{F} \to \mathbb{H}$ and γ_∞:

Theorem (4.5). *If there exists a bivariant Chern class* $\gamma : \mathbb{F} \to \mathbb{H}$, *then there exist canonical transformations*

$$\epsilon_{\mathbb{F}} : \mathbb{F} \to \mathbb{F}_\infty, \quad \epsilon_{\mathbb{H}} : \mathbb{H} \to \mathbb{H}_\infty$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\gamma} & \mathbb{H} \\ {\scriptstyle \epsilon_{\mathbb{F}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\mathbb{H}}} \\ \mathbb{F}_\infty & \xrightarrow[\gamma_\infty]{} & \mathbb{H}_\infty . \end{array}$$

Here $\epsilon_{\mathbb{F}} : \mathbb{F}(X \to Y) \to \mathbb{F}_\infty(X \to Y)$ *is injective for any morphism* $f : X \to Y$.

Proof. Let $\alpha \in \mathbb{F}$. Then, for any $g \in \mathcal{R}_Y$ we have $g^\star\alpha \in \mathbb{F}_g(X \to Y)$, which induces the canonical homomorphism $\epsilon_{\mathbb{F}} : \mathbb{F}(X \to Y) \to \mathbb{F}_\infty(X \to Y)$, which is defined by

$$\epsilon_{\mathbb{F}}(\alpha) := [g^\star\alpha]$$

for $g \in \mathcal{R}_Y$. Since $[g^\star\alpha] = [h^\star\alpha]$ for $g, h \in \mathcal{R}_Y$, $\epsilon_{\mathbb{F}}$ is well-defined. Similarly $\epsilon_{\mathbb{H}} : \mathbb{H}(X \to Y) \to \mathbb{H}_\infty(X \to Y)$ is defined by

$$\epsilon_{\mathbb{H}}(\omega) := [g^\star\omega].$$

Then the above diagram follows. It is straightforward that $\epsilon_{\mathbb{F}} : \mathbb{F}(X \to Y) \to \mathbb{F}_\infty(X \to Y)$ is injective for any morphism $f : X \to Y$. □

This theorem is an answer for how to view all the different bivariant classes $\pi^\star\gamma(\alpha)$ constructed by using the resolutions of singularities as "the same" or "equivalent" in some sense.

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