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# On Graphs Having Exact Two Vertices with the Same Degree

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## Abstract

As well known, every graph has at least two vertices with the same degree. The purpose of this note is to determine the graphs having exact two vertices with the same degree and to state some properties of these graphs. Especially we show that for every integer  $n \geq 2$  there exists exactly one connected graph of order  $n$  having exact two vertices with the same degree.

**Key words:** Graph, Graph having exact two vertices with the same degree, Degree sequence, Graphical sequence.

## 1 Degree sequences

In this note we use freely the terminology and notation concerning graphs in G.Chartrand and L.Lesniak [1]. For any positive integer  $n$  and non-negative integer  $m$  with  $m < n$ , we use the following notation:

$$[n] := \{1, 2, 3, \dots, n\} \quad [m, n] := \{m, m+1, \dots, n\}.$$

A sequence  $s : s_1, s_2, \dots, s_n$  of nonnegative integers is said to be *graphical* if there exists a graph  $G, V(G) = \{v_1, v_2, \dots, v_n\}$ , of order  $n$  whose degree sequence is  $s$ , that is,  $\deg v_j = s_j$  for all  $j \in [n]$ .

In this section, for any integer  $n \geq 2$  we shall determine the graphical sequences  $s : s_1, s_2, \dots, s_n$  with the following property:

$$(1.1) \quad n-1 \geq s_1 > s_2 > \dots > s_{k-1} > s_k = s_{k+1} > s_{k+2} > \dots > s_n \geq 0$$

for some  $k \in [n-1]$ .

For the sake of brevity any sequence  $s : s_1, s_2, \dots, s_n$  of non-negative integers with the property (1.1) is said to be *(n,2)-admissible*. Let  $s : s_1, s_2, \dots, s_n$  be *(n,2)-admissible*. If  $s_1 = n-2$  then  $s_n = 0$ . Moreover if  $s$  is graphical and  $s_1 = n-1$  then  $s_n$  must be equal to 1. So to our aim it suffices to consider the following two types of *(n,2)-admissible* sequences:

$$(1.2) \quad n-1, n-2, \dots, k+1, k, k, k-1, \dots, 2, 1 \text{ for some } k \in [n-1],$$

$$(1.3) \quad n-2, n-3, \dots, k+1, k, k, k-1, \dots, 1, 0 \text{ for some } k \in [0, n-2].$$

We denote by  $s_n(n-1; k)$  and  $s_n(n-2; k)$  the *(n,2)-admissible* sequence given in the form (1.2) and (1.3) respectively. The next lemma, noted in [1] and [2], plays the essential role in our

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discussion.

**Lemma 1.1** *A sequence  $s : s_1, s_2, \dots, s_n$  of non-negative integer with  $s_1 \geq s_2 \geq \dots \geq s_n, n \geq 2, s_1 \geq 1$ , is graphical if and only if the following sequence  $h(s)$  with  $n - 1$  terms is graphical:*

$h(s) : s_2 - 1, s_3 - 1, \dots, s_{t+1} - 1, s_{t+2}, s_{t+3}, \dots, s_n$   
where  $t = s_1$ .

In general for any sequence  $s : s_1, s_2, \dots, s_n$  of integers with  $n$  terms,  $n \geq 2$ , we define the following four kinds of sequences  $c(s)$  with  $n$  terms,  $h(s)$  with  $n - 1$  terms,  $p(s)$  and  $z(s)$  with  $n + 1$  terms respectively:

$c(s) : n - 1 - s_n, n - 1 - s_{n-1}, \dots, n - 1 - s_2, n - 1 - s_1$   
 $h(s) : s_2 - 1, s_3 - 1, \dots, s_n - 1$   
 $p(s) : n, s_1 + 1, s_2 + 1, \dots, s_n + 1$   
 $z(s) : s_1, s_2, \dots, s_n, 0$ .

**Lemma 1.2**

- (1) *Any  $(n, 2)$ -admissible sequence  $s$  is graphical if and only if so is  $c(s)$ .*
- (2)  *$s_n(n - 1; k)$  is graphical if and only if so is  $s_n(n - 2; n - 1 - k)$ .*
- (3) *A sequence  $s : s_1, s_2, \dots, s_n$  of positive integers is graphical if and only if so is  $z(s)$ .*
- (4)  *$s_n(n - 1; k + 1) = p(s_{n-1}(n - 3; k))$  for any  $k \in [0, n - 3]$ .*
- (5)  *$s_n(n - 2; k) = z(s_{n-1}(n - 2; k))$  for any  $k \in [n - 2]$ .*

**Proof** (1) follows from the fact that if  $s$  is a degree sequence of a graph  $G$  of order  $n$  then  $c(s)$  is the degree sequence of the complement graph of  $G$ . Since  $c(s_n(n - 1; k)) = s_n(n - 2; n - 1 - k)$ , (2) is obvious from (1). (3) is a consequence of the fact that any vertex with degree 0 of a graph is isolated. (4) and (5) are obvious.  $\square$

**Lemma 1.3** *For any integer  $n \geq 3$ , we have*

- (1)  *$s_n(n - 1; n - 1)$  is not graphical.*
- (2)  *$s_n(n - 2; 0)$  is not graphical.*
- (3)  *$s = s_n(n - 1; k), k \in [n - 2]$ , is graphical if and only if  $h(s) = s_{n-1}(n - 3; k - 1)$  is graphical.*
- (4)  *$s = s_{n-1}(n - 2; k), k \in [n - 3]$  is graphical if and only if so is  $z(s) = s_n(n - 2; k)$ .*

**Proof** (1)  $h(s_n(n - 1; n - 1)) = (n - 2, n - 3, \dots, 2, 1, 0)$  is not graphical, because any finite sequence consisting of mutually distinct non-negative integers is not graphical. So (1) follows from Lemma 1.1. Since  $s_n(n - 2; 0) = c(s_n(n - 1; n - 1))$ , (2) is obvious from (1) and Lemma 1.2 (2). (3) and (4) follows immediately from Lemma 1.1 and Lemma 1.2(3) respectively.  $\square$

Now we denote by  $GS(n, 2)$  the set of all graphical  $(n, 2)$ -admissible sequences, and  $GS_n(n - 1)$  and  $GS_n(n - 2)$  be the set of all graphical sequences in  $\{s_n(n - 1; k); k \in [n - 1]\}$  and  $\{s_n(n - 2; k); k \in [0, n - 2]\}$  respectively. Then we have

$$GS(n, 2) = GS_n(n - 1) \cup GS_n(n - 2)$$

$$GS_n(n - 2) = \{c(s); s \in GS_n(n - 1)\}.$$

Combining Lemmas 1.2 and 1.3, we have

**Lemma 1.4** For any integer  $n \geq 3$ ,  $GS(n, 2)$  is completely determined from  $GS(n-1, 2)$  in the following way :

- (1)  $GS_n(n-1) = \{p(s); s \in GS_{n-1}(n-3)\}$ .
- (2)  $GS_n(n-2) = \{z(s); s \in GS_{n-1}(n-2)\}$ .

The next is seen easily.

**Lemma 1.5**  $GS(2, 2)$  is given as follows:

- (1)  $GS_2(1)$  and  $GS_2(0)$  consists of only one sequence  $s(2) = (1, 1)$  and  $c(s(2)) = (0, 0)$  respectively.
- (2) The complete graph  $K_2$  of order 2 is the only one graph with degree sequence  $s(2)$ .

From Lemmas 1.4 and 1.5, we can obtain explicitly sequences in  $GS(n, 2)$ .

**Theorem 1.6** For any integer  $n \geq 2$ ,  $GS(n, 2)$  consists of two sequences  $s(n) = s_n(n-1; m)$  and  $c(s(n)) = s_n(n-2; n-m-1)$ , where  $m = \text{the floor of } \frac{n}{2}, \text{ i.e.,}$

$$s(n) = \begin{cases} (2m, 2m-1, 2m-2, \dots, m+1, m, m, m-1, \dots, 1) & \text{for } n = 2m+1 \\ (2m-1, 2m-2, \dots, m+1, m, m, m-1, \dots, 1) & \text{for } n = 2m. \end{cases}$$

**Remark 1.7**  $GS(3, 2)$  consists of the following two sequences:

$$s(3) = p(c(s(2))) = (2, 1, 1) \quad c(s(3)) = z(s(2)) = (1, 1, 0).$$

We note that the path  $P_3$  of order 3 is the only one graph whose degree sequence is  $s(3)$ .

**Remark 1.8** Let  $n \geq 4$ . From Lemma 1.4 and Theorem 1.6 we have

$$(1.4) \quad s(n) = p(z(s(n-2))).$$

## 2 Graphs corresponding to $s(n)$

In this section let  $n$  be any integer with  $n \geq 2$  and we construct the graphs whose degree sequence is  $s(n)$  given in Theorem 1.6. At first we note that this graph is uniquely determined by  $s(n)$ . Let  $s(n) : s_1, s_2, \dots, s_n$  and let  $G$  and  $H$  be any graphs with degree sequence  $s(n)$ . The vertex sets  $V(G) = \{v_k; k \in [n]\}$  and  $V(H) = \{u_k; k \in [n]\}$  are labeled as  $s_k = \deg v_k = \deg u_k$  for all  $k \in [n]$ . We define the degree preserving map  $\phi_n$  from  $V(G)$  onto  $V(H)$  by  $\phi_n(v_k) = u_k, k \in [n]$ . Then we have

**Lemma 2.1**  $\phi_n$  is an isomorphism from  $G$  to  $H$ .

**Proof** We show by the induction on  $n$ . As noted in Lemma 1.5 and Remark 1.7,  $\phi_2$  and  $\phi_3$  are isomorphic. Let  $n \geq 4$ . From (1.4) we see that the subgraphs  $G' = (G - v_1) - v_n$  and  $H' = (H - u_1) - u_n$  admit the degree sequence  $s(n-2)$ . Moreover the restriction of  $\phi_n$  to  $V(G')$  is identical with  $\phi_{n-2}$ . By the inductive hypothesis  $\phi_{n-2}$  is an isomorphism from  $G'$  to  $H'$ . Since  $\deg v_1 = \deg u_1 = n-1$ , and  $v_1$  and  $u_1$  are adjacent to all the other vertices in  $G$  and  $H$  respectively, we conclude that  $\phi_n$  is isomorphic.  $\square$

**Theorem 2.2** For every integer  $n \geq 2$  there exists exactly one graph  $G$  whose degree sequence is  $s(n)$ . Namely the graphs of order  $n$  admitting exact two vertices with the same degree are  $G$  and its complement.

We denote by  $G_n$  the graph with degree sequence  $s(n)$ . For any graphical sequence  $s : s_1, s_2, \dots, s_n$  let  $G$  be a graph of order  $n$  whose degree sequence is  $s$ . Then the union  $G \cup N_1$  of  $G$  and the empty graph  $N_1$  of order 1 has the degree sequence  $z(s)$ , and the join  $G + N_1$  of  $G$  and  $N_1$  has the degree sequence  $p(s)$ . Hence by virtue of (1.4) and Theorem 2.2,  $G_n$  is constructed inductively in the following way.

**Theorem 2.3**

- (1)  $G_2 = K_2$  and  $G_3 = P_3$ .
- (2)  $G_n = (G_{n-2} \cup N_1) + N_1$  for every integer  $n \geq 4$ .
- (3) The complement graph of  $G_n$  is  $G_{n-1} + N_1$  for every integer  $n \geq 3$ .

**Theorem 2.4**  $G_n$  is a connected graph and its size  $q(G_n)$  is

$$q(G_n) = \frac{1}{2} \left( \frac{n(n-1)}{2} + m \right), \text{ i.e.,}$$

$$q(G_{2m}) = m^2, \quad q(G_{2m+1}) = m^2 + m.$$

In what follows let the vertex set  $V(G_n) = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$  be labeled as:

$$\deg v_k = \begin{cases} k & \text{for } k \in [m] \\ k-1 & \text{for } k \in [m+1, n] \end{cases}$$

where  $m$  is the floor of  $\frac{n}{2}$ . Then from Theorem 2.3 we have

**Theorem 2.5** The adjacency matrix  $A(G_n) = (a_{ij})$  of  $G_n$  is given as follows:

$$a_{ij} = \begin{cases} 0 & \text{for } i = j \in [n] \\ 0 & \text{for } 1 \leq j \leq n-i, i \in [n] \\ 1 & \text{for } n-i < j \leq n, j \neq i, i \in [n]. \end{cases}$$

**Remark 2.6** Two vertices  $v_m, v_{m+1}$  of degree  $m$  are adjacent if and only if  $n$  is even.

**Example**  $A(G_n)$ , for  $n = 6$  and  $7$ , are given in the following form:

$$A(G_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad A(G_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

### 3 Properties of $G_n$

In this section we state some properties of  $G_n$  without proof, which are obtained from Theorem 2.3. Throughout this section let  $n$  be any integer with  $n \geq 2$ , and  $m$  be the floor of

$\frac{n}{2}$ . We denote by  $K(v; u_1, u_2, \dots, u_n)$  the star graph  $K_{1,n}$  with the center vertex  $v$  and the end vertices  $u_1, u_2, \dots, u_n$ , and by  $N(V)$  the empty graph with the vertex set  $V$ .

### 3.1 Subgraphs

The maximal complete subgraph of  $G_n$  is  $K_{m+1}$ . So  $G_n$  is planar if and only if  $n \leq 7$ .  $G_n$  contains the star  $K(v_n; v_1, v_2, \dots, v_{n-1})$  and the path  $P_n$  as its spanning trees. Moreover  $G_n$  is decomposed into mutually edge-disjoint spanning subgraphs  $F_1, F_2, \dots, F_m$  defined by

$$F_k = K(v_{n+1-k}; v_k, v_{k+1}, \dots, v_{n-k}) \cup N(v_1, v_n, v_2, v_{n-1}, \dots, v_{k-1}, v_{n+2-k})$$

for  $k \in [m]$ .

The center of  $G_n = N(v_n)$ , and the periphery of  $G_n$  is the union of  $G_{n-2}$  and  $N(v_1)$ , which is the complement of  $G_{n-1}$ . We see that the radius of  $G_n$  is equal to 1 and the diameter of  $G_n$  is equal to 2.

### 3.2 Colorings

The chromatic number  $\chi$  and the edge chromatic number  $\chi_1$  of  $G_n$  are given by

$$\chi(G_n) = m + 1 \text{ and } \chi_1(G_n) = n - 1.$$

The chromatic polynomial  $P(G_n, k)$  is given in the following form:

$$P(G_{2m}, k) = k((k-1)(k-2) \times \dots \times (k-m+1))^2(k-m),$$

$$P(G_{2m+1}, k) = k((k-1)(k-2) \times \dots \times (k-m+1)(k-m))^2.$$

### 3.3 Coverings

The vertex covering number  $\alpha$  and the independence number  $\beta$  of  $G_n$  are given by

$$\alpha(G_n) = m \text{ and } \beta(G_{2m}) = m, \quad \beta(G_{2m+1}) = m + 1.$$

The edge covering number  $\alpha_1$  and the edge independence number  $\beta_1$  of  $G_n$  are given by

$$\alpha_1(G_{2m}) = m, \quad \alpha_1(G_{2m+1}) = m + 1 \text{ and } \beta_1(G_n) = m.$$

Especially  $G_{2m}$  contains a 1-factor.

### 3.4 Characteristic polynomial of $A(G_n)$

We put  $P_n(\lambda) = \det(\lambda E - A(G_n))$ . Then we have

$$P_2(\lambda) = \lambda^2 - 1 \text{ and } P_3(\lambda) = \lambda^3 - 2\lambda,$$

$$P_4(\lambda) = \lambda^4 - 4\lambda^2 - 2\lambda + 1,$$

$$P_5(\lambda) = \lambda^5 - 6\lambda^3 - 4\lambda^2 + 2\lambda.$$

In general we have the following recurrence formula:

$$P_n(\lambda) = (2\lambda^2 + 2\lambda - 1)P_{n-2}(\lambda) - \lambda^2(\lambda + 1)^2P_{n-4}(\lambda) \text{ for } n \geq 6.$$

We see that  $\lambda = -1$  [resp.  $\lambda = 0$ ] is a proper value of  $A(G_n)$  for even [resp. odd]  $n$ .

### References

- [1] C.Chartrand and L. Lesniak: Graphs & Digraphs, Chapman & Hall, London, 1996.
- [2] N.Hartsfield and G. Ringel: Pearls in graph theory, Academic Press, 1990.