ON THE WOLVERTON AND WAGNER'S ASYMPTOTICALLY OPTIMAL DISCRIMINANT FUNCTION

著者	YAMATO Hajime
journal or publication title	鹿児島大学理学部紀要. 数学・物理学・化学
volume	3
page range	3-11
別言語のタイトル	WOLVERTON と WAGNER の漸近的に最適な判別函数について
URL	http://hdl.handle.net/10232/6304

Rep. Fac. Sci. Kagoshima Univ., (Nath. Phys. Chem.) No. 3, p. 3-11, 1970

ON THE WOLVERTON AND WAGNER'S ASYMPTOTICALLY OPTIMAL DISCRIMINANT FUNCTION

By

Hajime YAMATO

(Received September 30, 1970)

1. Introduction

Suppose we have an observation x, which may be a scalar or a vector, and we know apriori that it should have come from either of two populations π_1 and π_2, which have the probability density functions $f_1(x)$ and $f_2(x)$ respectively and apriori probabilities q_1 and q_2 respectively. We assume that the losses due to two kinds of misclassification are same, where one misclassification is that if the observation is actually from π_1 we classify it as coming from π_2 and the other is that if the observation is actually from π_2 we classify it as coming from π_1. Then according to the Bayes procedure, if

$$\frac{q_1 f_1(x)}{q_1 f_1(x) + q_2 f_2(x)} \geqq \frac{q_2 f_2(x)}{q_1 f_1(x) + q_2 f_2(x)}$$

then we decide that the observation has come from π_1, and otherwise we decide that the observation has come from π_2. Equivalently if

$$D(x) = q_1 f_1(x) - q_2 f_2(x) \geqq 0$$

then we decide that it has come from π_1 and otherwise we decide that it has come from π_2.

The purpose of this paper is to discuss the statistical properties of the estimate of $D(x)$ constructed by Wolverton and Wagner [6] by using the results in Yamato [7].

Let $X_1^1, X_2^1, X_3^1, \ldots.$ and $X_1^2, X_2^2, X_3^2, \ldots.$ be sequences of independent, identically distributed m-dimensional random vectors in the m-dimensional Euclidian space E_m, which have the probability density functions $f_1(x)$ and $f_2(x)$ respectively. Let $\rho_1, \rho_2, \rho_3 \ldots.$ be a sequence of independent identically distributed random variables with

Pr $(\rho_i=1)=q_1$ and Pr $(\rho_i=0)=q_2$ $(i=1,2,3,\ldots.)$.

We assume that X_i^1, X_j^2, ρ_k are mutually independent for all $i=1, 2, \ldots.$, $j=1, 2, \ldots.$ and $k=1, 2, \ldots.$ In this paper we consider a sequential estimation of $D(x)$ with a scheme that we observe X_i^1 when $\rho_i=1$ and X_i^2 when $\rho_i=0$. Wolverton and Wagner [6] considered an estimate of $D(x)$ under the same sampling scheme given by

$$D_n(x) = \frac{1}{n} \sum_{j=1}^{n} \left[\rho_j \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right) - (1 - \rho_j) \frac{1}{h_j^m} K\left(\frac{x - X_j^2}{h_j}\right)\right]$$

and showed that under a certain condition on $f_1(x)$, $f_2(x)$, h_n, $k(\cdot)$

$$\int_{Em} |D_n(x) - D(x)|^2 \, dx$$

converges to 0 in probability (with probability 1) and then $P_{D_n}(e)$ converges to $P_D(e)$ in probability (with probability 1), where $P_d(e)$ denote the probability of misclassification by using a discriminant function $d(x)$.

In the following sections we shall discuss the asymptotic unbiasedness, asymptotically uniform unbiasedness, consistency, uniform consistency and asymptotic normality of $D_n(x)$ by using Yamato[7]. Concerning its asymptotically uniform unbiasedness, Wolverton and Wagner[6] proved it in Lemma 2 under the assumption that $f_1(x)$ and $f_2(x)$ are uniformly continuous. In section 2 we shall, however, generalize it for continuous probability density functions $f_1(x)$ and $f_2(x)$ and moreover at the continuous point x of $f_1(x)$ and $f_2(x)$.

In section 3 we shall treat the limits of the variance and the mean square error of $D_n(x)$ and the limit of nh_n^m Var $[D_n(x)]$.

In section 4 we shall treat the uniform consistency of $D_n(x)$.

In section 5 we shall treat the limit distribution of $D_n(x)$.

The author expresses his hearty thanks to Professor A. Kudo of Kyushu University for his kind encouragements and advices, and also to Professor S. Kano of Kyushu University for his kind suggestions. The author also expresses his hearty thanks to Professor M. Okamoto of Osaka University for his kind comments.

2. Asymtotic unbiasedness

Theorem 1. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are continuous and that $\{h_n\}$ is a sequence of monotone decreasing positive numbers such that*

$$\lim_{n\to\infty} h_n = 0 \tag{2.1}$$

Let $K(y)$ be a measurable function satisfying

$$\sup_{y \in Em} |K(y)| < \infty \tag{2.2}$$

$$\int_{Em} K(y)\, dy = 1 \tag{2.3}$$

$$\int_{Em} |K(y)|\, dy < \infty \tag{2.4}$$

where E_m denotes the m-dimensional Euclidian space and let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section 1. *Then*

$$D_n(x) = \frac{1}{n} \sum_{j=1}^{n} \left[\rho_j \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right) - (1 - \rho_j) \frac{1}{h_j^m} K\left(\frac{x - X_j^2}{h_j}\right)\right] \tag{2.5}$$

is an asymptotically unbiased estimate of $D(x)$.

The following corollary can be found in Wolverton and Wagner [6], which we need to prove Theorem 5. In the following corollary, we assume the uniform continuity of the probability density function, which is satisfied when a population characteristic function is absolutely integrable.

Corollary 1. *If we assume the uniform continuity of the probability density functions* $f_1(x)$ *and* $f_2(x)$ *in Theorem* 1, *then we have*

$$\lim_{n\to\infty} \sup_{x\in E_m} |E D_n(x) - D(x)| = 0 \tag{2.6}$$

Theorem 2. *We suppose that* $\{h_n\}$ *is a sequence of monotone decreasing positive numbers satisfying* (2. 1) *and that the measurable function* $K(y)$ *satisfies* (2.2), (2.3), (2.4) *and*

$$\lim_{y\to\infty} |y|^m |K(y)| = 0 \tag{2.7}$$

Let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ *be mutually independent sequences of random variables and vectors as described in section* 1. *Then* $D_n(x)$ *is asymptotically unbiased at the point* x *such that both* $f_1(x)$ *and* $f_2(x)$ *are continuous.*

By applying Theorem 1, Corollary 1 and Theorem 2 in Yamato[7] on an inequality

$$\begin{aligned} &|E D_n(x) - D(x)| \\ &\leqq q_1 |E \hat{f}_1(x) - f_1(x)| + q_2 |E \hat{f}_2(x) - f_2(x)| \end{aligned} \tag{2.8}$$

we can easily obtain Theorem 1, Corollary 1 and Theorem 2, where

$$\hat{f}_1(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right) \tag{2.9}$$

$$\hat{f}_2(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{h_j^m} K\left(\frac{x - X_j^1}{h_j}\right). \tag{2.10}$$

3. Consistency

Theorem 3. *We suppose that the probability density functions* $f_1(x)$ and $f_2(x)$ *are continuous and that* $\{h_n\}$ *is a sequence of monotone decreasing positive numbers satisfying* (2.1) *and*

$$\lim_{n\to\infty} n h_n^m = \infty . \tag{3.1}$$

Let the measurable function $K(y)$ *satisfy* (2.2) *and* (2.4) *and let* $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ *be mutually independent sequences of random vectors and variables as described in section* 1. *Then we have*

$$\lim_{n\to\infty} \mathrm{Var}[D_n(x)] = 0 \qquad \text{at all points } x \in E_m. \tag{3.2}$$

Furthermore if $K(y)$ satisfy (2.3), then we have

(3.3) $$\lim_{n\to\infty} E|D_n(x)-D(x)|^2=0.$$

Proof. We shall note at first that

(3.4) $$\mathrm{Var}[D_n(x)] = E\,\mathrm{I}_1^2+E\,\mathrm{I}_2^2+E\,\mathrm{I}_3^2+E\,\mathrm{I}_4^2-2E\,\mathrm{I}_2\mathrm{I}_4$$

where

(3.5)
$$\mathrm{I}_1=\frac{1}{n}\sum_{j=1}^{n}\rho_j\frac{1}{h_j^m}\left[K\left(\frac{x-X_j^1}{h_j}\right)-E\,K\left(\frac{x-X_j^1}{h_j}\right)\right]$$
$$\mathrm{I}_2=\frac{1}{n}\sum_{j=1}^{n}(\rho_j-q_1)\frac{1}{h_j^m}E\,K\left(\frac{x-X_j^1}{h_j}\right)$$
$$\mathrm{I}_3=\frac{1}{n}\sum_{j=1}^{n}(1-\rho_j)\frac{1}{h_j^m}\left[K\left(\frac{x-X_j^2}{h_j}\right)-E\,K\left(\frac{x-X_j^2}{h_j}\right)\right]$$
$$\mathrm{I}_4=\frac{1}{n}\sum_{j=1}^{n}(1-\rho_j-q_2)\frac{1}{h_j^m}E\,K\left(\frac{x-X_j^2}{h_j}\right).$$

We can show easily that

(3.6)
$$E\,\mathrm{I}_1^2=q_1\,\mathrm{Var}[\hat{f}_1(x)]$$
$$E\,\mathrm{I}_2^2\leqq\frac{1}{n}q_1q_2\,||f_1||\left\{\int_{E_m}|K(y)|d\,y\right\}^2$$
$$E\,\mathrm{I}_3^2=q_2\,\mathrm{Var}[\hat{f}_2(x)]$$
$$E\,\mathrm{I}_4^2\leqq\frac{1}{n}q_1q_2\,||f_2||\left\{\int_{E_m}|K(y)|d\,y^2\right\}^2.$$

where $||f_1||=\max f_1(x)$ and $||f_2||=\max f_2(x)$, whose existence is secured by the continuity of $f_1(x)$ and $f_2(x)$. By applying (3.6) and the Schwarz's inequality on (3.4), we have

(3.7) $$\mathrm{Var}[D_n(x)]=q_1\,\mathrm{Var}[\hat{f}_1(x)]+q_2\,\mathrm{Var}[\hat{f}_2(x)]+0\left(\frac{1}{n}\right).$$

Theorem 3 in Yamato[7] implies that the right side of (3.7) tends to zero as n tends to ∞. Thus (3.2) was established.

Next, it is obvious that

(3.8) $$E|D_n(x)-D(x)|^2=\mathrm{Var}[D_n(x)]+|E\,D_n(x)-D(x)|^2.$$

The combination of (3.2), (3.8) and Theorem 1 leads us to (3.3), thus proving the theorem.

This theorem furnishes a sufficient condition for $D_n(x)$ to be consistent. In Theorem 3, if we assume furthermore that $K(y)$ satisfies (2.7), then we have that both $\mathrm{Var}[D_n(x)]$ and $E|D_n(x)-D(x)|^2$ converge to zero at all points x at which both probability density funcions $f_1(x)$ and $f_2(x)$ are continuous.

Theorem 4. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are continuous and that for the sequence of monotone decreasing positive numbers $\{h_n\}$* satisfying (2.1) *there exists a limit with*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{h_n^m}{h_j^m}=\alpha \qquad (0\leqq\alpha\leqq 1). \tag{3.9}$$

Let the measurable function $K(y)$ satisfy (2.2) *and* (2.4) *and let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section* 1. *Then for $D_n(x)$ defined by* (2.5) *we have*

$$\lim_{n\to\infty} n\,h_n^m\,\mathrm{Var}\,[D_n(x)] = \alpha\,\{q_1 f_1(x)+q_2 f_2(x)\}\int_{Em}|K(y)|^2\,dy\,. \tag{3.10}$$

Proof. It follows from (3.7) that

$$n\,h_n^m\,\mathrm{Var}\,[D_n(x)] = q_1\cdot n\,h_n^m\,\mathrm{Var}\,[\hat{f}_1(x)] + q_2\,n\,h_n^m\,\mathrm{Var}\,[\hat{f}_2(x)] + 0\,(h_n^m)\,. \tag{3.11}$$

Hence by applying Theorem 4 in Yamato[7] on (3.11) we have (3.10). Thus the theorem is proved.

4. Uniform consistency

Theorem 5. *We suppose that the probability density functions $f_1(x)$ and $f_2(x)$ are uniformly continuous and that a sequence of monotone decreasing postive numbers $\{h_n\}$ satisfy* (2.1) *and*

$$\lim_{n\to\infty} n^{1/2}\,h_n^m=\infty\,. \tag{4.1}$$

Let the measurable function $K(y)$ satisfy (2.3) *and* (2.4), *its Fourier transform*

$$k(u)=\int_{Em} e^{iu'y}\,K(y)\,dy \tag{4.2}$$

be absolutely integrable and $k(u)$ be nondecreasing in negative part and nonincreasing in positive part for each argument.

Let $\{X_i^1\}$, $\{X_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section 1. *Then for $D_n(x)$ defined by* (2.5) *we have*

(4.3) $$\sup_x |D_n(x) - D(x)| \xrightarrow{P} 0$$

where (4.2) *denotes that* $\sup|D_n(x)-D(x)|$ *converges to zero in probability as* n *tends to* ∞.

Proof. In terms of $k(u)$, the Fourier transform of $K(y)$, we have

(4.4) $$D_n(x) - E\,D_n(x)$$
$$= \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n}\sum_{j=1}^{n}[\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)]\,k(h_j u)\right\}e^{-iu'x}\,du$$
$$- \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n}\sum_{j=1}^{n}[(1-\rho_j)e^{iu'X_j^2} - q_2\varphi_2(u)]\,k(h_j u)\right\}e^{-iu'x}\,du$$

where $\varphi_1(u)$ and $\varphi_2(u)$ are the characteristic functions of $f_1(x)$ and $f_2(x)$ respectively. Therefore we have

(4.5) $$\sup_x |D_n(x) - E\,D_n(x)|$$
$$\leqq \frac{1}{(2\pi)^m}\int_{Em}\left|\frac{1}{n}\sum_{j=1}^{n}[\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)]\,k(h_j u)\right|du$$
$$+ \frac{1}{(2\pi)^m}\int_{Em}\left|\frac{1}{n}\sum_{j=1}^{n}[(1-\rho_j)e^{iu'X_j^2} - q_2\varphi_2(u)]\,k(h_j u)\right|du\,.$$

By applying the Schwartz's inequality on (4.5)

(4.6) $$E\sup_x |D_n(x) - E\,D_n(x)|$$
$$\leqq \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n^2}\sum_{j=1}^{n}E|\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)|^2\cdot|k(h_j u)|^2\right\}^{1/2}du$$
$$+ \frac{1}{(2\pi)^m}\int_{Em}\left\{\frac{1}{n^2}\sum_{j=1}^{n}E|(1-\rho_j)e^{iu'X_j^2} - q_2\varphi_2(u)|^2\cdot|k(h_j u)|^2\right\}^{1/2}du\,.$$

Since
$$E|\rho_j e^{iu'X_j^1} - q_1\varphi_1(u)|^2 \leqq 1\,,$$
$$E|(1-\rho_j)e^{iu'X_j^2} - q_2\varphi_2(u)|^2 \leqq 1\,,$$

$\{h_n\}$ is the sequence of monotone decreasing positive numbers and $k(u)$ is nondecreasing in negative part and nonincreasing in positive part for each argument, by (4.5) we have

(4.6) $$E\sup_x |D_n(x) - E\,D_n(x)|$$
$$\leqq \frac{1}{(2\pi)^m n}\int_{Em}\left\{n|k(h_n u)|^2\right\}^{1/2}du$$

$$+\frac{1}{(2\pi)^m n}\int_{E_m}\left\{n|k(h_n\,\boldsymbol{u})|^2\right\}^{1/2}d\,\boldsymbol{u}$$

$$=\frac{2}{n^{1/2}\,h_n^m\,(2\pi)^m}\int_{E_m}|k(\boldsymbol{u})|\,d\,\boldsymbol{u}\,.$$

By applying (4.1) on (4.6), we have

$$\lim_{n\to\infty} E\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\,\boldsymbol{D}_n(\boldsymbol{x})|=0\,. \tag{4.7}$$

It follows from (4.7) and Markov's inequality that

$$\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\,\boldsymbol{D}_n(\boldsymbol{x})|\xrightarrow{P}0\,. \tag{4.8}$$

Finally we remark the inequality

$$\begin{aligned}&\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-\boldsymbol{D}(\boldsymbol{x})|\\&\leqq\sup_{\boldsymbol{x}}|\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})|+\sup_{\boldsymbol{x}}|E\boldsymbol{D}_n(\boldsymbol{x})-\boldsymbol{D}(\boldsymbol{x})|\,.\end{aligned} \tag{4.9}$$

By applying Corollay 1 and (4.8) on (4.9), we have (4.3). Thus the theorem is proved.

5. Asymptotic normality

Theorem 6. *We suppose that the probability density functions $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ are continuous and that for the sequence of monotone decreasing positive numbers $\{h_n\}$ satisfying* (2.1) *and* (3.1) *there exists a non zero limit with* (3.9). *Let the measurable function $K(\boldsymbol{y})$ satisfy* (2.2) *and* (2.4) *and let $\{\boldsymbol{X}_i^1\}$, $\{\boldsymbol{X}_i^2\}$, $\{\rho_i\}$ be mutually independent sequences of random variables and vectors as described in section* 1. *Then for $\boldsymbol{D}_n(\boldsymbol{x})$ defined by* (2.5) *the distribution function of*

$$\frac{\boldsymbol{D}_n(\boldsymbol{x})-E\boldsymbol{D}_n(\boldsymbol{x})}{\sqrt{\mathrm{Var}[\boldsymbol{D}_n(\boldsymbol{x})]}} \tag{5.1}$$

converges to the standardized normal distribution function at all points $\boldsymbol{x}$.

$h_n=1/n^{r/m}$ $(0<r<1/2)$ and $h_n=1/(\log n)^{1/m}$ are examples of sequences of monotone decreasing positive numbers satisfying (2.1), (3.1), (4.1) and (3.9) with $\alpha=1/(r+1)$ and $\alpha=1$ respectively.

Proof. If we put for any fixed $\boldsymbol{x}$

$$\begin{aligned}V_j^1&=\frac{1}{h_j^m}\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^1}{h_j}\right)\quad(j=1,2,3,\cdots)\\V_j^2&=\frac{1}{h_j^m}\,K\left(\frac{\boldsymbol{x}-\boldsymbol{X}_j^2}{h_j}\right)\quad(j=1,2,3,\cdots)\end{aligned} \tag{5.2}$$

then $\{\rho_j V_j^1-(1-\rho_j)V_j^2\}$ $(j=1, 2, 3, \ldots.)$ is a sequence of independent random variables and we have

(5.3)
$$\frac{D_n(x)-ED_n(x)}{\sqrt{\mathrm{Var}[D_n(x)]}}$$
$$=\frac{\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2-q_1 EV_j^1+q_2 EV_j^2\}}{\sqrt{\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]}}$$

Therefore by virtue of Lyapunov's condition it is enough to show that

(5.4)
$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n} E|\rho_j V_j^1-(1-\rho_j)V_j^2-q_1EV_j^1+q_2EV_j^2|^3}{\left(\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]\right)^{3/2}}=0\,.$$

From Theorem 4 we have

(5.5)
$$\frac{h_n^m}{n}\mathrm{Var}\left[\sum_{j=n}^{n}\{\rho_j V_j^1-(1-\rho_j)V_j^2\}\right]$$
$$=n h_n^m\,\mathrm{Var}\,[D_n(x)]$$
$$\to a\,\{q_1 f_1(x)+q_2 f_2(x)\}\int_{Em} K^2(y)dy\,(n\to\infty)\,.$$

On the other hand, by an inequality

(5.6)
$$(a+b+c+d)^3 \leqq 16\,(a^3+b^3+c^3+d^3) \text{ for } a, b, c, d\geqq 0$$

we have

(5.7)
$$\sum_{j=1}^{n} E|\rho_j V_j^1-(1-\rho_j)V_j^2-q_1EV_j^1+q_2 EV_j^2|^3$$
$$\leqq\sum_{j=1}^{n} E\{|\rho_j(V_j^1-EV_j^1)|+|(\rho_j-q_1)EV_j^1|$$
$$+|(1-\rho_j)(V_j^2-EV_j^2)|+|(1-\rho_j-q_2)EV_j^2|\}^3$$
$$\leqq 16\left\{q_1\sum_{j=1}^{n} E|V_j^1-EV_j^1|^3+q_2\sum_{j=1}^{n} E|V_j^2-EV_j^2|^3\right.$$
$$\left.+q_1 q_2(q_1^2+q_2^2)\left(\sum_{j=1}^{n} E|V_j^1|^3+\sum_{j=1}^{n} E|V_j^2|^3\right)\right\}\,.$$

By (5.6) and (5.9) in Yamato[7], it turns out that the right hand side of (5.7) is smaller than

$$64\{q_1\|f_1\|+q_2\|f_2\|\}\frac{n}{h_n^{2m}}\int_{Em}|K(z)|^3dz$$

$$+64n\{q_1\|f_1\|^3+q_2\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3$$

$$+16q_1q_2(q_1^2+q_2^2)\cdot n\cdot\{\|f_1\|^3+\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3 .$$

Hence we have

$$\text{(5.9)}\qquad \lim_{n\to\infty}\frac{\sum_{j=1}^{n}E|\rho_jV_j^2-(1-\rho_j)V_j^2-q_1EV_j^1+q_2EV_j^2|^3}{\left(\mathrm{Var}\left[\sum_{j=1}^{n}\left\{\rho_jV_j^1-(1-\rho_j)V_j^1\right\}\right]\right)^{3/2}}$$

$$\leqq \lim_{n\to\infty}\frac{1}{\left(\frac{h_n^m}{n}\mathrm{Var}\left[\sum_{j=1}^{n}\{\rho_jV_j^1-(1-\rho_j)V_j^2\}\right]\right)^{3/2}}$$

$$\times\Big(64\{q_1\|f_1\|+q_2\|f_2\|\}\frac{1}{(nh_n^m)^{1/2}}\int_{Em}|K(z)|^3dz$$

$$+64\frac{h_n^{3m/2}}{n^{1/2}}\{q_1\|f_1\|^3+q_2\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3$$

$$+16q_1q_2(q_1^2+q_2^2)\frac{h_n^{3m/2}}{n^{1/2}}\{\|f_1\|^3+\|f_2\|^3\}\left\{\int_{Em}|K(z)|dz\right\}^3\Big).$$

By applying (2.1), (2.2), (2.4), (3.1) and (5.5) on (5.9) we have (5.4), which leads us to the completion of the theorem.

Thus we have obtained the asymptotic normality of $\boldsymbol{D}_n(\boldsymbol{x})$. We considered its property under the assumption that for $\{h_n\}$ there exists a non zero limit with (3.9) and the auther wishes to develop the asymptotic normality of $\boldsymbol{D}_n(\boldsymbol{x})$ without this assumption on another occasion.

References

[1] T.W. Anderson (1958), An Introduction to Multivariate Statistical Analysis, John Wiley and Sons.
[2] S. Bochner and K. Chandrasekharan (1949), Fourier Transforms, Princeton University Press.
[3] M. Loéve (1960), Probability Theory, Van Nostrand, Princeton.
[4] E. Parzen (1960), Modern Probability Theory and Its Applciations, John Wiley and Sons.
[5] E. Parzen (1962), On Estimation of A Probability Density Function and Mode, Ann. Math. Statist., vol. 33, pp. 1065–1076.
[6] C.T. Wolverton and T.J. Wagner (1969), Asymptotically Optimal Discriminant Functions for Pattern Classification, IEEE Trans. Information Theory, vol. IT-15, No. 2, pp. 258–265.
[7] H. Yamato, On Sequential Estimation of A Continuous Probability Density Function and Its Mode (to appear in Bull. Math. Statist.)

Errata

Page	Line	Wrong	Corrected
13	↑9	pifferentiable	differentiable