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1. Summary and Introduction

A Steiner system $S(t, k, v)$ is a set of v points and some blocks, with an incidence relation, such that (i) every block has exactly k points, (ii) every set of t distinct points is on exactly one block, and (iii) two blocks with the same points are equal.

In this paper we prove the following theorems.

THEOREM 1. *Let B be a block of S $(2, k, v)$. If there are exactly n blocks distinct from B each of which has no point in common with B, then $4n \equiv 0 \pmod{(k-1)}$.*

By this theorem we have the following theorem, which gives a characterization of the Witt system $S(4, 5, 11)$.

THEOREM 2. *In an $S(4, k, v)$, let B be a block of S. If there are exactly 20 blocks distinct from B each of which has exactly 2 points in common with B, then $S = S(4, 5, 11)$.*

2. Definitions and Lemmas

A Steiner system $S(t, k, v)$ reduces to the trivial case if $t = k$. In what follows we consider non-trivial Steiner systems which consist of more than one block. For a Steiner system $S = S(t, k, v)$ we use b to denote the number of blocks and $\lambda_i (1 \leq i \leq t)$ the number of blocks which contain the given i points of S. Then, with the convention $b = \lambda_0$, we have

$$\lambda_i = \frac{(v-i)(v-i-1)\cdots(v-t+1)}{(k-i)(k-i-1)\cdots(k-t+1)} \quad (0 \leq i \leq t). \tag{1}$$

For a block B of S we use $x_i (0 \leq i \leq t-1)$ to denote the number of blocks distinct from B each of which has exactly i points in common with B. The number x_i depends on S, but as Mendelsohn has shown, is independent of the choice of a block B in S (see Lemma 2). These numbers are called intersection numbers of S. If S is a Steiner system $S(t, k, v)$ with $t \geq 2$, and α is a point of S, then S_α is defined to be the set of all points of S except α and of all blocks of S which contain α. Then S_α is a system $S(t-1, k-1, v-1)$ and is called a contraction of S.

LEMMA 1 (Witt [1]). *There exists a unique Steiner system $S(t, k, v)$ with the following parameters; $(t, k, v) = (2, 3, 9)$ or $(4, 5, 11)$.*

LEMMA 2 (Mendelsohn [2]). *Let S be a Steiner system $S(t, k, v)$ and B a block of S. Let $x_i(0 \leqq i \leqq t\text{-}1)$ denote the number of blocks distinct from B each of which has exactly i points in common with B. Then the following equations hold:*

$$x_0 + x_1 + \cdots\cdots\cdots\cdots\cdots + x_{t-1} = \lambda_0 - 1,$$

$$x_1 + 2x_2 + \cdots\cdots\cdots + (t-1)\, x_{t-1} = (\lambda_1 - 1) \binom{k}{1},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$x_i + \binom{i+1}{i} x_{i+1} + \cdots + \binom{t-1}{i} x_{t-1} = (\lambda_i - 1) \binom{k}{i},$$

$$\cdots\cdots\cdots\cdots$$

$$x_{t-1} = (\lambda_t - 1) \binom{k}{t-1}.$$

Therefore x_i's are independent of the choice of a block B in S.

3. Proofs of Theorem 1 and Theorem 2

First of all a proof of Theorem 1 will be given.   Let S be a Steiner system $S(2, k, v)$. Let x_i's $(0 \leqq i \leqq 1)$ be intersection numbers of S.   By Lemma 2 we have

$$x_0 + x_1 = b - 1,$$

$$x_1 = (\lambda_1 - 1)\, k.$$

From these equations it follows that

$$x_0 = \frac{(v-k)(v+k-1-k^2)}{k(k-1)}.$$

We put $n = x_0$, then by a simple calculation

$$v^2 - (k^2 + 1)\, v + k^3 - (n+1)\, k^2 + (n+1)\, k = 0.$$

Hence

$$v = \frac{(k^2 + 1) \pm \sqrt{k^4 - 4k^3 + (4n+6)k^2 - 4(n+1)\, k + 1}}{2}.$$

Since λ_1 is an integer, by (1) we have

$$\frac{v-1}{k-1} = \frac{(k^2 - 1) \pm \sqrt{k^4 - 4k^3 + (4n+6)\, k^2 - 4(n+1)\, k + 1}}{2(k-1)}$$

is an integer.
Therefore

$$\frac{k^4 - 4k^3 + (4n+6)\, k^2 - 4(n+1)\, k + 1}{(k-1)^2}$$

is an integer.

Since

$$k^4 - 4k^3 + (4n+6)\,k^2 - 4(n+1)\,k + 1$$
$$= (k-1)^2\,(k^2 - 2k + 4n + 1) + 4n(k-1)\,,$$

we deduce that

$$\frac{4n}{(k-1)} \quad \text{is an integer.}$$

So $4n \equiv 0 \pmod{(k-1)}$.

To prove the Theorem 2, we need the following lemma.

LEMMA 3. *In Theorem 1, if $n=2$, then $S=S(2,3,9)$.*

PROOF. Since $S(2,k,v)$ is non-trivial and by Theorem 1, $k-1$ divides 8, so $k=3, 5,$ or 9. On the other hand

$$v = \frac{(k^2+1) \pm \sqrt{k^4 - 4k^3 + 14k^2 - 12k + 1}}{2}\,. \tag{2}$$

If $k=5$ or 9, then $k^4 - 4k^3 + 14k^2 - 12k + 1$ is not a square number. This is a contradiction.

By Lemma 1 and (2), $S=S(2,3,9)$.

We prove the Theorem 2. Let S be a Steiner system $S(4,k,v)$.

Let x_i's $(0 \leqq i \leqq 3)$, y_i's $(0 \leqq i \leqq 2)$, and z_i's $(0 \leqq i \leqq 1)$ be intersection number of S, S_α and $S_{\alpha\beta}$, respectively. Assume that $x_2 = 20$. By Lemma 2, we have

$$\begin{cases} x_0 + x_1 + x_2 + x_3 = b-1 \\[1mm] x_1 + 2x_2 + 3x_3 = (\lambda_1 - 1)\dbinom{k}{1}, \\[1mm] x_2 + 3x_3 = (\lambda_2 - 1)\dbinom{k}{2}, \\[1mm] x_3 = (\lambda_3 - 1)\dbinom{k}{3}, \end{cases} \tag{3}$$

$$\begin{cases} y_0 + y_1 + y_2 = \lambda_1 - 1\,, \\[1mm] y_1 + 2y_2 = (\lambda_2 - 1)\dbinom{k-1}{1}, \\[1mm] y_2 = (\lambda_3 - 1)\dbinom{k-1}{2}, \end{cases} \tag{4}$$

and

$$\begin{cases} z_0 + z_1 = \lambda_2 - 1\,, \\[1mm] z_2 = (\lambda_3 - 1)\dbinom{k-2}{1}. \end{cases} \tag{5}$$

T. ATSUMI

By (3), (4) and (5) we have

$$x_2 = \frac{1}{2} k y_1, \qquad y_1 = (k-1) z_0.$$

So
$$z_0 = \frac{40}{k(k-1)}.$$

Since $k > 4$ and z_0 is an integer, $k=5$. Hence $z_0=2$.
$S_{\alpha\beta}=S(2, 3, 9)$ by Lemma 3. So $S=S(4, 5, 11)$.

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References

[1] Witt, E., Uber Steinersche Systeme, Abh. Math. Sem. Univ. Hamburg **12** (1938), 265–275.
[2] Mendelsohn, N.S., A theorem on Steiner systems, Canadian J. Math. **22** (1970), 1010–1015.