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Rep. Fac. Sci., Kagoshima Univ. (Math., Phys. & Chem.),
No. 11, p. 25-31, 1978

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(Received September 30, 1978)

1. Introduction

Suppose a bivariate normal random vector (X, Y) with the unknown mean vector (θ_1, θ_2) and a unit variance and a known correlation coefficient ρ. We shall consider the following two problems of testing hypothesis. The first is to test the null hypothesis that the mean vector lies on the boundary of a positive orthant, namely, H_0: $(\theta_1 \geqq 0$ and $\theta_2 = 0)$ or $(\theta_1 = 0$ and $\theta_2 \geqq 0)$, against the alternative that the mean vector lies in the interior of a positive orthant, namely, K_0: $(\theta_1 > 0$ and $\theta_2 > 0)$, and the second is to test the null hypothesis that the mean vector lies either on the boundary of a positive orthant or a negative orthant, namely, H_1: $(-\infty \leqq \theta_1 \leqq \infty$ and $\theta_2 = 0)$ or $(\theta_1 = 0$ and $-\infty \leqq \theta_2 \leqq \infty)$, against the alternative that the mean vector lies either in the interior of a positive orthant or a negative orthant, namely, K_1: $(\theta_1 > 0$ and $\theta_2 > 0)$ or $(\theta_1 < 0$ and $\theta_2 < 0)$. We shall call the former the one-sided boundary test of bivariate normal mean, the latter the two-sided boundary test of bivariate normal mean. The purpose of this paper is to give the likelihood ratio test of these testing hypothesis problems. This type of hypothesis has not been investigated so far as the present author is aware.

Some related problems have been considered by many authors. For a multivariate normal distribution with the known covariance matrix, the problem of testing the null hypothesis that the mean vector is zero against the alternative that it is non-zero and all the components are non-negative was treated by Kudô (1963) and independently by Nüesch (1966). In the two-sided version of this problem the null hypothesis remains the same but the alternative is replaced by the one that the mean vector is non-zero with all the components simultaneously non-negative or non-positive, was first treated by Kudô and Fujisawa (1964) in bivariate case and the difficulty in multivariate generalization was demonstrated in Kudô and Fujisawa (1965). In multivariate case Yeh (1968) treated the same with a unit variance matrix and in bivariate case Inada, Tsukamoto and Yamauchi (1977) treated the same with the unknown covariance matrix which was factored as a product of an unknown scalar and a known matrix. Bartholomew demonstrated several problems of testing ordered alternatives in his

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** This research was supported by the Grant in Aid to Encouraging Research (A) of the Minstry of Education, Japan.

papers and all these were discussed in details in the book of him and others (1972).

2. One-sided boundary test

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with the unknown mean vector $\theta=(\theta_1,\theta_2)$ and the known covariance matrix which has a unit variance and a correlation coefficient ρ, and $(\bar{X}, \bar{Y})$ be a sample mean vector. In this section we shall consider a problem of testing the null hypothesis H_0: $H_{01} \cup H_{02}$, where H_{01}: $\theta_1 \geqq 0$ and $\theta_2=0$, and H_{02}: $\theta_1=0$ and $\theta_2 \geqq 0$, against the alternative K_0: $\theta_1>0$ and $\theta_2>0$.

At first we shall derive the maximum likelihood estimates (MLE) $\hat{\theta}$ of θ under H_0 and $H_0 \cup K_0$ respectively. To do this let us consider the following transformation in the two dimensional Euclidean space R^2:

$$\xi = x\,, \qquad \eta = (1-\rho^2)^{-1/2}(\rho x - y) \tag{1}$$

or

$$x = \xi\,, \qquad y = \rho\xi - (1-\rho^2)^{1/2}\eta\,. \tag{2}$$

(ξ, η), the random vector corresponding to (X, Y), is distributed as a bivariate normal with a mean

$$\{\theta_1, (1-\rho^2)^{-1/2}(\rho\theta_1-\theta_2)\} = (\phi_1, \phi_2) = \phi$$

and a common variance 1 and a covariance 0.

H_0, H_{01}, H_{02} and K_0 are transformed to H_0': $H_{01}' \cup H_{02}'$, H_{01}': $\phi_1 \geqq 0$ and $\phi_2=\rho(1-\rho^2)^{-1/2}\phi_1$, H_{02}': $\phi_1=0$ and $\phi_2 \leqq 0$ and K_0': $\phi_1>0$ and $\phi_2<\rho(1-\rho^2)^{-1/2}\phi_1$.

The following factorization of the likelihood is convenient to derive the MLE.

$$\begin{aligned} L(\theta_1, \theta_2) &= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x, y)\right]\exp\left[-\frac{n}{2}(\bar{x}-\theta_1)^2\right] \\ &\quad \times \exp\left[-\frac{n}{2(1-\rho^2)}\{\bar{y}-\rho\bar{x}-(\theta_2-\rho\theta_1)\}^2\right] \qquad (3) \\ &= \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2}Q(x, y)\right]\exp\left[-\frac{n}{2}(\bar{y}-\theta_2)^2\right] \\ &\quad \times \exp\left[-\frac{n}{2(1-\rho^2)}\{\bar{x}-\rho\bar{y}-(\theta_1-\rho\theta_2)\}^2\right] \end{aligned}$$

where

$$Q(x, y) = \sum_{i=1}^{n}\{(x_i-\bar{x})^2-2\rho(x_i-\bar{x})(y_i-\bar{y})+(y_i-\bar{y})^2\}/(1-\rho^2)\,.$$

Thus we have easily the MLE $\hat{\phi}$ and ϕ under $H_0' \cup K_0'$ as follows.

If $\bar{\xi} \geqq 0$ and $\bar{\eta} \leqq \rho(1-\rho^2)^{-1/2}\bar{\xi}$, $\hat{\phi}_1 = \bar{\xi}$ and $\hat{\phi}_2 = \bar{\eta}$, (4)

if $\bar{\eta} > \rho(1-\rho^2)^{-1/2}\bar{\xi}$ and $\bar{\eta} > -\rho^{-1}(1-\rho^2)^{1/2}\bar{\xi}$, (5)

$$\hat{\phi}_1 = (1-\rho^2)\{\bar{\xi}+\rho(1-\rho^2)^{-1/2}\bar{\eta}\} \quad\text{and}\quad \hat{\phi}_2 = \rho(1-\rho^2)^{1/2}\{\bar{\xi}+\rho(1-\rho^2)^{-1/2}\bar{\eta}\}\,,$$

if $\bar{\eta} \leqq -\rho^{-1}(1-\rho^2)^{1/2}\bar{\xi}$ and $\bar{\eta} > 0$, $\hat{\phi}_1 = 0$ and $\hat{\phi}_2 = 0$ (6)

and

if $\bar{\eta} \leqq 0$ and $\bar{\xi} > 0$, $\hat{\phi}_1 = 0$ and $\hat{\phi}_2 = \bar{\eta}$. (7)

Transforming back to the original variables, the MLE $\hat{\theta}$ under $H_0 \cup K_0$ and its maximum likelihood, $\underset{H_0 \cup K_0}{\operatorname{Max}} L(\theta_1, \theta_2)$, are given as follows.

If $\bar{X} \geqq 0$ and $\bar{Y} \geqq 0$, $\hat{\theta}_1 = \bar{X}$, $\hat{\theta}_2 = \bar{Y}$ and

$$\underset{H_0 \cup K_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right], \tag{8}$$

if $\bar{Y} < 0$ and $\rho\bar{Y} < \bar{X}$, $\hat{\theta}_1 = \bar{X} - \rho\bar{Y}$, $\hat{\theta}_2 = 0$ and

$$\underset{H_0 \cup K_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right] \exp\left[-\frac{n}{2} \bar{y}^2\right], \tag{9}$$

if $\rho\bar{Y} \geqq \bar{X}$ and $\bar{Y} < \rho\bar{X}$, $\hat{\theta}_1 = 0$, $\hat{\theta}_2 = 0$ and

$$\underset{H_0 \cup K_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right] \times \exp\left[-\frac{n}{2(1-\rho^2)} (\bar{x}^2 - 2\rho\bar{x}\bar{y} + \bar{y}^2)\right] \tag{10}$$

and

if $\bar{Y} \geqq \rho\bar{X}$ and $\bar{X} < 0$, $\hat{\theta}_1 = 0$, $\hat{\theta}_2 = \bar{Y} - \rho\bar{X}$ and

$$\underset{H_0 \cup K_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right] \exp\left[-\frac{n}{2} \bar{x}^2\right]. \tag{11}$$

As the space of H_0 is the boundary of that of K_0, the MLE and the maximum likelihood under H_0 differ when $\bar{X}$ and $\bar{Y}$ are both positive.

If $\bar{X} \geqq \bar{Y} \geqq 0$, $\hat{\theta}_1 = \bar{X} - \rho\bar{Y}$, $\hat{\theta}_2 = 0$ and

$$\underset{H_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right] \exp\left[-\frac{n}{2} \bar{y}^2\right] \tag{12}$$

and

if $\bar{Y} > \bar{X} \geqq 0$, $\hat{\theta}_1 = 0$, $\hat{\theta}_2 = \bar{Y} - \rho\bar{X}$ and

$$\underset{H_0}{\operatorname{Max}} L(\theta_1, \theta_2) = \left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right)^n \exp\left[-\frac{1}{2} Q(x, y)\right] \exp\left[-\frac{n}{2} \bar{x}^2\right]. \tag{13}$$

The likelihood ratio test can be easily derived and the rigion of rejection is given by the following suprisingly simple form:

$$\begin{cases} \sqrt{n}\,\bar{Y} \geqq B_0 & \text{if } \bar{X} \geqq \bar{Y} \geqq 0, \\ \sqrt{n}\,\bar{X} \geqq B_0 & \text{if } \bar{Y} > \bar{X} \geqq 0 \end{cases} \tag{14}$$

or equivalently

$$\sqrt{n}\,\mathrm{Min}\,(\bar{X}, \bar{Y}) \geqq B_0 \qquad \text{if} \quad \bar{X} \geqq 0\,, \quad \bar{Y} \geqq 0 \tag{15}$$

where B_0 is chosen so that the probability of (14) when the null hypothesis is true is equal to the significance level α.

In order to determine B_0 for the significance level α, the probability of rejecting H_0 when the population mean vector is (θ_1, θ_2) is denoted by

$$\alpha(\theta_1, \theta_2) = P(\bar{X} \geqq B_0/\sqrt{n}\,, \bar{Y} \geqq B_0/\sqrt{n}\,) \tag{16}$$

and the constant B_0 is to be determined by the relation

$$\alpha = \mathrm{Max}\,\{\sup_{\theta \geqq 0} \alpha(\theta, 0),\ \sup_{\theta \geqq 0} \alpha(0, \theta)\}\,. \tag{17}$$

Noticing the relation

$$\alpha(\theta, 0) = \alpha(0, \theta) \qquad \text{for} \qquad \theta \geqq 0$$

and

$$\begin{aligned}\alpha(\theta, 0) &= L(B_0 - \sqrt{n}\,\theta, B_0; \rho) \\ &\leqq \sup_{\theta \geqq 0} L(B_0 - \sqrt{n}\,\theta, B_0; \rho) \\ &= Q(B_0)\end{aligned}$$

where

$$L(h, k; \rho) = \int_h^\infty \int_k^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right] dv\,du$$

and

$$Q(m) = \int_m^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}u^2\right] du\,.$$

Taking the above properties into consideration, the relation (17) is reduced to

$$\alpha = Q(B_0)\,. \tag{18}$$

Therefore the desired B_0 can be found easily from the table of a univariate normal distribution [7].

3. Two-sided boundary test

In this section we shall consider a kind of two-sided version of the test in the previous section. The null hypothesis is $H_1: H_{11} \cup H_{12}$ where $H_{11}: -\infty \leqq \theta_1 \leqq \infty$ and $\theta_2 = 0$, and $H_{12}: \theta_1 = 0$ and $-\infty \leqq \theta_2 \leqq \infty$, and the alternative is $K_1: K_{11} \cup K_{12}$ where $K_{11}: \theta_1 > 0$ and $\theta_2 > 0$ and $K_{12}: \theta_1 < 0$ and $\theta_2 < 0$.

Making use of the same transformation (1) and applying the method similar to the one used in the previous section, the MLE $\hat{\theta}$ of θ under $H_1 \cup K_1$ and its maximum likelihood, $\mathrm{Max}_{H_1 \cup K_1} L(\theta_1, \theta_2)$, are given as follows.

If $\bar{X}$ and $\bar{Y}$ are of the same sign, $\hat{\theta}_1=\bar{X}$, $\hat{\theta}_2=\bar{Y}$ and

$$\operatorname*{Max}_{H_1\cup K_1} L(\theta_1, \theta_2)=\Big(\frac{1}{2\pi\sqrt{1-\rho^2}}\Big)^n \exp\Big[-\frac{1}{2}Q(x,y)\Big], \tag{19}$$

if $\bar{X}$ and $\bar{Y}$ are of the different sign and $|\bar{X}|\geqq|\bar{Y}|$, $\hat{\theta}_1=\bar{X}-\rho\bar{Y}$, $\hat{\theta}_2=0$ and

$$\operatorname*{Max}_{H_1\cup K_1} L(\theta_1, \theta_2)=\Big(\frac{1}{2\pi\sqrt{1-\rho^2}}\Big)^n \exp\Big[-\frac{1}{2}Q(x,y)\Big]\exp\Big[-\frac{n}{2}\bar{y}^2\Big] \tag{20}$$

and

if $\bar{X}$ and $\bar{Y}$ are of the different sign and $|\bar{X}|<|\bar{Y}|$, $\hat{\theta}_1=0$, $\hat{\theta}_2=\bar{Y}-\rho\bar{X}$ and

$$\operatorname*{Max}_{H_1\cup K_1} L(\theta_1, \theta_2)=\Big(\frac{1}{2\pi\sqrt{1-\rho^2}}\Big)^n \exp\Big[-\frac{1}{2}Q(x,y)\Big]\exp\Big[-\frac{n}{2}\bar{x}^2\Big]. \tag{21}$$

As the space of H_1 is the boundary of that of K_1, the MLE and the maximum likelihood under H_1 differ when $\bar{X}$ and $\bar{Y}$ are of the same sign.

If $|\bar{X}|\geqq|\bar{Y}|$, $\hat{\theta}_1=\bar{X}-\rho\bar{Y}$, $\hat{\theta}_2=0$ and

$$\operatorname*{Max}_{H_1} L(\theta_1, \theta_2)=\Big(\frac{1}{2\pi\sqrt{1-\rho^2}}\Big)^n \exp\Big[-\frac{1}{2}Q(x,y)\Big]\exp\Big[-\frac{n}{2}\bar{y}^2\Big] \tag{22}$$

and

if $|\bar{X}|<|\bar{Y}|$, $\hat{\theta}_1=0$, $\hat{\theta}_2=\bar{Y}-\rho\bar{X}$ and

$$\operatorname*{Max}_{H_1} L(\theta_1, \theta_2)=\Big(\frac{1}{2\pi\sqrt{1-\rho^2}}\Big)^n \exp\Big[-\frac{1}{2}Q(x,y)\Big]\exp\Big[-\frac{n}{2}\bar{x}^2\Big]. \tag{23}$$

The likelihood ratio test can be easily derived and the rigion of rejection is given by the following surprisingly simple form:

$$\begin{cases} \sqrt{n}\,\bar{Y}\geqq B_1 & \text{if} \quad \bar{X}\geqq\bar{Y}\geqq 0, \\ \sqrt{n}\,\bar{Y}\leqq -B_1 & \text{if} \quad \bar{X}\leqq\bar{Y}\leqq 0, \\ \sqrt{n}\,\bar{X}\geqq B_1 & \text{if} \quad \bar{Y}>\bar{X}\geqq 0, \\ \sqrt{n}\,\bar{X}\leqq -B_1 & \text{if} \quad \bar{Y}<\bar{X}\leqq 0 \end{cases} \tag{24}$$

or equivalently

$$\sqrt{n}\operatorname{Min}(|\bar{X}|, |\bar{Y}|)\geqq B_1 \text{ and } \bar{X} \text{ and } \bar{Y} \text{ are of the same sign} \tag{25}$$

where B_1 is chosen so that the probability of (24) when the null hypothesis is true is equal to the significance level α.

In order to determine B_1 for the significance level α, the probability of rejecting H_1 when the population mean vector is (θ_1,θ_2) is denoted by

$$\alpha(\theta_1, \theta_2; \rho)=P(\bar{X}\geqq B_1/\sqrt{n}\,,\ \bar{Y}\geqq B_1/\sqrt{n}\,) \\ +P(\bar{X}\leqq -B_1/\sqrt{n}\,,\ \bar{Y}\leqq -B_1/\sqrt{n}\,) \tag{26}$$

and the constant B_1 is to be determined by the relation

$$\alpha = \mathrm{Max}\{ \sup_{-\infty \leqq \theta \leqq \infty} \alpha(\theta, 0; \rho), \sup_{-\infty \leqq \theta \leqq \infty} \alpha(0, \theta; \rho)\} . \tag{27}$$

The above equation can be simplified by the property, $\alpha(\theta,0;\rho)=\alpha(0,\theta;\rho)$, as follows

$$\alpha = \sup_{-\infty \leqq \theta \leqq \infty} \alpha(\theta, 0; \rho) \tag{28}$$

and by simple calculation we have

$$\alpha(\theta, 0; \rho) = L(B_1 - \sqrt{n}\,\theta, B_1; \rho) + L(B_1 + \sqrt{n}\,\theta, B_1; \rho) .$$

In case when $\rho \leqq 0$, we have

$$\alpha = \sup_{0 \leqq \theta \leqq \infty} \alpha(\theta, 0; \rho) = Q(B_1) . \tag{29}$$

However when $\rho > 0$, we have to compute $\sup_{0 \leqq \theta \leqq \infty} \alpha(\theta,0; \rho)$ for each values of ρ.

Therefore the constant B_1 can be determined the equation, $\alpha = Q(B_1)$, in case when $\rho \leqq 0$. But in case when $\rho > 0$ we must determine the constant B_1 directly from the following equation.

$$\alpha = \sup_{0 \leqq \theta \leqq \infty} \alpha(\theta, 0; \rho) . \tag{30}$$

4. Application

Suppose we have samples from three normal distributions with different means μ_1, μ_2, μ_3 and a known common variance, and we want to test the hypothesis H_0: ($\mu_1 = \mu_2 \geqq \mu_3$ or $\mu_1 \geqq \mu_2 = \mu_3$) against the alternative K_0: ($\mu_1 > \mu_2 > \mu_3$) or to test the hypothesis H_0': ($\mu_1 = \mu_2 \geqq \mu_3$ or $\mu_1 = \mu_3 \geqq \mu_2$) against the alternative K_0': ($\mu_1 > \mu_2$, $\mu_1 > \mu_3$). In the first case the two differences in sample means, $y_1 = \bar{x}_1 - \bar{x}_2$, $y_2 = \bar{x}_2 - \bar{x}_3$, will have a bivariate normal distribution with a known covariance matrix and the means are both non-negative and at least one of them is zero under the null hypothesis and both are positive under the alternative. Similarly we can work on $y_1 = \bar{x}_1 - \bar{x}_2$, $y_2 = \bar{x}_1 - \bar{x}_3$ in the second case. The correlation between y_1 and y_2 is negative in case of H_0 and K_0 and positive in case of H_0' and K_0'. Therefore we can legitimately apply the one-sided boundary test of bivariate normal mean discussed in this paper.

Futhermore we can apply the two-sided boundary test of normal mean to the problems of testing the hypothesis H_1: ($\mu_1 = \mu_2$ or $\mu_2 = \mu_3$) against the alternative K_1: ($\mu_1 > \mu_2 > \mu_3$ or $\mu_1 < \mu_2 < \mu_3$) and of testing the hypothesis H_1': ($\mu_1 = \mu_2$ or $\mu_1 = \mu_3$) against the alternative K_1': ($\mu_1 > \mu_2$, $\mu_1 > \mu_3$ or $\mu_1 < \mu_2$, $\mu_1 < \mu_3$).

The author is deeply indebted to Professor A. Kudô of Kyushu University for his helpful advices and critical readings of the original manuscript. The author is also grateful to Dr. H. Yamato of Kagashima University for his advices and encouragments.

References

[1] Barlow, R.E., Bartholomew, D.J., Bremner, J.M. and Brunk, H.D. (1972). Statistical Inference under Order Restrictions. New York: Wiley.

[2] Inada, K., Tsukamoto, K. and Yamauchi, K. (1977). A bivariate analogue of the two-sided t test. Tamkang Journal of Mathematics, Vol. 8, No. 1, 111–121.

[3] Kudô, A. (1963). A multivariate analogue of the one-sided test. Biometrika 50, 403–418.

[4] Kudô, A. and Fujisawa, H. (1964). A bivariate normal test with two-sided alternative. Memoirs of the Faculty of Science, Kyushu University, Ser. A, 18, 104–108.

[5] Kudô, A. and Fujisawa, H. (1965). Some multivariate tests with restricted alternative hypotheses. Multivariate Analysis edited by Krishnaish, P.R., Academic Press Inc. New york, 73–85.

[6] Nüesch, P.E. (1966). On the problem of testing location in multivariate populations for restricted alternatives. Ann. Math. Statist. 37, 113–119.

[7] Yamauti, Z. (1972). Statistical Tables and Formulas with Computer Applications JSA–1972. Japanese Standards Association.

[8] Yeh, Neng-che. (1968). A multivariate normal test with two-sided alternative. Bull. Math. Statist. 13, 85–88.