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Polynomial splines are employed, experimentally, to approximate to the solution of a simple two-point boundary value problem for a nonlinear ordinary differential equation. Checked by comparison with the analytical solution, the results are encouraging.

1. Introduction and Description of Method

Cubic and quintic splines are of much use for approximating solutions of two-point boundary value problems for both linear and nonlinear ordinary differential equations. For a spline $s(t)$ of order $n+3$ defined on a partition $\pi: \{t_0 < t_1 < \cdots < t_n;\ t_i = ih = i/n\}$, we may represent this in the form

$$s(t) = \sum \alpha_i Q_{n+4}(t/h + n + 4 - i) \tag{1}$$

with undetermined coefficient $(\alpha_1, \alpha_2, \cdots, \alpha_{2n+3})$.
Thus the use of polyonomial splines entails a determination of these *too many* parameters. In the present paper, we shall give a simple and efficient selection of the additional conditions.

Now suppose that the differential equation is

$$x'' = f(t, x, x') \qquad (0 \le t \le 1) \tag{2}$$

with the boundary conditions

$$a_0 x(0) - b_0 x'(0) = c_0, \tag{3}$$

$$a_1 x(1) + b_1 x'(1) = c_1. \tag{4}$$

The number of coefficients in (1) is $(2n+3)$. The conditions (3) and (4) give us two equations towards the determination of these. There remain $(2n+1)$ to be determined, and we notice that there are $(n+1)$ nodes: the satisfaction of the differential equation by collocation at these nodes and mid-points gives us precisely the requisite number of equations. The equation to be satisfied at $t = ph/2$ $(p = 0, 1, \cdots, 2n)$ is:

$$\begin{aligned} &\sum \alpha_j \{Q_{n+2}(k-j) - 2Q_{n+2}(k-j-1) + Q_{n+2}(k-j-2)\}/h^2 \\ &\quad = f(ph/2, \sum \alpha_j Q_{n+4}(k-j), \sum \alpha_j \{Q_{n+3}(k-j) - Q_{n+3}(k-j-1)\}/h) \\ &\qquad\qquad (k = p/2 + n + 4). \end{aligned} \tag{5}$$

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To these equations we add those from the boundary conditions (3) and (4):

$$\sum \alpha_j \left[a_0 Q_{n+4}(k_1-j)-b_0\{Q_{n+3}(k_1-j)-Q_{n+3}(k_1-j-1)\}/h\right] = c_0\,, \qquad (6)$$

$$\sum \alpha_j \left[a_1 Q_{n+4}(k_2-j)+b_1\{Q_{n+3}(k_2-j)-Q_{n+3}(k_2-j-1)\}/h\right] = c_1 \qquad (7)$$

$$(k_1, k_2) = (n+4, 2n+4)\,.$$

Now we consider the application of the stated method by the sample equations.

2. Numerical Illustration

Example 1 ([2]). Consider the linear two-point boundary value problem:

$$x'' = 4x+4\cosh(1)\,, \qquad x(0)=x(1)=0\,.$$

The problem has a unique solution

$$x(t) = \cosh(2t-1)-\cosh(1)\,.$$

Table 1.

n	e (0.25)	e (0.5)	e (0.75)
4	1.65 (− 8)	1.51 (− 8)	1.65 (− 8)
8	1.35 (−13)	1.21 (−13)	1.37 (−13)

We use 1.65 (−8) to denote 1.65×10^{-8}.

Example 2 ([4]). Now we exmaine a nonlinear boundary value problem:

$$x'' = (e^{2x}+x'^2)/2\,,$$

$$x(0)-x'(0)=1, \qquad x(1)+x'(1) = -\ln(2)-0.5\,.$$

The solution is $x(t)=-\log(1+t)$.

Table 2.

n	e (0)	e (0.5)	e (1)
2	1.82 (− 4)	1.15 (− 4)	7.77 (− 5)
4	5.53 (− 7)	3.80 (− 7)	2.86 (− 7)
8	1.32 (−10)	8.53 (−11)	5.74 (−11)

Example 3 ([4]). We now take as our next example the boundary value problem:

$$x'' = (x^2+x'^2)/2e^t\,,$$

$$x(0)-x'(0)=0\,, \qquad x(1)+x'(1)=2e\,.$$

The solution is $x(t)=e^t$.

Table 3.

n	e (0)	e (0.5)	e (1)
2	1.02 (− 5)	9.87 (− 6)	1.25 (− 5)
4	4.27 (− 9)	4.04 (− 9)	4.59 (− 9)
8	9.57 (−13)	8.57 (−13)	9.06 (−13)

For a more efficient approximation, we shall consider the spline function $\phi_i(t)$ ($i=1, 2,\cdots, n$) of the form

$$\phi_i(t) = \sum \alpha_{ij} Q_{m+4}\{(t-t_i)/h_1+m+4-j\}$$

such that

$$\phi_i''(t) = f(t, \phi_i(t), \phi_i'(t)) \qquad (t = t_{i-1}+jh_1/2; j = 0, 1,\cdots, 2m, h_1 = h/m),$$

$$\phi_i(t_i) = \phi_{i+1}(t_i), \qquad \phi_i'(t_i) = \phi_{i+1}'(t_i) \qquad (i = 1, 2,\cdots, n-1),$$

$$a_0\phi_1(0)-b_0\phi_1'(0) = c_0, \qquad a_1\phi_n(1)+b_1\phi_n'(1) = c_1.$$

Here we notice that the coefficient matrix of the unknown α_{ij} is a band one.

Example 4. We shall consider the same problem in Example 1.

Table 4.

	$n=1$	2	4
$m=2$	3.50(− 6)	8.26(− 6)	5.17(− 7)
4	1.51(− 8)	3.24(− 9)	4.94(−11)
8	1.20(−13)	1.09(−13)	2.71(−13)

In Examples 4 and 5, the errors denote the maximum differences between the approximations and the solution at the joints.

This method can be also applied to the following singular boundary value problem:

$$x'' = f(t, x, x') \;(= -kx'/t+g(t, x)) \qquad (0 < t \leq 1),$$

$$x'(0) = 0, \qquad x(1) = c_1.$$

In this case, we shall consider the spline function $\phi_i(t)$ of the form

$$\phi_i(t) = \sum \alpha_{ij} Q_{m+2}\{(t-t_i)/h_1+m+2-j\}$$

such that

$$\phi_i'' = f(t, \phi_i(t), \phi_i'(t)) \qquad (t = t_{i-1}+jh_1/2 : j = 1, 2,\cdots, 2m-1),$$

$$\phi_i(t_i) = \phi_{i+1}(t_i), \qquad \phi_i'(t_i) = \phi_{i+1}'(t_i) \qquad (i = 1, 2,\cdots, n-1),$$

$$\phi_1'(0) = 0, \qquad \phi_n(1) = c_1.$$

Example 5 ([3]). Let us consider the linear boundary problem:

$$x''+2x'/t-4x = -2, \qquad x'(0) = 0, \qquad x(1) = 5.5.$$

The solution is given by

$$x(t) = 0.5+(5/t)\sinh(2t)/\sinh(2).$$

Table 5.

	$n=1$	2	4
$m=2$	4.65(−3)	3.21(− 3)	9.13(− 4)
4	2.59(−5)	9.27(− 6)	6.65(− 7)
8	3.76(−9)	2.00(−10)	1.82(−10)

These experiments show that the method is potentially useful. Work is proceeding on an examination of more complicated cases and on an error analysis. The gain to be achieved by *un*equal order of spline $\phi_i(t)$ could also be explored.

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