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Rep. Fac. Sci. Kagoshima Univ., (Math., Phys. & Chem.),
No. 13, p. 11–13, 1980

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(Received Feb. 4, 1980)

Abstract

The parameters which determine quintic spline are used to give more accurate approximations than those from the quintic spline with little additional computational effort. A selection of numerical results is presented in Tables 1–3.

1. Introduction and description of method

Let s be a quintic spline, with equally spaced knots t_i ($t_i=ih$; $nh=1$) interpolating to the given function y at the knots. Since there are $(n+5)$ parameters, the determination of the spline entails the use of special four equations (end conditions). In the present paper we shall consider the following end conditions:

$$(1)\qquad \begin{aligned} &s_0^{(4)}+\alpha_1 s_1^{(4)}+\beta_1 s_2^{(4)}=c_0\,, \quad s_0^{(4)}+\gamma_1 s_1^{(4)}+\delta_1 s_2^{(4)}+\eta_1 s_3^{(4)}=c_1\,,\\ &s_n^{(4)}+\alpha_2 s_{n-1}^{(4)}+\beta_2 s_{n-2}^{(4)}=c_n, \quad s_n^{(4)}+\gamma_2 s_{n-1}^{(4)}+\delta_2 s_{n-2}^{(4)}+\eta_2 s_{n-3}^{(4)}=c_{n-1}\,. \end{aligned}$$

Letting θ and $\kappa(|\theta|>|\kappa|>1)$ be the roots of the quartic polynomial $t^4+26t^3+66t^2+26t+1=0$, $p_i(t)=1+\alpha_i t+\beta_i t^2$ and $q_i(t)=1+\gamma_i t+\delta_i t^2+\eta_i t^3$, we have

THEOREM 1 ([3]) *Let s be an interpolatory quintic spline which agrees with the smooth function y at the uniform knots and satisfies the conditions* (1). *If $p_i(1/\theta)q_i(1/\kappa)-p_i(1/\kappa)q_i(1/\theta)\neq 0$ we have in the interval bounded away from the end points $t=0$, 1*

$$s_i'=y_i''+(h^6/5040)\,y_i^{(7)}+O(h^8)$$
$$s_i''=y_i''+(h^4/720)\,y_i^{(6)}-(h^6/3360)\,y_i^{(8)}+O(h^8)\,.$$

PROOF. From the relationship between the function values and the fourth derivatives of the quintic spline:

$$\begin{aligned}&(1/120)\,(s_{i+2}^{(4)}+26s_{i+1}^{(4)}+66s_i^{(4)}+26s_{i-1}^{(4)}+s_{i-2}^{(4)})\\ &\quad=(1/h^4)(s_{i+2}-4s_{i+1}+6s_i-4s_{i-1}+s_{i-2})\,,\\ &(1/120)\,\{(s_{i+2}^{(4)}-y_{i+2}^{(4)})+26\,(s_{i+1}^{(4)}-y_{i+1}^{(4)})+66\,(s_i^{(4)}-y_i^{(4)})+26\,(s_{i-1}^{(4)}-y_{i-1}^{(4)})\\ &\quad+(s_{i-2}^{(4)}-y_{i-2}^{(4)})\}=-(h^2/12)\,y_i^{(6)}-(h^4/60)\,y_i^{(8)}+O\,(h^6)\,.\end{aligned}$$

Hence we have the asymptotic expansion:

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$$s_i^{(4)} = y_i^{(4)} - (h^2/12)\, y_i^{(6)} + (h^4/240)\, y_i^{(8)} + O(h^6)$$

in the interval bounded away from the end points.

Since $h^2 s_i'' = (2s_i - 5s_{i+1} + 4s_{i+2} - s_{i+3}) + h^4(18s_i^{(4)} + 65s_{i+1}^{(4)} + 26s_{i+2}^{(4)} + s_{i+3}^{(4)})/120$

$$h s_i' = (-11s_i + 3s_{i+1} - 3s_{i+2} + s_{i+3}) + h^4(-19s_i^{(4)} - 108s_{i+1}^{(4)} - 51s_{i+2}^{(4)} - 2s_{i+3}^{(4)})/720\,,$$

we have the desired result (Hoskins and McMaster [1972]).

By arguments similar to those of Theorem 1 we arrive at

THEOREM 2. *Let s be an interpolatory quintic spline subject to the conditions:* $\Delta^r s_0^{(4)} = \Delta^{r+1} s_0^{(4)} = 0$ *and* $\nabla^r s_n^{(4)} = \nabla^{r+1} s_n^{(4)} = 0$ $(r = 6, 7, \cdots)$. *Then we have the same asymptotic expansion of Theorem 1 for all i* $(0 \le i \le n)$.

Using Taylor series we obtain

COROLLARY. *Let the hypotheses of Theorem 2 hold. Then we have*

(2) $$(1/5040)(-s_{i+3}' + 6s_{i+2}' - 15s_{i+1}' + 5060s_i' - 15s_{i-1}' + 6s_{i-2}' - s_{i-3}') = y_i' + O(h^8)$$

(3) $$(1/720)(-s_{i+2}'' + 4s_{i+1}'' + 714s_i'' + 4s_{i-1}'' - s_{i-2}'') = y_i'' + O(h^6)$$

(4) $$(1/7560)(4s_{i+3}'' - 34.5s_{i+2}'' + 102s_{i+1}'' + 7417s_i'' + 102s_{i-1}'' - 34.5s_{i-2}'' + 4s_{i-3}'') = y_i'' + O(h^8)\,.$$

In order to obtain a coefficient matrix of band width five, we shall require to rewrite the end condition $\Delta^r s_0^{(4)} = 0$ in the form:

$$s_0^{(4)} + a_r s_1^{(4)} + b_r s_2^{(4)} + c_r s_3^{(4)} = \cdots$$

where

	$r=5$	6	7	8	$\cdots$
a_r	27	26	8229/317	59805/2304	$\cdots$ $26+1/\theta$
b_r	67	65	20571/317	149490/2304	$\cdots$ $-\theta(\theta+26)$
c_r	25	304/13	7363/317	53469/2304	$\cdots$ $-\theta$

In using (2), (3) and (4), the end conditions $\Delta^r s_0^{(4)} = \nabla^r s_n^{(4)} = 0$ $(r = 7, 8)$ would give rise to the better approximations.

2. Numerical Illustration

In this section we shall consider the application of the above stated method by the sample functions under the end conditions:

$$\Delta^7 s_0^{(4)} = \Delta^8 s_0^{(4)} = 0 \text{ and } \nabla^7 s_n^{(4)} = \nabla^8 s_n^{(4)} = 0\,.$$

Table 1 (sin$t$, $n=16$)

	s'-y'		s''-y''		(4)
	(2)		(3)		
1/8			−2.64(−9)	−3.97(−12)	
2/8	−1.15(−11)	−1.83(−14)	−5.25(−9)	−7.81(−12)	−1.24(−14)
3/8	−1.10(−11)	−1.02(−14)	−7.77(−9)	−1.15(−11)	5.97(−14)
4/8	−1.04(−11)	−1.80(−14)	−1.02(−9)	−1.51(−11)	1.38(−14)
5/8	−9.60(−12)	−1.39(−14)	−1.24(−8)	−1.58(−11)	−5.73(−14)
6/8	−8.66(−12)	−1.58(−14)	−1.45(−8)	−2.14(−11)	3.73(−14)
7/8			−1.63(−8)	−2.42(−11)	

Table 2 (log(1+t), $n=16$)

	s'-y'		s''-y''		(4)
	(2)		(3)		
1/8			−1.21(−6)	6.90(− 8)	
2/8	1.68(− 9)	−1.68(−10)	−6.51(−7)	2.80(− 8)	−7.22(−11)
3/8	8.84(−10)	−5.99(−11)	−3.69(−7)	1.27(− 8)	−2.37(−10)
4/8	4.86(−10)	−2.51(−11)	−2.20(−7)	6.25(− 9)	−1.49(−10)
5/8	2.79(−10)	−1.20(−11)	−1.36(−7)	3.27(− 9)	−8.48(−11)
6/8	1.65(−10)	−7.75(−12)	−8.76(−8)	1.76(− 9)	−8.25(−11)
7/8			−5.83(−8)	7.71(−10)	

Table 3 (exp (t), $n=16$)

	s'-y'		s''-y''		(4)
	(2)		(3)		
1/8			2.40(−8)	−3.58(−11)	
2/8	1.52(−11)	−2.07(−14)	2.72(−8)	−4.05(−11)	6.04(−14)
3/8	1.72(−11)	−2.86(−14)	3.08(−8)	−4.59(−11)	1.62(−14)
4/8	1.95(−11)	−2.84(−14)	3.49(−8)	−5.20(−11)	8.02(−14)
5/8	2.21(−11)	−3.80(−14)	3.96(−8)	−5.89(−11)	3.73(−14)
6/8	2.50(−11)	−4.11(−14)	4.48(−8)	−6.68(−11)	2.71(−14)
7/8			5.08(−8)	−7.60(−11)	

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