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Polynomial Approximations Based on Iterated Cubic Splines and their Applications

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Abstract

We consider an application of iterated cubic splines to Hermite intepolation which is of much use for development of numerical integration formulas of singular integrals. Some numerical examples are given to illustrate the usefulness of our methods.

1. Introduction and description of the methods

Iterated splines are of much use for order-preserving approximations to a given function. There is computational evidence that these give better results than a single spline ([1], [4], [6]). For $n \geq 1$ and a sufficiently smooth function f defined on [0, 1], we consider an application of the iterated cubic splines to the Hermite interpolant p_{2m+1} of the function f at two points x_i and x_{i+1} $(0 \leq i \leq n-1)$, i.e.,

(1) $$p_{2m+1}^{(k)}(x_j) = f^{(k)}(x_j) \quad (j=i,\ i+1; 0 \leq k \leq m-1)$$

where $x_j = jh (= j/n,\ nh=1)$.
Note that polynomial p_{2m+1} of degree $2m+1$ is given in ([1], [2]):

(2) $$p_{2m+1}(x) = f_i\, T_{m,0}(\theta) + f_{i+1}\, T_{m,0}(1-\theta) + h\{f_i'\, T_{m,1}(\theta) - f_{i+1}'\, T_{m,1}(1-\theta)\} + \cdots$$
$$+ h^m \{f_i^{(m)} T_{m,m}(\theta) + (-1)^m f_{i+1}^{(m)}\, T_{m,m}(1-\theta)\} \quad (x = x_i + \theta h,\ 0 \leq \theta \leq 1)$$

with

(3) $$k!\, T_{m,k}(\theta) = \theta^k - \sum_{j=0}^{m} (-1)^j \binom{m-k+j}{j} \binom{2m-k+1}{m-j} \theta^{m+s+1}.$$

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The iterated cubic spline s_m $(m\geq 0)$ can be recursively defined as follows. Let s_0 and s_m $(m\geq 1)$ be the usual cubic spline interpolants of f and s'_{m-1}, respectively, i.e.,

(4) $$s_{0,j}(=s_0(x_j))=f_j,\quad s_{m,j}(=s_m(x_j))=s'_{m,j}\quad (0\leq j\leq n)$$

subject to

(5) $$\Delta^k s'_{m,0}=\nabla^k s'_{m,n}=0\quad (m\geq 0)$$

where $0\leq k\leq n$ and Δ (∇) is the forward (backward) difference operator. For the periodic function f, end conditions (5) are to be replaced by

(6) $$s^{(r)}_{m,0}=s^{(r)}_{m,n}\quad (0\leq r\leq 2,\ m\geq 0).$$

Here we have to notice that the coefficient matrices for determining the iterated cubic splines s_m $(m\geq 0)$ are exactly the same, i.e., s_m $(m\geq 1)$ are easily obtained with little additional effort ([1], page 14). For practical treatment of end conditions (5) (in order to get a tridiagonal system for determining the iterated cubic spline s_m), see [4]. For example, $\Delta^9 s'_{m,0}=0$ (which will be used in numerical examples) can be equivalently rewritten as follows:

(7) $$s'_{m,0}+(265/71)s'_{m,1}=(92017d_1-24637d_2+6567d_3-1715d_4 +419d_5-87d_6+13d_7-d_8)/15336$$

where d_j is the right hand side of the consistency relation for the cubic spline s_m:

(8) $$(s'_{m,j+1}+4s'_{m,j}+s'_{m,j-1})/6=(s_{m,j+1}-s_{m,j-1})/2h.$$

To get an idea of how the m-th iterated cubic spline s_m approximates the m-th derivative $f^{(m)}$, we give the following two lemmas based on results in [4] (non-periodic case) and [6] (periodic case) where the term $O(h^p)$ $(p>0)$ denotes a quantity whose size is proportional to h^p or possibly smaller and $C^q_p[0,1]$ is the set of functions in $C^q(-\infty,\infty)$ which are periodic with period one.

Lemma 1. *For* $1\leq m\leq 9$ *and* $f\in C^9_p[0,1]$ (*periodic case*) *and* $1\leq m\leq k\leq 9$ *and* $f\in C^9[0,1]$ (*non-periodic case*), *the iterated cubic spline* s_m $(m\geq 0)$ *can be uniquely and recursively determined for sufficiently small* h *under* (4) *and* (5) (*or* (6)), *and*

$$s_{m,j}=f_j^{(m)}-m\left\{\frac{h^4}{180}f_j^{(m+4)}-\frac{h^6}{1512}f_j^{(m+6)}\right\}+O(h^L)$$

with $L=9-m$ in the periodic case and $L=\min(9-m, k+1-m)$ in the non periodic case, where for $m+4\geq 9$ and $m+6\geq 9$, the terms $f_j^{(m+4)}$ and $f_j^{(m+6)}$ are absorbed in the order term $O(h^L)$, respectively.

Proof. The non periodic cases $m=1$, 2 were covered in [4] while the periodic cases for all m were done in [6]. The similar technique in [4] gives the desired relations for $m\geq 3$. Just use $m+4$ and $m+6$ in place of m in the above lemma to obtain

$$(9)\quad \begin{aligned} &\text{(i)}\quad h^4 s_{m+4,j}=h^4 f_j^{(m+4)}+O(h^L)\quad (m\geq 4)\\ &\text{(ii)}\quad h^6 s_{m+6,j}=h^6 f_j^{(m+6)}+O(h^L)\quad (m\leq 2). \end{aligned}$$

Combining the above asymptotic relations (9) with Lemma 1, we have

Lemma 2. *For $m\geq 1$, under* (4) *and* (5) (*or* (6))

$$f_j^{(m)}=s_{m,j}+m\left\{\frac{h^4}{180}s_{m+4,j}-\frac{h^6}{1512}s_{m+6,j}\right\}+O(h^L)\quad (0\leq j\leq n).$$

On making use of the above Lemma 2, we can obtain useful approximations to the derivatives $f^{(r)}$ $(1\leq r\leq 3)$:

$$(10)\quad \begin{aligned} &\text{(i)}\quad \text{for } k\geq 3,\ f_j'=s_{1,j}+O(h^3)\\ &\text{(ii)}\quad \text{for } k\geq 5,\ f_j'=s_{1,j}+\frac{h^4}{180}s_{5,j}+O(h^5),\ f_j''=s_{2,j}+O(h^4)\\ &\text{(iii)}\quad \text{for } k\geq 7,\ f_j'=s_{1,j}+\frac{h^4}{180}s_{5,j}-\frac{h^6}{1512}s_{7,j}+O(h^7)\\ &\qquad f_j''=s_{2,j}+\frac{h^4}{90}s_{6,j}+O(h^6),\ f_j^{(3)}=s_{3,j}+\frac{h^4}{60}s_{7,j}+O(h^5). \end{aligned}$$

Hence we have the following theorem where $\tilde{p}_{2m+1}$ of degree $2m+1$ is the polynomial defined by (2) with the derivatives $\tilde{f}_j^{(r)}$ $(j=i,\ i+1,\ 1\leq r\leq m)$ approximated by using (10) (i)-(iii) for $m=1$, 2 and 3, respectively.

Theorem 1. *For $k\geq 2m+1$ $(1\leq m\leq 3)$ (The restriction on k, defined in* (4), *is not necessary in the periodic case) and $f\in C_p^9[0, 1]$ or $f\in C^9[0, 1]$, then*

$$\tilde{p}_{2m+1}(x)-f(x)=O(h^{2m+2})\quad (0\leq x\leq 1)$$

Next, for an approximation of the derivative function f', we can use Lemma 2 again to obtain more accurate approximations to $f_j^{(r)}$ $(1\leq r\leq 4)$ at node points:

$$(11)\quad \begin{aligned}
&\text{(i)}\quad \text{for } k\geq 4,\ f_j'=s_{1,j}+O(h^4),\ f_j''=s_{2,j}+O(h^3)\\
&\text{(ii)}\quad \text{for } k\geq 6,\ f_j'=s_{1,j}+\frac{h^4}{180}s_{5,j}+O(h^6),\ f_j''=s_{2,j}+\frac{h^4}{90}s_{7,j}+O(h^5)\\
&\qquad\qquad f_j^{(3)}=s_{3,j}+O(h^4)\\
&\text{(iii)}\quad \text{for } k\geq 8,\ f_j'=s_{1,j}+\frac{h^4}{180}s_{5,j}-\frac{h^6}{1512}s_{7,j}+O(h^8)\\
&\qquad\qquad f_j''=s_{2,j}+\frac{h^4}{90}s_{6,j}-\frac{h^6}{756}s_{8,j}+O(h^7)\\
&\qquad\qquad f_j^{(3)}=s_{3,j}+\frac{h^4}{6}s_{7,j}+O(h^6),\ f_j^{(4)}=s_{4,j}+\frac{h^4}{45}s_{8,j}+O(h^5).
\end{aligned}$$

Hence as an approximation of the first derivative, we have a polynomial $\tilde{q}_{2m+1}$ of degree $2m+1$ which is the Hermite interpolant of f' given by (2) with the derivatives $f_j^{(r)}$ $(j=i,\ i+1,\ 1\leq r\leq m+1)$ approximated by use of the right hand sides of 11 (i)-(iii) without the order terms.

Theorem 2. *For* $k\geq 2m+2$ $(1\leq m\leq 3)$ (*This restriction on* k, *defined in* (4), *is not necessary in the periodic case*) *and* $f\in C_p^9[0,\ 1]$ *or* $f\in C^9[0,\ 1]$,

$$\tilde{q}_{2m+1}(x)-f'(x)=O(h^{2m+2})\quad (0\leq x\leq 1).$$

For an approximation of the second derivative f'', there exists a polynomial $\tilde{r}_{2m+1}$ of degree $2m+1$ which is the Hermite interplant of f'' given by (2) with the derivatives $f_j^{(r)}$ $(j=i,\ i+1,\ 2\leq r\leq m+2)$ approximated by using Lemma 2:

$$(12)\quad \begin{aligned}
&\text{(i)}\quad \text{for } k\geq 5,\ f_j''=s_{2,j}+O(h^4),\ f_j^{(3)}=s_{3,j}+O(h^3)\\
&\text{(ii)}\quad \text{for } k\geq 7,\ f_j''=s_{2,j}+\frac{h^4}{90}s_{6,j}+O(h^6),\ f_j^{(3)}=s_{3,j}+\frac{h^4}{60}s_{7,j}+O(h^5)\\
&\qquad\qquad f_j^{(4)}=s_{4,j}+O(h^4)\\
&\text{(iii)}\quad \text{for } k\geq 9,\ f_j''=s_{2,j}+\frac{h^4}{90}s_{6,j}-\frac{h^6}{756}s_{8,j}+O(h^8),\\
&\qquad\qquad f_j^{(3)}=s_{3,j}+\frac{h^4}{60}s_{7,j}-\frac{h^6}{504}s_{9,j}+O(h^7),\ f_j^{(4)}=s_{4,j}+\frac{h^4}{45}s_{8,j}+O(h^6)\\
&\qquad\qquad f_j^{(5)}=s_{5,j}+\frac{h^4}{36}s_{9,j}+O(h^5).
\end{aligned}$$

Theorem 3. *For* $k\geq 2m+3$ $(1\leq m\leq 3)$ (*This restriction on* k, *defined in* (4), *is not necessary in the periodic case*) *and* $f\in C_p^9[0,\ 1]$ *or* $f\in C^9[0,\ 1]$,

$$\tilde{r}_{2m+1}(x)-f''(x)=O(h^{2m+2})\quad (0\leq x\leq 1).$$

Finally we note that polynomials $\tilde{p}_{2m+1}$, $\tilde{q}_{2m+1}$ and $\tilde{r}_{2m+1}$ of degree $2m+1$ are all m-times continuously differentiable functions on [0, 1] and order-preserving, i.e., $O(h^{2m+2})$-approximations to $f^{(r)}$ $(0\leq x\leq 2)$, respectively.

2. Numerical integration formulas for singular integrals

We consider an application of the above stated polynomial approximation to numerical integration formulas for singular integrals of the form:

(13) $$\int_{xj}^{xj+1} w(x)f(x)\,dx \approx \text{" a linear combination of } f_i \ (0\leq i\leq n-1) \text{ "}$$

for *some or all* j $(0\leq j\leq n-1)$ where $w(x)=x^{\sigma}$ $(\sigma>-1)$ or $\log(x)$. Now use of the above stated polynomial approximation $\tilde{p}_{2m+1}$ to f leads to following numerical integration formulas $I_{m,j}(h)$ $(1\leq m\leq 3)$:

(14) $$I_{m,j}(h)=\int_{xj}^{xj+1} w(x)p_{2m+1}(x)\,dx\left(=\sum_{i=0}^{m} h^i\{\alpha_j^{(i)} f_j^{(i)}+(-1)^i\beta_j^{(i)} f_{j+1}^{(i)}\}\right)$$

with

(15) $$\alpha_j^{(i)}=h\int_0^1 w(x_j+\theta h)\,T_{m,i}(\theta)\,d\theta \text{ and } \beta_j^{(i)}=h\int_0^1 w(x_j+\theta h)\,T_{m,i}(1-\theta)\,d\theta.$$

For the error in $I_{m,j}(h)$, by means of Theorems 1–3, it is straightforward to show the following theorem.

Theorem 4. *For* $k\geq 2m+1$ $(1\leq m\leq 3)$ (*This restriction on* k *is not necessary in the periodic case*),

$$\int_{xj}^{xj+1} w(x)f(x)\,dx=I_{m,j}(h)+O(h^{2m+2})\int_{xj}^{xj+1}|w(x)|\,dx \quad (0\leq j\leq n-1)$$

Corollary. *Under the same restriction on* m *and* k *in* Theorem 4,

$$\int_0^1 w(x)f(x)\,dx=\sum_{j=0}^{n-1} I_{m,j}(h)+O(h^{2m+2})$$

while the error in compound Simpson rule is $O(h^4 \ln h)$ ([2]).

For the calculation of the weights $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ $(0\leq i\leq m)$, define the following auxiliary integrals:

(16) $$c_j^{(i)}=h\int_0^1 \theta^i\, w(x_j+h\theta)\,d\theta \quad (0\leq i\leq 2m+1)$$

which can be evaluated by means of the following recurrence formulas:

(17)

(i) for $w(x)=x^\alpha$ $(\alpha>-1)$, $(1+\alpha)c_j^{(0)}=x_{j+1}^{(1+\alpha)}-x_j^{(1+\alpha)}$

$$(i+1+\alpha)c_j^{(i)}=x_{j+1}^{(1+\alpha)}-ijc_j^{(i-1)} \quad (1\leq i\leq 2m+1)$$

(ii) for $w(x)=\log(x), c_j^{(0)}=x_{j+1}\log(x_{j+1})-x_j\log(x_j)-h$

$$(i+1)c_j^{(i)}=x_{j+1}\log(x_{j+1})-h-ij\,c_j^{(j-1)}+ih/(i+1) \quad (1\leq i\leq 2m+1).$$

Then, the weights in the integration formulas $I_{m,j}(h)$ $(1\leq m\leq 3)$ can be represented in terms of $c_j^{(i)}$ $(0\leq i\leq 2m+1)$:

(18) $$i!\,\alpha_j^{(i)}=\sum_{r=m+1}^{2m+1}\mu_m(i,r)\,c_j^{(i)} \quad \text{and} \quad i!\,\beta_j^{(i)}=\sum_{r=m+1}^{2m+1}\lambda_m(i,r)\,c_j^{(i)}.$$

Here values of $\mu_m(i,r)$ and $\lambda_m(i,r)$ $(1\leq m\leq 3;\ i<m+1\leq r\leq 2m+1)$ are given in Table 1.

Table 1. Values of μ_m and λ_m $(1\leq m\leq 3)$.

	$\mu_1(i,r)$		$\lambda_1(i,r)$		$\mu_2(i,r)$			$\lambda_2(i,r)$		
$i\backslash r$	2	3	2	3	3	4	5	3	4	5
0	−3	2	3	−2	−10	15	−6	10	−15	6
1	−2	1	1	−1	−6	8	−3	4	−7	3
2					−3	3	−1	1	−2	1

Table 1 (continued)

	$\mu_3(i,r)$				$\lambda_3(i,r)$			
$i\backslash r$	4	5	6	7	4	5	6	7
0	−35	84	−70	20	35	−84	70	−20
1	−20	45	−36	10	15	−39	34	−10
2	−10	20	−15	4	5	−14	13	−4
3	−4	6	−4	1	1	−3	3	−1

3. Numerical Examples

First we consider an application of the above stated methods by taking some examples $f(x)=1/(1+25x^2)$ $(-1\leq x\leq 1)$ (or $1/(1+100(x-1/2)^2)$: $0\leq x\leq 1$) and $\sin(4\pi x)$ $(0\leq x\leq 1)$. In Table 2–4, we give the observed maximum absolute errors of the function, first and second derivative values at mid-points where $a-b=a\times 10^{-b}$. Methods I–III mean the ones by use of $\tilde{p}_{2m+1}$, $\tilde{q}_{2m+1}$ and $\tilde{r}_{2m+1}$ $(1\leq m\leq 3)$, respectively. Rates are the observed ones obtained from the numerical results with $n=64$ and 128 while the figures in parentheses are the predicted ones given in Theorems 1–3.

Table 2

The observed maximum absolute errors of the function values at mid-points.

	$1/(1+25x^2)$: $-1\leq x\leq 1$			$\sin(4\pi x)$: $0\leq x\leq 1$		
n\Method	I	II	III	I	II	III
16	3.79−2	5.67−2	3.94−2	1.06−3	5.41−5	8.17−6
32	6.47−4	2.02−4	1.55−4	6.31−5	1.11−6	2.99−8
64	4.02−5	1.37−6	3.09−7	3.89−6	1.83−8	1.15−10
128	2.38−6	2.57−8	1.76−9	2.42−7	2.90−10	4.47−13
rate	4.1(4)	5.7(6)	7.5(8)	4.0(4)	6.0(6)	8.0(8)

Table 3

The observed maximum absolute errors of the first derivatives at mid-points.

	$1/(1+25x^2)$: $-1\leq x\leq 1$			$\sin(4\pi x)$: $0\leq x\leq 1$		
n\Method	I	II	III	I	II	III
32	1.04−1	7.27−2	5.56−2	2.45−3	8.75−5	3.14−5
64	5.69−3	1.26−3	3.00−4	1.52−4	1.37−6	8.31−9
128	3.12−4	1.97−5	1.47−6	9.53−6	2.14−8	3.24−11
rate	4.2(4)	6.0(6)	7.7(8)	4.0(4)	6.0(6)	8.0(8)

Table 4

The observed maximum absolute errors of the second derivatives at mid-points.

	$1/(1+25x^2)$: $-1\leq x\leq 1$			$\sin(4\pi x)$: $0\leq x\leq 1$		
n\Method	I	II	III	I	II	III
64	5.78−1	8.29−2	4.71−2	3.22−3	6.05−6	1.03−7
128	3.69−2	8.41−4	1.80−4	2.01−4	9.26−8	4.00−10
rate	4.0(4)	6.6(6)	8.0(8)	4.0(4)	6.0(6)	8.0(8)

Next we consider an application of the numerical integration formulas $I_{m,j}(h)$ $(1 \leq m \leq 3)$ for the weight function $w(x)=1/\sqrt{x}$ or $\log(x)$. In the following Tables 5–6, we give the observed absolute errors of the integration formulas on subintervals and the whole interval [0, 1] in the case when $f(x)=\exp(5x)$. The rates are the ones from the numerical results with $n=32$ and 64 while figures in parentheses are the theoretical ones given in Theorem 4. Note that the observed maximum absolute errors of the integration formulas on subintervals occurred at node points bounded away from $x=0$. In comparison with the proposed methods in [3], the errors in the cases when $(w, f)=(1/\sqrt{x}, \exp(x))$ and $(\log(x), \exp(x))$ are 9.32−7 and 8.89−7 $(n=400)$, while the errors of our methods $(1 \leq m \leq 3)$ are 5.94−8, 2.93−11, 3.15−14 and 2.76−8, 1.36−11, 1.47−14 $(n=16)$, respectively. Taking into account of these results, our methods would be of much use in the case when a finer mesh is not acceptable.

Table 5

The observed maximum absolute errors of the integration formulas on subintervals.

	$w(x)=1/\sqrt{x}$			$w(x)=\log(x)$		
$n\backslash m$	1	2	3	1	2	3
16	1.03−4	1.08−6	2.90−7	9.92−6	1.30−7	1.10−7
32	3.55−6	1.08−8	3.02−10	3.16−7	9.85−10	3.27−11
64	1.16−7	8.93−11	2.50−13	9.91−9	7.66−12	1.60−14
rate	4.9(5)	6.9(7)	10.2(9)	5.0(5)	7.0(7)	11.0(9)

Table 6

The observed absolute errors of the integration formulas on the whole interval [0, 1].

	$w(x)=1/\sqrt{x}$			$w(x)=\log(x)$		
$n\backslash m$	1	2	3	1	2	3
16	4.38−4	5.23−6	5.04−7	9.85−5	1.14−6	1.12−7
32	2.81−5	8.77−8	6.55−10	6.25−6	1.95−8	1.14−10
64	1.77−6	1.37−9	1.65−12	3.93−7	3.03−10	3.43−13
rate	4.0(4)	6.0(6)	8.6(8)	4.0(4)	6.0(6)	8.4(8)

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