On the Propagation of Error in Numerical Integrations

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On the Propagation of Error in Numerical Integrations

NAKASHIMA Masaharu
We consider the first order differential equation

\[ \begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases} \]  

And in [5] we have tried to approximate the equation by some difference equation, which is of open type, and have considered the propagation of error of the formula. Here we consider the closed type approximation formula, i.e., Adams-Bashforth type, and investigate the propagation of error of the formula. We denote the value of the solution of the differential equation (1) at the point \( x_n \) by \( y(x_n) \) and the \( i \)-th approximation to \( y(x_n) \) by \( y^{(i)}_{n+i} \). We shall now try to approximate the equation (1) by the difference equation

\[ \begin{align*} y(x_{n+1}) &= y(x_n) + hf(x_{n+1}, y(x_{n+1})) + T_n(\sigma = \frac{1}{2} h^2 y''(x_n + \theta_n h), \ 0 \leq \theta_n \leq 1), \\ y^{(i)}_{n+i+1} &= y_n + hf(x_{n+i+1}, y^{(i)}_{n+i}), \quad (n = 0, 1, \ldots) \end{align*} \]

where we denote the truncation error of \( n \)-step by \( T_n \). And the calculated value of \( y_n \) will be given by the formula:

\[ Y^{(i+1)}_{n+1} = Y^{(i+1)}_n + hf(x_{n+1}, Y^{(i)}_{n+1}) - R^{(i+1)}_n, \]

where \( R^{(i+1)}_n \) is the round-off error of the \((i+1)\)-th iteration of \( n \)-step. And if the difference

\[ L^{(i+1)}_n = Y^{(i+1)}_n - Y^{(i)}_n \]

is smaller than the constant \( L \):

\[ |L^{(i+1)}_n| \leq L, \]

then we set

\[ Y_{n+1} = Y^{(i+1)}_{n+1}. \]

From the equations (2), (3), we obtain the relation

\[ y(x_{n+1}) - Y^{(i+1)}_{n+1} = y(x_n) - y_n + hf(x_{n+1}, y(x_{n+1})) \]

\[ - hf(x_{n+1}, Y^{(i)}_{n+1}) + T_n + R^{(i+1)}_n. \]

And if \( \frac{\partial f}{\partial y} (x, y) \) exists, we have the relations
\[ f(x_{n+1}, \eta(x_{n+1})) - f(x_{n+1}, Y_{n+1}^{i+1}) = \frac{\partial f}{\partial y} (x_{n+1}, \eta_{n+1}) (y(x_{n+1}) - Y_{n+1}^{i+1}), \]

for some \( \eta_{n+1} \) which lies between \( Y_{n+1}^{i+1} \) and \( y(x_{n+1}) \), and

\[ Y_{n+1}^{i+1} - y(x_{n+1}) = Y_{n+1} - y(x_{n+1}). \]

If we set

\[ e_n = y(x_n) - Y_n, \quad e_0 = 0, \]

\[ W_n^{(i)} = T_n + R_n^{(i)} + h \frac{\partial f}{\partial y} (x_{n+1}, \eta_{n+1}) (Y_n - Y_n^{(i)}), \]

then we obtain the relation

\[ e_{n+1} = e_n + h \frac{\partial f}{\partial y} (x_{n+1}, \eta_{n+1}) e_{n+1} + W(x+(n+1)h). \]

And by using the backward difference operator, the above equation may be written in the form (for some constant \( \rho \))

\[ \nabla e_{n+1} = \rho e_{n+1} + (h \frac{\partial f}{\partial y} (x_{n+1}, \eta_{n+1}) - \rho) e_{n+1} + W_n^{(i)}. \]

Here we discuss the asymptotic behavior of the difference equations (4), (5).

**Theorem 1** Under the following assumptions,

(1) \[ |f_y(x_0+\nu h, \eta_\nu)| \leq \Phi(x) \quad (-\infty < y < +\infty) \]

where \( \Phi(x) \) is a continuous function satisfying the following condition

\[ K > 0, \quad h > 0, \quad \sum_{\nu=1}^{\infty} h \Phi(x_0+\nu h) \leq K \quad \text{for} \quad 0 < h \leq h_0, \quad \text{and} \quad \Phi(x) \leq M, \]

(2) \[ \sum_{\nu=1}^{\infty} |W_\nu| \leq E \quad \text{for} \quad 0 < h \leq h_1, \]

we have

\[ |e_n| \leq C \quad \text{for} \quad 0 < h \leq \min\left\{h_0, h_1, \frac{C-CK-E}{CM}\right\}. \]

**Proof.**

The proof is derived by mathematical induction.

Let us assume

\[ |e_\nu| \leq C \quad (\nu = 1, 2, 3, \ldots n-1), \]

and we shall show

\[ |e_n| \leq C. \]
From (4), we have the inequality
\[
|(1 - hf(x_0 + nh, \eta_n)) e_n| < h \sum_{v=1}^{n-1} |f(x_0 + nh, \eta_v) e_v| + \sum_{v=1}^{n} |W_v|,
\]
and taking the constant \( h \) small, we have
\[
|e_n| \leq C.
\]

Next we shall investigate the propagation of error more explicitly.

**Lemma [1]**

**The solution of the equation**

\[
\begin{align*}
\nabla z(x_0 + nh) &= A_n z(x_0 + nh) + B(x_0 + nh) z(x_0 + nh) + W(x_0 + nh) \quad (n = 1, 2, 3, \ldots), \\
z(x_0) &= z_0, \quad (A_n \neq 1: n = 1, 2, \ldots),
\end{align*}
\]

is
\[
z(x_0 + nh) = z_0 Y(x_0 + nh) + Y(x_0 + nh) \sum_{v=1}^{n-1} \left[ B(x_0 + nh) Y^{-1}(x_0 + nh) z(x_0 + nh) \right] \\
+ Y(x_0 + nh) \sum_{v=0}^{n-1} Y^{-1}(x_0 + nh) W(x_0 + nh),
\]

where \( Y(x) \) is the solution of the following difference equation
\[
\begin{align*}
\nabla Y(x_0 + nh) &= A_n Y(x_0 + nh) \quad (v = 1, 2, \ldots), \\
Y(x_0) &= 1.
\end{align*}
\]

**Proof.**

The proof is derived by the well-known method, namely the variation of parameters. Let
\[
z(x_0 + nh) = Y(x_0 + nh) u(x_0 + nh),
\]
then
\[
\nabla z(x_0 + nh) = \nabla \left[ Y(x_0 + nh) u(x_0 + nh) \right]
\]
\[
= (\nabla Y(x_0 + nh)) u(x_0 + nh) + Y(x_0 + nh) \nabla u(x_0 + nh)
\]
\[
= A_n Y(x_0 + nh) u(x_0 + nh) + Y(x_0 + nh) \nabla u(x_0 + nh)
\]
\[
= A_n z(x_0 + nh) + B(x_0 + nh) z(x_0 + nh) + W(x_0 + nh).
\]

Thus
\[
\nabla u(x_0 + nh) = Y^{-1}(x_0 + nh) \left[ B(x_0 + nh) z(x_0 + nh) + W(x_0 + nh) \right],
\]
and hence
\[ u(x_0 + nh) = u(x_0) + \sum_{v=1}^{n} Y^{-1}(x_0 + (v-1) h) B(x_0 + vh) z(x_0 + vh) \]
\[ + \sum_{v=1}^{n} Y^{-1}(x_0 + (v-1) h) W(x_0 + vh) , \]
where
\[ z(x_0) = Y(x_0) u(x_0) \]
\[ = u(x_0). \]

Thus we have the solution
\[ z(x_0 + nh) = Y(x_0 + nh) u(x_0 + nh) \]
\[ = z(x_0) Y(x_0 + nh) + Y(x_0 + nh) \sum_{v=0}^{n-1} \{ Y^{-1}(x_0 + vh) B(x_0 + (v+1) h) z(x_0 + (v+1) h) \} \]
\[ + Y(x_0 + nh) \sum_{v=0}^{n-1} Y^{-1}(x_0 + vh) W(x_0 + (v+1) h) . \]

Q.E.D.

**Lemma [2]**

In the equation (6), if there exist constants \( \rho_n, C, \lambda, L_1, L_2 \), which satisfy the following conditions

(I) \(| Y(x_0 + vh) | \leq C \) (\( C \); constant),

where \( Y(x) \) is a solution of the difference equation

\[ \begin{cases} \nabla Y(x_0 + vh) = \rho_n Y(x_0 + vh) , & (\rho_n \neq 1). \\ Y(x_0) = 1 , \end{cases} \]

(II) \( \sum_{v=0}^{\infty} | B(x_0 + vh) | \leq L_1 , \)

(III) \( | W(x_0 + vh) | \leq \alpha_v e^{\lambda_1 + \lambda_2 + \cdots + \lambda_{n+1}} \) \( (\nu = 0, 1, 2, \cdots) \),

where \( \sum_{v=0}^{\infty} \alpha_v \leq L_2 , \)

\( L_1 + L_2 < |1 - \rho_n - B(x_0 + nh)| \) \((n = 1, 2, \cdots) \) and

\( |(1 - \rho_n)|^{-1} \leq e^{\lambda_n+1} \) \((n = 1, 2, 3, \cdots) \),

then we have

\[ |z(x_0 + mh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \] \((n = 1, 2, 3, \cdots) \).

**Proof.**

The proof is derived by mathematical induction.

Let us assume

\[ |z(x_0 + mh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \cdots + \lambda_m} \] \((m = 1, 2, \cdots n-1) \),

and we shall show that the above inequality holds for \( m=n \).

From Lemma [1], we have
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\[
[1 - Y(x_0 + nh) Y^{-1}(x_0 + (n-1)h) B(x_0 + nh)] z(x_0 + nh)
\]

\[
= z(x_0) Y(x_0 + nh) + Y(x_0 + nh) \sum_{\nu=1}^{n-2} B(x_0 + (\nu + 1)h) Y^{-1}(x_0 + \nu h) z(x_0 + (\nu + 1)h)
\]

\[
+ Y(x_0 + nh) \sum_{\nu=1}^{n-1} Y^{-1}(x_0 + \nu h) W(x_0 + (\nu + 1)h).
\]

And hence we have the inequality

\[
|1 - \rho_n - B(x_0 + nh)| |z(x_0 + nh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} (L_1 + L_2),
\]

and from the condition (3), we have

\[
|z(x_0 + nh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.
\]

Q.E.D.

From the above Lemma we may derive the next theorem.

**Theorem 2** Consider the difference equation (5). If the following conditions are satisfied:

(I) \[ |W_0(x) - \cdots - W_{n-1}(x)| \leq a_\nu e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \] for \(0 < h \leq h_1\) \((\nu = 1, 2, \ldots)\),

where \(\sum_{\nu=1}^{\infty} |a_\nu| \leq L_1\), \(\sum_{\nu=1}^{\infty} |\rho_\nu| \leq L_2\), \(|1 - (1 - \rho_n)^{-1}| \leq e^{\lambda_{n+1}}\)

and \(L_1 + L_2 < 1\),

(II) \[ |f_\nu(x_0 + \nu h, \eta_\nu)| \leq \Phi(x) \] \((-\infty < y < +\infty)\),

where \(\Phi(x)\) is continuous satisfying following conditions

\[ \sum_{\nu=0}^{\infty} \hbar \Phi(x_0 + \nu h) \leq 1 - L_1 - L_2 \] for \(0 < h \leq h_0\) and \(\Phi(x) \leq K\)

then we have

\[ |e_n| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \] for \(0 < h \leq \min\left[h_0, h_1, \frac{1}{K}\right]\).

References