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Approximation of Fourier Transforms by Piecewise Linear Functions

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Abstract

An approximation theorem of Fourier transform by piecewise linear functions is proved.

Key words: Fourier transform, piecewise linear function.

1. Introduction

We denote Banach spaces of all integrable functions and all regular complex finite measures on the real line by \(L^1(R)\) and \(M(R)\), respectively. For a function \(f \in L^1(R)\), the Fourier transform is

\[
f(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \, dx \quad (\xi \in R),
\]

and for a measure \(\mu \in M(R)\), the Fourier-Stieltjes transform is

\[
\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} d\mu(x) \quad (\xi \in R).
\]

We define

\[
A(R) = \{ \hat{f} : f \in L^1(R) \}, \quad \| f \|_{A(R)} = \| f \|_{L^1(R)}
\]

and

\[
B(R) = \{ \hat{\mu} : \mu \in M(R) \}, \quad \| \hat{\mu} \|_{B(R)} = \| \mu \|_{M(R)}.
\]

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We denote Banach spaces of all integrable functions on one dimensional torus $T$ by $L^1(T)$. For a function $f \in L^1(T)$, the Fourier coefficients are

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx$$

for integers $n$.

We define

$$A(T) = \{ \varphi : \varphi \in L^1(T) \}.$$  

Herz [2] has proved that Cantor's ternary set is a spectral synthesis set. Kahane [3] has proved this result using an approximation of functions in $A(T)$ by piecewise linear functions: if $\varphi \in A(T)$ and each $\varphi_n$ is a continuous function on $T$ to equal $\varphi$ at the multiples of $2\pi/n$ and linear on the remainder, then $\varphi_n \in A(T)$ and $\| \varphi_n - \varphi \|_{A(T)} \to 0$ ($n \to \infty$) (See also Kahane and Salem [4] and Benedetto [1]). In this paper, we prove that a similar result holds for $A(R)$.

2. Approximation of Fourier transforms

For a continuous function $\varphi$ on $R$ and a positive integer $n$, we define $\varphi_n$ to be the function equal $\varphi$ at the points $2\pi k/n$, $k = 0, \pm 1, \pm 2, \ldots$, and linear on the remainder of $R$. For a continuous function $\varphi$ on $T$, we define $\varphi_n$ similarly.

Lemma. If $\varphi \in B(R)$, then $\varphi_n \in B(R)$ and $\| \varphi_n \|_{B(R)} \leq \| \varphi \|_{B(R)}$ for all $n$.

Proof. For all $x \in R$, define the function $e_x$ by $e_x(x) = e^{ix}$ (\(x \in R\)). We now show that if $x$ is a rational number, then $(e_{-x})_n \in B(R)$ and $\| (e_{-x})_n \|_{B(R)} = 1$. Suppose that $x$ is a rational number. Take a positive integer $r$ and an integer $s$ such that $x = s/r$. We see that $(e_{-x})_n(\xi) = (e_{-s})_n(r\xi)$ for all $\xi \in R$. Since $e_{-x}$ is $2\pi$-periodic, regarding $e_{-x}$ as a function on $T$, we have $e_{-x} \in A(T)$. It follows that $(e_{-s})_n \in A(T)$ and $\| (e_{-s})_n \|_{A(T)} = 1$ (See Benedetto [1, p.168]. Hence $(e_{-s})_n \in B(R)$ and $\| (e_{-s})_n \|_{B(R)} = 1$. Let $\xi_1, \ldots, \xi_m \in R$ and $c_1, \ldots, c_m$ be complex numbers such that $\| \sum_{k=1}^{m} c_k e^{-\xi_k} \| = 1$. To prove $\varphi_n \in B(R)$ and $\| \varphi_n \|_{B(R)} \leq \| \varphi \|_{B(R)}$, it suffices to show that

$$\left| \sum_{k=1}^{m} c_k \varphi_n(\xi_k) \right| \leq \| \varphi \|_{B(R)}.$$

Let $\varphi = \tilde{\mu}$, $\mu \in M(R)$, and $\varepsilon > 0$. Take $\delta > 0$ so that $2(1 + \| \varphi \|_{B(R)}) \delta \sum_{k=1}^{m} |c_k| < \varepsilon$. Since
Approximation of Fourier Transforms by Piecewise Linear Functions

(e_-\pi n) (\xi_i) is uniformly continuous as a function of \( x \) for each \( k \), there is a positive number \( \eta \) such that \( |(e_-\pi n (\xi_k) - (e_-\pi n (\xi_i))| < \delta \) for all \( x, y \) and \( k \) such that \( |x - y| < \eta \). Since \( \mu \in M(R) \), there are \( t_0, ..., t_p \in R \) so that \( t_0 < ... < t_p, t_i - t_{i-1} = \eta \) for all \( i \), and \( \mu \left( (t_0, t_p) \right) < \delta \). Choose rational numbers \( x_1, ..., x_p \) such that \( x_i \in (t_{i-1}, t_i) \) \((i=1, ..., p)\). It follows that

\[
\phi_n(\xi) = \sum_{j=n}^{\infty} \phi(\frac{2\pi j}{n}) \Delta(\xi - \frac{2\pi j}{n}) = \sum_{j=n}^{\infty} (e_-\pi) \left( \frac{2\pi j}{n} \right) d\mu(x) \Delta(\xi - \frac{2\pi j}{n}) = \int_{-\infty}^{\infty} (e_-\pi n) (\xi) d\mu(x),
\]

where \( \Delta(\xi) = \max \left(1 - \frac{n}{2\pi} |\xi|, 0 \right) \) for \( \xi \in R \). This implies that

\[
\left| \sum_{k=1}^{m} c_k \phi_n(\xi_i) \right| = \left| \sum_{k=1}^{m} c_k \int_{-\infty}^{\infty} (e_-\pi n) (\xi) d\mu(x) \right| \\
\leq \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_-\pi n) (\xi) d\mu(x) + \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_-\pi n) (\xi) d\mu(x) \\
\leq \sum_{k=1}^{m} c_k \left| \int_{(t_{i-1}, t_i)} (e_-\pi n) (\xi) d\mu(x) \right| + \sum_{k=1}^{m} |c_k| \delta \| \phi \|_{B(R)} \\
+ \sum_{k=1}^{m} |c_k| \mu \left( (t_0, t_p) \right) \\
\leq \sum_{k=1}^{m} \int_{(t_{i-1}, t_i)} d\mu(x) \left| \sum_{k=1}^{m} c_k (e_-\pi n) (\xi) \right| + \varepsilon.
\]

Since \((e_-\pi n) \in B(R)\) and \( \| (e_-\pi n) \|_{B(R)} = 1 \) for every rational numbers \( x \), it follows that

\[
\left| \sum_{k=1}^{m} c_k (e_-\pi n) (\xi_i) \right| \leq 1 \quad (i=1, ..., p).
\]

Consequently, we have

\[
\left| \sum_{k=1}^{m} c_k \phi_n(\xi_i) \right| \leq \| \phi \|_{B(R)} + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this completes the proof of the lemma.

**Theorem.** If \( \varphi \in A(R) \), then \( \varphi_n \in A(R) \) for all \( n \) and \( \left\| \varphi_n - \varphi \right\|_{A(R)} \to 0 \) \((n \to \infty)\).

**Proof.** Let \( V_m \) be de la Vallée Poussin’s kernel; that is,

\[
\tilde{V}_m(\xi) = \begin{cases} 
1 & (|\xi| \leq m) \\
2 - |\xi|/m & (m \leq |\xi| \leq 2m) \\
0 & (2m \leq |\xi|)
\end{cases} \quad (\xi \in R).
\]
By Lemma, we have \((\varphi \tilde{V}_m)_n \in B(R)\). Since each \((\varphi \tilde{V}_m)_n\) has compact support, this means that \((\varphi \tilde{V}_m)_n \in A(R)\). Since \(\| \varphi - \varphi \tilde{V}_m \|_{A(R)} \to 0 \ (m \to \infty)\), Lemma shows that \((\varphi \tilde{V}_m)_m\) is a Cauchy sequence in \(A(R)\). Let \(g\) denote the limit of \((\varphi \tilde{V}_m)_n\) in \(A(R)\). The value \(\varphi_n(x)\) is equal to \(g(x)\) since it is the limit of \((\varphi \tilde{V}_m)_n(x)\) for each \(x \in R\). Thus we have \(\varphi_n \in A(R)\).

To prove that \(\| \varphi_n - \varphi \|_{A(R)} \to 0 \ (n \to \infty)\), assume first that \(\varphi\) has compact support. Let \(E\) be the subset of \(A(R)\) consisting of all functions \(h\) with compact support contained in \((-3\pi/4, 3\pi/4)\). Then there exists a linear mapping \(S\) of \(E\) into \(A(T)\) and constant \(C\) such that \(Sh = h\) on \((-\pi, \pi)\) and \(\| h \|_{A(R)} \leq C \| Sh \|_{A(T)}\) for all \(h \in E\) (See Rudin [5, p.56]). It follows that \(\| h_n - h \|_{A(R)} \leq C \| Sh_n - Sh \|_{A(T)}\) for all \(n \geq 8\) and all \(h \in A(R)\) with compact support contained in \((-\pi/2, \pi/2)\). To show that \(\| \varphi_n - \varphi \|_{A(R)} \to 0 \ (n \to \infty)\), let \(\varepsilon > 0\) and \(p\) be a positive integer such that the support of \(\varphi\) is contained in \((-\pi p/2, \pi p/2)\). Let \(T\) be a mapping defined by \(T\varphi(\xi) = f(\varphi(\xi))\) for \(f \in A(R)\). Then \(T\varphi \in A(R)\) and the support of \(T\varphi\) is contained in \((-\pi p/2, \pi p/2)\), it follows that \(ST\varphi \in A(T)\). Therefore there is a positive integer \(N\) such that \(\| (ST\varphi)_n - ST\varphi \|_{A(T)} < \varepsilon/\pi\) for every \(n \geq N\) (See Benedetto [1, p.168]). For \(n > \max(8/p, N)\), we have

\[
\| \varphi_n - \varphi \|_{A(R)} = \| T\varphi_n - T\varphi \|_{A(R)} = \| (T\varphi)_n - T\varphi \|_{A(R)} \leq C \| S(T\varphi)_n - ST\varphi \|_{A(T)} < \varepsilon/\pi.
\]

In the general case, we may use the fact that the functions with compact supports are dense in \(A(R)\). Let \(\varepsilon > 0\), and let \(\varphi\) be a function with compact support such that \(\varphi - \varphi \|_{A(R)} < \varepsilon/3\). By Lemma, we have \(\| \varphi - \varphi\|_{A(R)} \leq \| \varphi - \varphi \|_{A(R)} \leq \| \varphi - \varphi \|_{A(R)} \leq \| \varphi - \varphi \|_{A(R)}\). Since \(\varphi\) has compact support, there is a positive integer \(N\) such that \(\| \varphi_n - \varphi \|_{A(R)} < \varepsilon/3\) for all \(n > N\). It follows that

\[
\| \varphi_n - \varphi \|_{A(R)} \leq \| \varphi_n - \varphi \|_{A(R)} + \| \varphi_n - \varphi \|_{A(R)} + \| \varphi - \varphi \|_{A(R)} < \varepsilon/3.
\]

This completes the proof of the theorem.

References