<table>
<thead>
<tr>
<th>著者</th>
<th>SUDA Katsuhiro, KAWAI Toru</th>
</tr>
</thead>
<tbody>
<tr>
<td>発行所</td>
<td>鹿児島大学理学部紀要 数学・物理学・化学</td>
</tr>
<tr>
<td>発行年月</td>
<td>なし</td>
</tr>
<tr>
<td>別言語のタイトル</td>
<td>区分的線形関数によるフーリエ変換の近似</td>
</tr>
<tr>
<td>リンク</td>
<td><a href="http://hdl.handle.net/10232/00000501">http://hdl.handle.net/10232/00000501</a></td>
</tr>
</tbody>
</table>
Approximation of Fourier Transforms by Piecewise Linear Functions

Katsuhiro SUDA\(^{1)}\) and Toru KAWAI\(^{1)}\)

(Received September 12, 1994)

Abstract

An approximation theorem of Fourier transform by piecewise linear functions is proved.

Key words: Fourier transform, piecewise linear function.

1. Introduction

We denote Banach spaces of all integrable functions and all regular complex finite measures on the real line by \(L^1(R)\) and \(M(R)\), respectively. For a function \(f \in L^1(R)\), the Fourier transform is

\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \, dx \quad (\xi \in R),
\]

and for a measure \(\mu \in M(R)\), the Fourier-Stieltjes transform is

\[
\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} d\mu(x) \quad (\xi \in R).
\]

We define

\[
A(R) = \{ f : f \in L^1(R) \}, \quad \| f \|_{A(R)} = \| f \|_{L^1(R)},
\]

and

\[
B(R) = \{ \hat{\mu} : \mu \in M(R) \}, \quad \| \hat{\mu} \|_{B(R)} = \| \mu \|_{M(R)}.
\]
We denote Banach spaces of all integrable functions on one dimensional torus $T$ by $L^1(T)$. For a function $f \in L^1(T)$, the Fourier coefficients are

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx$$

for integers $n$.

We define

$$A(T) = \{ \varphi: \sum_{n=-\infty}^{\infty} |\varphi(n)| < \infty \}, \quad ||\varphi||_{A(T)} = \sum_{n=-\infty}^{\infty} |\varphi(n)|.$$

Herz [2] has proved that Cantor's ternary set is a spectral synthesis set. Kahane [3] has proved this result using an approximation of functions in $A(T)$ by piecewise linear functions: if $\varphi \in A(T)$ and each $\varphi_n$ is a continuous function on $T$ to equal $\varphi$ at the multiples of $2\pi/n$ and linear on the remainder, then $\varphi_n \in A(T)$ and $||\varphi_n - \varphi||_{A(T)} \to 0$ ($n \to \infty$) (See also Kahane and Salem [4] and Benedetto [1]). In this paper, we prove that a similar result holds for $A(R)$.

## 2. Approximation of Fourier transforms

For a continuous function $\varphi$ on $R$ and a positive integer $n$, we define $\varphi_n$ to be the function equal $\varphi$ at the points $2\pi k/n$, $k=0, \pm 1, \pm 2, \ldots$, and linear on the remainder of $R$. For a continuous function $\varphi$ on $T$, we define $\varphi_n$ similarly.

**Lemma.** If $\varphi \in B(R)$, then $\varphi_n \in B(R)$ and $||\varphi_n||_{B(R)} \leq ||\varphi||_{B(R)}$ for all $n$.

**Proof.** For all $x \in R$, define the function $e_x$ by $e_x(\xi) = e^{ix\xi}$ ($\xi \in R$). We now show that if $x$ is a rational number, then $(e_x)_n \in B(R)$ and $||(e_x)_n||_{B(R)} = 0$. Suppose that $x$ is a rational number. Take a positive integer $r$ and an integer $s$ such that $x = s/r$. We see that $(e_x)_n(\xi) = e^{i\xi/s}$ for all $\xi \in R$. Since $e_x$ is $2\pi$-periodic, regarding $e_x$ as a function on $T$, we have $e_x \in A(T)$. It follows that $(e_x)_n \in A(T)$ and $||(e_x)_n||_{A(T)} = 1$ (See Benedetto [1, p.168]. Hence $(e_x)_n(\xi) \in B(R)$ and $||(e_x)_n||_{B(R)} = 1$. Let $\xi_1, \ldots, \xi_m \in R$ and $c_1, \ldots, c_m$ be complex numbers such that $||\sum_{k=1}^{m} c_k e^{-i\xi_k}||_1 \leq 1$. To prove $\varphi_n \in B(R)$ and $||\varphi_n||_{B(R)} \leq ||\varphi||_{B(R)}$, it suffices to show that

$$||\sum_{k=1}^{m} c_k \varphi_n(\xi_k)|| \leq ||\varphi||_{B(R)}.$$

Let $\varphi = \tilde{\mu}$, $\mu \in M(R)$, and $\varepsilon > 0$. Take $\delta > 0$ so that $2(1 + ||\varphi||_{B(R)}) \delta \sum_{k=1}^{m} |c_k| < \varepsilon$. Since
(e_\omega)^n(\xi) is uniformly continuous as a function of x for each k, there is a positive number \eta such that \(|(e_\omega)^n(\xi_k) - (e_\omega)^n(\xi)\) < \delta for all x, y and k such that \(|x-y| < \eta\). Since \(\mu \in M(R)\), there are \(t_0, \ldots, t_p \in R\) so that \(t_0 < \ldots < t_p, t_{i-1} - t_i = \eta\) for all i, and \(|\mu((t_0, t_p))| < \delta\).

Choose rational numbers \(x_1, \ldots, x_p\) such that \(x_i \in (t_{i-1}, t_i)\) \((i = 1, \ldots, p)\). It follows that

\[
\varphi_n(\xi) = \sum_{j=-\infty}^{\infty} \varphi\left(\frac{2\pi j}{n}\right) \Delta\left(\xi - \frac{2\pi j}{n}\right) = \sum_{j=-\infty}^{\infty} (e_\omega)^n\left(\frac{2\pi j}{n}\right) d\mu(x) \Delta\left(\xi - \frac{2\pi j}{n}\right)
= \int_{-\infty}^{\infty} (e_\omega)^n(\xi) d\mu(x),
\]

where \(\Delta(\xi) = \max\left(1 - \frac{n}{2\pi} |\xi|, 0\right)\) for \(\xi \in R\). This implies that

\[
\left| \sum_{k=1}^{m} c_k \varphi_n(\xi) \right| = \left| \sum_{k=1}^{m} c_k \int_{-\infty}^{\infty} (e_\omega)^n(\xi) d\mu(x) \right|
\leq \left| \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_\omega)^n(\xi) d\mu(x) \right| + \left| \sum_{k=1}^{m} c_k \int_{(t_i, t_{i+1})} (e_\omega)^n(\xi) d\mu(x) \right|
\leq \left| \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_\omega)^n(\xi) d\mu(x) \right| + \left| \sum_{k=1}^{m} c_k \int_{(t_i, t_{i+1})} (e_\omega)^n(\xi) d\mu(x) \right|
\leq \left| \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_\omega)^n(\xi) d\mu(x) \right| + \left| \sum_{k=1}^{m} c_k \int_{(t_{i-1}, t_i)} (e_\omega)^n(\xi) d\mu(x) \right|
+ \left| \sum_{k=1}^{m} c_k \int_{(t_i, t_{i+1})} (e_\omega)^n(\xi) d\mu(x) \right| + \varepsilon.
\]

Since \((e_\omega)^n \in B(R)\) and \(\|(e_\omega)^n\|_{B(R)} = 1\) for every rational numbers x, it follows that

\[
\left| \sum_{k=1}^{m} c_k (e_\omega)^n(\xi) \right| \leq 1 \quad (i = 1, \ldots, p).
\]

Consequently, we have

\[
\left| \sum_{k=1}^{m} c_k \varphi_n(\xi) \right| \leq \| \varphi \|_{B(R)} + \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, this completes the proof of the lemma.

**Theorem.** If \(\varphi \in A(R)\), then \(\varphi_n \in A(R)\) for all n and \(\| \varphi_n - \varphi \|_{A(R)} \to 0 \quad (n \to \infty)\).

**Proof.** Let \(V_m\) be de la Vallée Poussin's kernel; that is,

\[
V_m(\xi) = \begin{cases} 
1 & (|\xi| \leq m) \\
2 - |\xi|/m & (m \leq |\xi| \leq 2m) \\
0 & (2m \leq |\xi|)
\end{cases} \quad (\xi \in R).
\]
By Lemma, we have \((\varphi V_m)_n \in B(\mathbb{R})\). Since each \((\varphi V_m)_n\) has compact support, this means that \((\varphi V_m)_n \in A(\mathbb{R})\). Since \(\| \varphi - \varphi V_m \|_{A(\mathbb{R})} \to 0 \) \((m \to \infty)\), Lemma shows that \(((\varphi V_m)_n)_m\) is a Cauchy sequence in \(A(\mathbb{R})\). Let \(g\) denote the limit of \((\varphi V_m)_n\) in \(A(\mathbb{R})\). The value \(\varphi_n(x)\) is equal to \(g(x)\) since it is the limit of \((\varphi V_m)_n(x)\) for each \(x \in \mathbb{R}\). Thus we have \(\varphi_n \in A(\mathbb{R})\).

To prove that \(\| \varphi_n - \varphi \|_{A(\mathbb{R})} \to 0 \) \((n \to \infty)\), assume first that \(\varphi\) has compact support. Let \(E\) be the subset of \(A(\mathbb{R})\) consisting of all functions \(h\) with compact support contained in \((-3\pi/4, 3\pi/4)\). Then there exists a linear mapping \(S\) of \(E\) into \(A(T)\) and constant \(C\) such that \(Sh = h\) on \((-\pi, \pi)\) and \(\| h \|_{A(\mathbb{R})} \leq C \| Sh \|_{A(T)}\) for all \(h \in E\) (See Rudin \[5, p.56\]). It follows that \(\| h_n - h \|_{A(\mathbb{R})} \leq C \| Sh_n - Sh \|_{A(T)}\) for all \(n \geq 8\) and all \(h \in A(\mathbb{R})\) with compact support contained in \((-\pi/2, \pi/2)\). To show that \(\| \varphi_n - \varphi \|_{A(\mathbb{R})} \to 0 \) \((n \to \infty)\), let \(\varepsilon > 0\) and \(p\) be a positive integer such that the support of \(\varphi\) is contained in \((-\pi p/2, \pi p/2)\). Let \(T\) be a mapping defined by \(Tf(\xi) = f(p\xi)\) for \(f \in A(\mathbb{R})\) and \(\xi \in \mathbb{R}\). Since \(T\varphi \in A(\mathbb{R})\) and the support of \(T\varphi\) is contained in \((-\pi/2, \pi/2)\), it follows that \(ST\varphi \in A(T)\). Therefore there is a positive integer \(N\) such that \(\|(ST\varphi)_n - ST\varphi\|_{A(T)} < \varepsilon/C\) for every \(n \geq N\) (See Benedetto \[1, p.168\]). For \(n > \max(8/p, N)\), we have

\[
\| \varphi_n - \varphi \|_{A(\mathbb{R})} = \| T\varphi_n - T\varphi \|_{A(\mathbb{R})} = \|(T\varphi)_n - T\varphi\|_{A(\mathbb{R})} \leq C \| S(T\varphi)_n - ST\varphi \|_{A(T)}
= C \| (ST\varphi)_n - ST\varphi \|_{A(T)} < \varepsilon.
\]

In the general case, we may use the fact that the functions with compact supports are dense in \(A(\mathbb{R})\). Let \(\varepsilon > 0\), and let \(\psi\) be a function with compact support such that \(\| \varphi - \psi \|_{A(\mathbb{R})} < \varepsilon/3\). By Lemma, we have \(\| \varphi_n - \varphi \|_{A(\mathbb{R})} \leq \| \varphi - \psi \|_{A(\mathbb{R})}\). Since \(\psi\) has compact support, there is a positive integer \(N\) such that \(\| \varphi_n - \psi \|_{A(\mathbb{R})} < \varepsilon/3\) for all \(n > N\). It follows that

\[
\| \varphi_n - \varphi \|_{A(\mathbb{R})} \leq \| \varphi_n - \varphi_n \|_{A(\mathbb{R})} + \| \varphi_n - \psi \|_{A(\mathbb{R})} + \| \psi - \varphi \|_{A(\mathbb{R})} < \varepsilon.
\]

This completes the proof of the theorem.

References