Approximation of Fourier Transforms by Piecewise Linear Functions

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Approximation of Fourier Transforms by Piecewise Linear Functions

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Abstract

An approximation theorem of Fourier transform by piecewise linear functions is proved.

Key words: Fourier transform, piecewise linear function.

1. Introduction

We denote Banach spaces of all integrable functions and all regular complex finite measures on the real line by $L^1(\boldsymbol{R})$ and $M(\boldsymbol{R})$, respectively. For a function $f \in L^1(\boldsymbol{R})$, the Fourier transform is

$$\hat{f}(\xi)=\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi x} f(x)\,dx \quad (\xi \in \boldsymbol{R}),$$

and for a measure $\mu \in M(\boldsymbol{R})$, the Fourier-Stieltjes transform is

$$\hat{\mu}(\xi)=\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi x}\,d\mu(x) \quad (\xi \in \boldsymbol{R}).$$

We define

$$A(\boldsymbol{R})=\{\hat{f}\colon f \in L^1(\boldsymbol{R})\}, \ \|\hat{f}\|_{A(R)}=\|f\|_{L^1(R)}$$

and

$$B(\boldsymbol{R})=\{\hat{\mu}\colon \mu \in M(\boldsymbol{R})\}, \ \|\hat{\mu}\|_{B(R)}=\|\mu\|_{M(R)}.$$

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We denote Banach spaces of all integrable functions on one dimensional torus $\boldsymbol{T}$ by $L^1(\boldsymbol{T})$. For a function $f \in L^1(\boldsymbol{T})$, the Fourier coefficients are

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx \quad \text{for integers } n.$$

We define

$$A(\boldsymbol{T}) = \{\varphi : \sum_{n=-\infty}^{\infty} |\hat{\varphi}(n)| < \infty\}, \quad \|\varphi\|_{A(\boldsymbol{T})} = \sum_{n=-\infty}^{\infty} |\hat{\varphi}(n)|.$$

Herz [2] has proved that Cantor's ternary set is a spectral synthesis set. Kahane [3] has proved this result using an approximation of functions in $A(\boldsymbol{T})$ by piecewise linear functions: if $\varphi \in A(\boldsymbol{T})$ and each φ_n is a continuous function on $\boldsymbol{T}$ to equal φ at the multiples of $2\pi/n$ and linear on the remainder, then $\varphi_n \in A(\boldsymbol{T})$ and $\|\varphi_n - \varphi\|_{A(\boldsymbol{T})} \to 0$ $(n \to \infty)$ (See also Kahane and Salem [4] and Benedetto [1]). In this paper, we prove that a similar result holds for $A(\boldsymbol{R})$.

2. Approximation of Fourier transforms

For a continuous function φ on $\boldsymbol{R}$ and a positive integer n, we define φ_n to be the function to equal φ at the points $2\pi k/n$, $k=0, \pm 1, \pm 2, \ldots$, and linear on the remainder of $\boldsymbol{R}$. For a continuous function φ on $\boldsymbol{T}$, we define φ_n similarly.

Lemma. *If $\varphi \in B(\boldsymbol{R})$, then $\varphi_n \in B(\boldsymbol{R})$ and $\|\varphi_n\|_{B(\boldsymbol{R})} \leq \|\varphi\|_{B(\boldsymbol{R})}$ for all n.*

Proof. For all $x \in \boldsymbol{R}$, define the function e_x by $e_x(\xi) = e^{ix\xi}$ $(\xi \in \boldsymbol{R})$. We now show that if x is a rational number, then $(e_{-x})_n \in B(\boldsymbol{R})$ and $\|(e_{-x})_n\|_{B(\boldsymbol{R})} = 1$. Suppose that x is a rational number. Take a positive integer r and an integer s such that $x = s/r$. We see that $(e_{-s})_{rn}(\xi) = (e_{-x})_n(r\xi)$ for all $\xi \in \boldsymbol{R}$. Since e_{-s} is 2π-periodic, regarding e_{-s} as a function on $\boldsymbol{T}$, we have $e_{-s} \in A(\boldsymbol{T})$. It follows that $(e_{-s})_{rn} \in A(\boldsymbol{T})$ and $\|(e_{-s})_{rn}\|_{A(\boldsymbol{T})} = 1$ (See Benedetto [1, p.168]. Hence $(e_{-x})_n \in B(\boldsymbol{R})$ and $\|(e_{-x})_n\|_{B(\boldsymbol{R})} = 1$. Let $\xi_1, \ldots, \xi_m \in \boldsymbol{R}$ and $c_1, \ldots, c_m$ be complex numbers such that $\|\sum_{k=1}^{m} c_k e_{-\xi_k}\|_\infty \leq 1$. To prove $\varphi_n \in B(\boldsymbol{R})$ and $\|\varphi_n\|_{B(\boldsymbol{R})} \leq \|\varphi\|_{B(\boldsymbol{R})}$, it suffices to show that

$$\left| \sum_{k=1}^{m} c_k \varphi_n(\xi_k) \right| \leq \|\varphi\|_{B(\boldsymbol{R})}.$$

Let $\varphi = \hat{\mu}$, $\mu \in M(\boldsymbol{R})$, and $\varepsilon > 0$. Take $\delta > 0$ so that $2(1 + \|\varphi\|_{B(\boldsymbol{R})})\delta \sum_{k=1}^{m} |c_k| < \varepsilon$. Since

$(e_{-x})_n(\xi_k)$ is uniformly continuous as a function of x for each k, there is a positive number η such that $|(e_{-x})_n(\xi_k)-(e_{-y})_n(\xi_k)|<\delta$ for all x, y and k such that $|x-y|<\eta$. Since $\mu\in M(\boldsymbol{R})$, there are $t_0, \ldots, t_p\in\boldsymbol{R}$ so that $t_0<\ldots<t_p$, $t_i-t_{i-1}=\eta$ for all i, and $|\mu|((t_0, t_p]^c)<\delta$. Choose rational numbers $x_1, \ldots, x_p$ such that $x_i\in(t_{i-1}, t_i]$ $(i=1, \ldots, p)$. It follows that

$$\begin{aligned}\varphi_n(\xi)&=\sum_{j=-\infty}^{\infty}\varphi\left(\frac{2\pi j}{n}\right)\Delta\left(\xi-\frac{2\pi j}{n}\right)=\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}(e_{-x})\left(\frac{2\pi j}{n}\right)d\mu(x)\Delta\left(\xi-\frac{2\pi j}{n}\right)\\&=\int_{-\infty}^{\infty}(e_{-x})_n(\xi)\,d\mu(x),\end{aligned}$$

where $\Delta(\xi)=\max\left(1-\frac{n}{2\pi}|\xi|, 0\right)$ for $\xi\in\boldsymbol{R}$. This implies that

$$\begin{aligned}\left|\sum_{k=1}^{m}c_k\varphi_n(\xi_k)\right|&=\left|\sum_{k=1}^{m}c_k\int_{-\infty}^{\infty}(e_{-x})_n(\xi_k)\,d\mu(x)\right|\\&\leq\left|\sum_{k=1}^{m}c_k\int_{(t_0,t_p]}(e_{-x})_n(\xi_k)\,d\mu(x)\right|+\left|\sum_{k=1}^{m}c_k\int_{(t_0,t_p]^c}(e_{-x})_n(\xi_k)\,d\mu(x)\right|\\&\leq\left|\sum_{k=1}^{m}c_k\sum_{i=1}^{p}\int_{(t_{i-1},t_i]}(e_{-x_i})_n(\xi_k)\,d\mu(x)\right|+\sum_{k=1}^{m}|c_k|\,\delta\,\|\varphi\|_{B(R)}\\&\quad+\sum_{k=1}^{m}|c_k|\,|\mu|((t_0, t_p]^c)\\&\leq\sum_{i=1}^{p}\int_{(t_{i-1},t_i]}d|\mu|(x)\left|\sum_{k=1}^{m}c_k(e_{-x_i})_n(\xi_k)\right|+\varepsilon.\end{aligned}$$

Since $(e_{-x})_n\in B(\boldsymbol{R})$ and $\|(e_{-x})_n\|_{B(R)}=1$ for every rational numbers x, it follows that

$$\left|\sum_{k=1}^{m}c_k(e_{-x_i})_n(\xi_k)\right|\leq 1\quad(i=1, \ldots, p).$$

Consequently, we have

$$\left|\sum_{k=1}^{m}c_k\varphi_n(\xi_k)\right|\leq\|\varphi\|_{B(R)}+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this completes the proof of the lemma.

Theorem. *If $\varphi\in A(\boldsymbol{R})$, then $\varphi_n\in A(\boldsymbol{R})$ for all n and* $\|\varphi_n-\varphi\|_{A(R)}\to 0$ $(n\to\infty)$.

Proof. Let V_m be de la Vallée Poussin's kernel; that is,

$$\hat{V}_m(\xi)=\begin{cases}1 & (|\xi|\leq m)\\ 2-|\xi|/m & (m\leq|\xi|\leq 2m)\\ 0 & (2m\leq|\xi|)\end{cases}\quad(\xi\in\boldsymbol{R}).$$

By Lemma, we have $(\varphi \hat{V_m})_n \in B(\boldsymbol{R})$. Since each $(\varphi \hat{V_m})_n$ has compact support, this means that $(\varphi \hat{V_m})_n \in A(\boldsymbol{R})$. Since $\| \varphi - \varphi \hat{V_m} \|_{A(R)} \to 0$ $(m \to \infty)$, Lemma shows that $\{(\varphi \hat{V_m})_n\}_m$ is a Cauchy sequence in $A(\boldsymbol{R})$. Let g denote the limit of $(\varphi \hat{V_m})_n$ in $A(\boldsymbol{R})$. The value $\varphi_n(x)$ is equal to $g(x)$ since it is the limit of $(\varphi \hat{V_m})_n(x)$ for each $x \in \boldsymbol{R}$. Thus we have $\varphi_n \in A(\boldsymbol{R})$. To prove that $\| \varphi_n - \varphi \|_{A(R)} \to 0$ $(n \to \infty)$, assume first that φ has compact support. Let E be the subset of $A(\boldsymbol{R})$ consisting of all functions h with compact support contained in $(-3\pi/4, 3\pi/4)$. Then there exists a linear mapping S of E into $A(\boldsymbol{T})$ and constant C such that $Sh = h$ on $(-\pi, \pi)$ and $\| h \|_{A(R)} \leq C \| Sh \|_{A(T)}$ for all $h \in E$ (See Rudin [5, p.56]). It follows that $\| h_n - h \|_{A(R)} \leq C \| Sh_n - Sh \|_{A(T)}$ for all $n \geq 8$ and all $h \in A(\boldsymbol{R})$ with compact support contained in $(-\pi/2, \pi/2)$. To show that $\| \varphi_n - \varphi \|_{A(R)} \to 0$ $(n \to \infty)$, let $\varepsilon > 0$ and p be a positive integer such that the support of φ is contained in $(-\pi p/2, \pi p/2)$. Let T be a mapping defined by $Tf(\xi) = f(p\xi)$ for $f \in A(\boldsymbol{R})$ and $\xi \in \boldsymbol{R}$. Since $T\varphi \in A(\boldsymbol{R})$ and the support of $T\varphi$ is contained in $(-\pi/2, \pi/2)$, it follows that $ST\varphi \in A(\boldsymbol{T})$. Therefore there is a positive integer N such that $\| (ST\varphi)_{pn} - ST\varphi \|_{A(T)} < \varepsilon / C$ for every $n \geq N$ (See Benedetto [1, p.168]). For $n > \max(8/p, N)$, we have

$$\begin{aligned} \| \varphi_n - \varphi \|_{A(R)} &= \| T\varphi_n - T\varphi \|_{A(R)} = \| (T\varphi)_{pn} - T\varphi \|_{A(R)} \leq C \| S(T\varphi)_{pn} - ST\varphi \|_{A(T)} \\ &= C \| (ST\varphi)_{pn} - ST\varphi \|_{A(T)} < \varepsilon. \end{aligned}$$

In the general case, we may use the fact that the functions with compact supports are dense in $A(\boldsymbol{R})$. Let $\varepsilon > 0$, and let ψ be a function with compact support such that $\| \varphi - \psi \|_{A(R)} < \varepsilon/3$. By Lemma, we have $\| \varphi_n - \psi_n \|_{A(R)} \leq \| \varphi - \psi \|_{A(R)}$. Since ψ has compact support, there is a positive integer N such that $\| \psi_n - \psi \|_{A(R)} < \varepsilon/3$ for all $n > N$. It follows that

$$\| \varphi_n - \varphi \|_{A(R)} \leq \| \varphi_n - \psi_n \|_{A(R)} + \| \psi_n - \psi \|_{A(R)} + \| \psi - \varphi \|_{A(R)} < \varepsilon.$$

This completes the proof of the theorem.

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