Mutually tangent spheres on the n-dimensional sphere

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1. Introduction

Let $\mathbf{S}^n$ be the n-dimensional unit sphere. On it we consider $n + 1$ spheres S_i ($i = 1, 2, \dots, n + 1$) which are of dimension (n - 1) and which contact each other. Then there are two spheres which are tangent to all of these $n + 1$ spheres, one of which is surrounded by all of them and the other of which surrounds all of them. We denote these two spheres by the same notation S_0. Let denote the radii of S_i by r_i for $i = 0, 1, 2, \dots, n + 1$. In case of the n-dimensional Euclidean space $\mathbf{R}^n$, there are a lot of studies on S_0 (for example, see [1], [2], [3], [4], and [5]), and it is known that the radii of $n + 2$ mutually tangent spheres enjoy the formula

$$\left(\sum_{i=0}^{n+1} \frac{1}{r_i}\right)^2 = n \sum_{i=0}^{n+1} \left(\frac{1}{r_i}\right)^2 . \tag{1}$$

In this paper we investigate a problem of finding an analogous formula to (1) which holds between these $n + 2$ radii of mutually tangent spheres on $\mathbf{S}^n$. As a result of our investigation, we obtain the following result.

Main Theorem.

$$\left(\sum_{i=0}^{n+1} \cot r_i\right)^2 = n \left(\sum_{i=0}^{n+1} \cot^2 r_i + 2\right) . \tag{2}$$

As far as the author has searched previous studies on this problem, it seems that this formula was first obtained in [4]. In [4] the formula (2) was proved by a direct computation. In this paper we present an alternative proof which reduces the problem on $\mathbf{S}^n$ to a corresponding one in $\mathbf{R}^n$ by a stereographic projection. In course of such a reduction, we find a somewhat interesting property about the stereographic projection which will be stated in Proposition 1 of section 4.

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2. Preliminary lemmas

In this section we prepare several elementary lemmas, which will be used in later sections.

Lemma 1. *Assume that an* $(n+1)$ $(n+1)$*- symmetric matrix* $A=(a_{ij})$ *is non-negative definite. Then there exist* $(n+1)$*-dimensional vectors* a_i $(i=1,2,\ldots,n+1)$ *such that* $a_{ij}=\mathrm{a}_i\cdot\mathrm{a}_j$ *for all i, j, where the notation "·" denotes for the inner product of two vectors. Furthermore, if A is singular, then there exist n-dimensional vectors* a_i $(i=1,2,\ldots,n+1)$ *which satisfy the same relation.*

Proof. Since we can prove the first part of the lemma in a similar way to the second part of it, we present only a proof for the second part. Since A is a singular non-negative matrix, there exists an orthogonal matrix P and non-negative numbers α_i $(i=1,2,\ldots,n)$ such that

$$P^T AP=\mathrm{diag}(\alpha_1,\alpha_2\ldots,\alpha_n,0),$$

where $\mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_n,0)$ denotes a diagonal matrix with its diagonal entries being $\alpha_1,\alpha_2\ldots,\alpha_n$ and 0, and P^T denotes the transpose of P. Now we consider the following multiplication of two matrices:

$$P\,\mathrm{diag}\left(\sqrt{\alpha_1},\sqrt{\alpha_2},\ldots,\sqrt{\alpha_n},0\right).$$

Since the $(n+1)$ -th column vector of this matrix equals zero, there exist n-dimensional row vectors a_i $(i=1,2,\ldots,n+1)$ such that

$$P\,\mathrm{diag}\left(\sqrt{\alpha_1},\sqrt{\alpha_2},\ldots,\sqrt{\alpha_n},0\right)=\begin{pmatrix}\mathbf{a}_1 & 0\\ \mathbf{a}_2 & 0\\ \vdots & \vdots\\ \mathbf{a}_{n+1} & 0\end{pmatrix}.$$

Accordingly we have

$$A=\begin{pmatrix}\mathbf{a}_1 & 0\\ \mathbf{a}_2 & 0\\ \vdots & \vdots\\ \mathbf{a}_{n+1} & 0\end{pmatrix}\begin{pmatrix}\mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_{n+1}^T\\ 0 & 0 & \cdots & 0\end{pmatrix},$$

which implies that $a_{ij}=\mathrm{a}_i\cdot\mathrm{a}_j$ for all i,j. Thus the proof is completed.

In the following sections we use two $(n+1) \times (n+1)$ -matrices $\tilde{A} = (\tilde{a}_{ij})$ and $A(u) = (a_{ij}(u))$ where

(3) $$\tilde{a}_{ij} = \begin{cases} 1+t_i^2 & \text{for } i=j \\ -1+t_it_j & \text{for } i \neq j \end{cases}$$

(4) $$a_{ij}(u) = \begin{cases} 1+2t_iu-u^2 & \text{for } i=j \\ -1+(t_i+t_j)u-u^2 & \text{for } i \neq j \end{cases} .$$

Then, putting

(5) $$T_1 = \sum_{i=1}^{n+1} t_i \quad \text{and} \quad T_2 = \sum_{i=1}^{n+1} t_i^2 ,$$

we can express the characteristic polynomials of these matrices as follows.

Lemma 2.

(6) $$\det(\tilde{A} - \omega I_{n+1}) = (2-\omega)^{n-1}\Big[\omega^2 - \omega\big\{T_2 - (n-3)\big\} + \big\{T_1^2 - (n-1)T_2 - 2(n-1)\big\}\Big]$$

(7) $$\det(A(u)) - \omega I_{n+1}) = (2-\omega)^{n-1}\Big[\omega^2 - \omega\big\{2uT_1 - (n-3) - (n+1)u^2\big\} - \big\{u^2((n+1)T_2 + 2(n+1) - T_1^2) - 4uT_1 + 2(n-1)\big\}\Big] .$$

Proof. As is easily seen, these characteristic polynomials are symmetric functions of variables t_i $(i = 1, 2, \ldots, n+1)$ and also they are quadratic polynomials of t_1. Acoordingly we can expresse them as

$$c_1T_1^2 + c_2T_2 + c_3T_1 + c_4 ,$$

where c_1, c_2, c_3 and c_4 are constants. Note that, in case of (7), these constants may depend on u. Then, setting appropriate particular values to variables t_i $(i = 1, 2, \ldots, n+1)$ several times, we can easily determine these constants. Thus the proof is completed.

Lemma 2 implies the following property about $\tilde{A}$.

Lemma 3. *In order that the matrix $\tilde{A}$ is non-negative definite, it is necessary and sufficient that the condition*

(8) $$T_1^2 \geq (n-1)T_2 + 2(n-1)\,.$$

holds.

Proof. Because of Lemma 2, the matrix $\tilde{A}$ has eigenvalues 2 with multiplicity n - 1 and moreover, eigenvalues which are equal to two (possibly identical) roots of the following quadratic equation of ω,

(9) $$\omega^2 - \omega(T_2 - (n-3)) + \left(T_1^2 - (n-1)T_2 - 2(n-1)\right) = 0\,.$$

The determinant of the quadratic equation (9) is equal to $(T_2 + n + 1)^2 - 4T_1^2$. Since $T_1^2 \leq (n+1)T_2$ by the Cauchy-Schwarz inequality, this determinant is always non-negative. Now, if the quadratic equation (9) has two non-negative roots, then the condition (8) obviously holds. Conversely, if the condition (8) holds, then $(n+1)T_2 \geq T_1^2 \geq (n-1)(T_2+2)$, from which follows $T_2 \geq n-1$. Accordingly, the quadratic equation (9) has two non-negative roots. Thus the proof is completed.

For the matrix $A(u)$ defined by (4), we have the following lemma.

Lemma 4. *Assume that the condition* (8) *holds and u satisfies a quadratic equation*

(10) $$u^2\left((n+1)T_2 + 2(n+1) - T_1^2\right) - 4uT_1 + 2(n-1) = 0\,.$$

Then, the matrix A(u) is singular and non-negative definite.

Proof. Because of Lemma 2 the matrix $A(u)$ has eigenvalues 2 with multiplicity n - 1 and moreover, eigenvalues which are equal to two (possibly identical) roots of the following quadratic equation of ω:

(11) $$\omega^2 - \omega\left\{2uT_1 - (n-3) - (n+1)u^2\right\}$$
$$-\left\{u^2((n+1)T_2 + 2(n+1) - T_1^2) - 4uT_1 + 2(n-1)\right\} = 0\,.$$

Since u satisfies the quadratic equation (10), the quadratic equation (11) reduces to

$$\omega^2 - \omega\left\{2uT_1 - (n-3) - (n+1)u^2\right\} = 0$$

and thus it has two roots 0 and $2uT_1 - (n-3) - (n+1)u^2$. Consequently, in order to prove the lemma, it suffices to show that

(12) $$2uT_1 - (n-3) - (n+1)u^2 \geq 0.$$

By the way, the determinant of the quadratic equation (10) is equal to

$$2(n+1)(T_1^2 - (n-1)T_2 - 2(n-1)),$$

which is non-negative by the assumption (8). Accordingly the quadratic equation (10) has two (possibly identical) positive roots. Denote the smaller root of it by u_1. Then, solving (10) explicitly, we have

$$\begin{aligned} 2u_1T_1 &= 2(n-1)\cdot\frac{2T_1}{2T_1+\sqrt{4T_1^2-2(n-1)(s+2(n+1))}} \\ &> 2(n-1)\cdot\frac{1}{2} \\ &> n-3, \end{aligned} \tag{13}$$

where $s = (n+1)T_2 - T_1^2$. Now we return to (12). Then, using (10), we get

$$2u_1T_1 - (n-3) - (n+1)u_1^2 = \frac{(2u_1T_1 - (n-3))s + 4(n+1)}{s+2(n+1)},$$

which is positive by (13). Thus we have completed the proof.

3. Condition for the existence of n+1 mutually tangent spheres

Since $\mathbf{S}^n$ has a finite volume, if radii of n + 1 mutually tangent spheres are too large, then it is impossible for these spheres to exist on $\mathbf{S}^n$. In this section we shall state a necessary and sufficient condition for the existence of them. Denote the centers of spheres S_i by $(\mathbf{a}_i, b_i)$ (i = 1, 2,..., n + 1), where $\mathbf{a}_i$'s are n-dimensional vectors and b_i 's are real numbers such that $|a_i|^2 + b_i^2 = 1$. Furthermore, letting $t_i = \cot r_i$, we introduce T_1 and T_2 which are defined by (5).

Theorem 1. *In order that there exist n + 1 mutually tangent spheres on* $\mathbf{S}^n$*, it is necessary and sufficient that the following condition holds:*

(14) $$T_1^2 \geq (n-1)T_2 + 2(n-1).$$

Proof. First suppose that n+1 mutually tangent spheres S_i (i = 1,..., n + 1) exist. Since they are tangent each other,

(15) $$\mathbf{a}_i\cdot\mathbf{a}_j + b_ib_j = \begin{cases} 1 & \text{for } i = j \\ \cos(r_i + r_j) & \text{for } i \neq j \end{cases}.$$

Now we introduce the matrix $\tilde{A}$ defined by (3). Since the condition (15) is obviously equivalent to

$$\tilde{a}_{ij} = \frac{\mathbf{a}_i}{s_i} \cdot \frac{\mathbf{a}_j}{s_j} + \frac{b_i}{s_i} \cdot \frac{b_j}{s_j}, \tag{16}$$

where $s_i = \sin r_i$, $\tilde{A}$ is non-negative definite. So that, from Lemma 3, the condition (14) follows.

Conversely, we assume the condition (14). Then, by Lemma 3, $\tilde{A}$ is non-negative definite. Accordingly, because of Lemma 1, there exist $(n+1)$ -dimensional vectors $(\mathbf{a}_i, b_i)$ $(i = 1,..., n+1)$ for which (16), or equivalently, (15) holds. Therefore the assertion of Theorem 1 is established.

Remark 1. It may happen that in the condition (14) the equality holds. For example, it happens when, on $\mathbf{S}_2$, centers of 3 circles S_1, S_2 and S_3 lie on an equator and their radii are all equal to $\frac{\pi}{3}$.

4. Relation between the radii of n+2 mutually tangent spheres

Returning to our problem stated in the section 1, we shall solve it by reducing it to a corresponding problem in $\mathbf{R}^n$. For this purpose, we introduce the stereographic projection f from $\mathbf{S}^n$ to $\mathbf{R}^n$. It can be defined explicitly by

$$\xi = f(\mathbf{x}, y) = \frac{\mathbf{x}}{1-y},$$

where ξ denotes a point in $\mathbf{R}^n$ and $(\mathbf{x}, y)$ denotes a point on $\mathbf{S}^n$, in other words, $\mathbf{x}$ is a n-dimensional vector and y is a real number for which $|\mathbf{x}|^2+y^2=1$ holds. Then we can see the following lemma easily.

Lemma 5. *Let K be a sphere on S^n with center at* $(\mathbf{a}, b)$ *and radius r, and assume that* $\cos r \neq b$. *Then the image f(K) is a sphere in* $\mathbf{R}^n$ *with center at* $\frac{\mathbf{a}}{\cos r - b}$ *and radius* $\frac{\sin r}{\cos r - b}$.

Let S_i $(i = 1, 2,..., n+1)$ be mutually tangent spheres, and denote centers and radii of the spheres $f(S_i)$ $(i = 1, 2,..., n+1)$ by γ_i and ρ_i Assuming $\cos r_i \neq b_i$ for all i, we have, by Lemma 5,

$$\gamma_i = \frac{\mathbf{a}_i}{\cos r_i - b_i} \text{ and } \rho_i = \frac{\sin r_i}{\cos r_i - b}. \tag{17}$$

The following proposotion will play a crucial role in solving our problem.

Proposition 1. *By moving the* $n+1$ *spheres* S_i $(i = 1, 2, \ldots, n+1)$ *appropriately on* S^n *while preserving their relative positions, we can make all* ρ_i $(i = 1, 2, \ldots, n+1)$ *to have a common value independent of i.*

Proof. First, assuming the consequence of the proposition to be true, and denoting the common value of ρ_i $(i = 1, 2, \ldots, n+1)$ simply by ρ, we shall determine this common value ρ. From (17), we have

$$\frac{b_i}{s_i} = t_i - \frac{1}{\rho}. \tag{18}$$

Now we introduce a $(n+1) \times (n+1)$ -matrix $A = (a_{ij})$ defined by

$$a_{ij} = \frac{\mathbf{a}_i}{s_i} \cdot \frac{\mathbf{a}_j}{s_j}. \tag{19}$$

Because of (15) and (18), the formula (19) is rewritten as

$$a_{ij} = \begin{cases} 1 + \dfrac{2t_i}{\rho} - \dfrac{1}{\rho^2} & \text{for } i = j \\ -1 + \dfrac{t_i + t_j}{\rho} - \dfrac{1}{\rho^2} & \text{for } i \neq j \end{cases}. \tag{20}$$

Thus the matrix A coinsides with $A(u)$ with $u = 1/\rho$, which was defined by (4) in the section 2. Since $\mathbf{a}_i$'s are n-dimensional vectors, A is a degenerate matrix, and so, $\det A = 0$. Accordingly, Lemma 2 implies that ρ must satisfy a quadratic equation

$$2(n-1)\rho^2 - 4T_1\rho + \left((n+1)T_2 + 2(n+1) - T_1^2\right) = 0. \tag{21}$$

By the way, from Theorem 1, it follows that the determinant of the quadratic equation (21) is nonnegative. Thus we see that the quadratic equation(21) has two positive roots.

Now, let ρ have the value which is equal to one of the positive roots of (21). If we consider a matrix $A(1/\rho)$, then Lemma 4 shows that $A(1/\rho)$ is singular and non-negative definite. Accordingly, by Lemma 1, there exist n-dimensional vectors a_i 's for which (19) hold. Setting b_i $(i = 1, 2, \ldots, n+1)$ by (18), we can derive (15). Thus the assertion of the proposition is established.

Now, turning our attention to the sphere S_0, we denote its center by (a_0, b_0),where a_0 is an n-dimensional vector and b_0 is a real number such that $|\mathbf{a}_0|^2 + b_0^2 = 1$. Moreover, denote the center of the sphere $f(S_0)$ by γ_0 and its radius ρ_0. By Lemma 5, we have

(22) $$\gamma_0 = \frac{\mathbf{a}_0}{\cos r_0 - b_0} \text{ and } \rho_0 = \frac{\sin r_0}{\cos r_0 - b_0}.$$

Then, from Proposition 1, we can deduce the following lemma.

Lemma 6. *Suppose that the* $n+1$ *spheres* S_i *(*$i = 1, 2, \ldots, n+1$*) have the configuration such that all* ρ_i *have a commom value p. Then*

(23) $$\frac{1}{\rho_0} = \frac{(n+1) \pm \sqrt{2n(n+1)}}{n-1} \frac{1}{\rho}$$

(24) $$|\gamma_0|^2 + 1 = \frac{\rho}{n+1}(2T_1 - (n-1)\rho)$$

Proof. Note that, since spheres S_i ($i = 0, 1, 2, \ldots, n+1$) are mutually tangent, spheres $f(S_i)$ ($i = 0, 1, 2, \ldots, n+1$) are also mutually tangent. Accordingly, using the formula (1), we obtain (23) immediately. Now we shall prove (24). Under the assumption of the lemma, γ_i ($i = 1, 2, \ldots, n+1$) 's form a system of vertices of a n-dimensional regular simplex. Accordingly we have

$$\gamma_0 = \frac{1}{n+1}\sum_{i=1}^{n+1}\gamma_i .$$

Note that $\gamma_i = \rho \mathbf{a}_i / s_i$ because of (17). Then, using (19) and (20), we can proceed as follows:

$$\begin{aligned}
|\gamma_0|^2 &= \frac{\rho^2}{(n+1)^2}\sum_{i=1}^{n+1}\sum_{j+1}^{n+1}\frac{\mathbf{a}_i}{s_i}\cdot\frac{\mathbf{a}_j}{s_j} \\
&= \frac{\rho^2}{(n+1)^2}\left\{\sum_{i=1}^{n+1}\left(1 + \frac{2t_i}{\rho} - \frac{1}{\rho^2}\right) + \sum_{i \neq j}\sum\left(-1 + \frac{t_i + t_j}{\rho} - \frac{1}{\rho^2}\right)\right\} \\
&= \frac{1}{n+1}\left(-(n-1)\rho^2 + 2\rho T_1 - (n+1)\right)
\end{aligned}$$

Hence (24) follows immediately.

Now we prove our main theorem.

Proof of Main Theorem. Using (22), we have

(25) $$2t_0 = \frac{|\gamma_0|^2 + 1}{\rho_0} - \rho_0 .$$

Then, substituting (23) and (24) into (25), we get

$$t_0 = \frac{T_1}{n-1}\left(1 \pm \sqrt{\frac{2n}{n+1}}\right) \mp \rho\sqrt{\frac{2n}{n+1}}. \tag{26}$$

Now, solving the quadratic equation (21) explicitly, we have as its smaller positive root

$$\rho = \frac{1}{n-1}\left\{T_1 - \sqrt{\frac{n+1}{2}(T_1^2 - (n-1)T_2 - 2(n-1))}\right\}. \tag{27}$$

Then, substituting (27) into (26), we obtain

$$t_0 = \frac{1}{n-1}\left\{T_1 \pm \sqrt{n(T_1^2 - (n-1)T_2 - 2(n-1))}\right\}. \tag{28}$$

From this last expression (28), we can easily derive the formula (2) of Main Theorem.

Remark 2. In the proof of Main Theorem, if we use the larger positive root of the quadratic equation (21) instead of the smaller one (27), then we obtain

$$t_0 = \frac{1}{n-1}\left\{T_1 \mp \sqrt{n(T_1^2 - (n-1)T_2 - 2(n-1))}\right\}.$$

This means that by the streographic projection f corresponding to the larger positive root, the smaller sphere S_0 is projected to the larger one $f(S_0)$, while by f corresponding to the smaller positive root, the smaller sphere is projected to the smaller one.

Remark 3. For mutually tangent $n+1$ spheres on the n-dimensional sphere with radius R, the formula (2) stated in Main Theorem needs to be modified as

$$\left(\sum_{i=0}^{n+1} \cot\frac{r_i}{R}\right)^2 = n\left(\sum_{i=0}^{n+1} \cot^2\frac{r_i}{R} + 2\right).$$

Obviously, when R tends to infinity, this formula reduces to (1).

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