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A Formula in Combinatorics Derived from the Zeta Functions of RLL(m,n) Shift Dynamical Systems

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Abstract. In this article we will compute the numbers of periodic points and the zeta functions of RLL(m,n) shift dynamical systems to obtain a formula in combinatorics. We will generalize this formula and prove it directly. It will turn out that the generalized formula is related to the numbers of periodic points of RLL(a_1,\ldots,a_k) shift dynamical systems which is a generalization of RLL(m,n) shift dynamical systems.

Keywords and Phrases. Number of periodic points of RLL(m,n) shift; Zeta function of Symbolic Dynamical System; A formula in Combinatorics

1 Preliminaries

Throughout this paper we will use the notation and terminology in [1]. We begin with recalling a minimum of them. Let \( \mathcal{A} \) be a finite set of symbols which are called the alphabet. Elements of alphabet are also called letters, and they will typically be donoted by \( a, b, c, \ldots \), or sometimes by digits like \( 0, 1, 2, \ldots \).

**Definition 1.1** If \( \mathcal{A} \) is a finite alphabet, then the full \( \mathcal{A} \)-shift \( \mathcal{A}^\mathbb{Z} \) is the collection of all bi-infinite sequences of symbols from \( \mathcal{A} \), i.e.,

\[
\mathcal{A}^\mathbb{Z} = \{ x = (x_i)_{i \in \mathbb{Z}} | x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z} \}.
\]

The full r-shift (or simply r-shift) is the full shift over the alphabet \( \{0, 1, \ldots, r-1\} \).

When writing a specific sequence in \( \mathcal{A}^\mathbb{Z} \), we need to specify which is the 0-th coordinate. This is conveniently done with a decimal point to separate the \( x_i \) with \( i \geq 0 \) from those with \( i < 0 \). For example,

\[
x = \cdots 010.1101 \ldots
\]

means that \( x_{-3} = 0, x_{-2} = 1, x_{-1} = 0, x_0 = 1, x_1 = 1, x_2 = 0, x_3 = 1, \) and so on.

**Definition 1.2** The shift map \( \sigma \) on the full shift \( \mathcal{A}^\mathbb{Z} \) maps a point \( x \) to the point \( y = \sigma(x) \) whose \( i \)-th coordinate is \( y_i = x_{i+1} \).

A block (or word) over \( \mathcal{A} \) is a finite sequence of symbols from \( \mathcal{A} \). It is convenient to include the sequence of no symbols, called the empty block (or empty word) and denoted \( \epsilon \). The length of a block \( u \) is the number of symbols it contains, and is denoted by \( |u| \). Thus if \( u = a_1 a_2 \cdots a_k \) is a nonempty block, then \( |u| = k \), while \( |\epsilon| = 0 \). A k-block is simply a block of length \( k \). The set of all k-blocks over \( \mathcal{A} \) is denoted by \( \mathcal{A}^k \). If \( x \) is a point in \( \mathcal{A}^\mathbb{Z} \) and \( i \leq j \), then the block of coordinates in \( x \) from position \( i \) to position \( j \) is denoted by

\[
x_{[i,j]} = x_i x_{i+1} \cdots x_j
\]

If \( i > j \), define \( x_{[i,j]} \) to be \( \epsilon \). If \( x \in \mathcal{A}^\mathbb{Z} \) and \( w \) is a block over \( \mathcal{A} \), it is said that \( w \) occurs in \( x \) if there are indices \( i \) and \( j \) so that \( w = x_{[i,j]} \). Note that the empty block \( \epsilon \) in every \( x \), since \( \epsilon = x_{[1,0]} \). Let \( \mathcal{F} \) be a collection of blocks over \( \mathcal{A} \), which is thought of as being the forbidden blocks. For any such \( \mathcal{F} \), define \( X_\mathcal{F} \) to be the set of subsequences in \( \mathcal{A}^\mathbb{Z} \) which do not contain any block in \( \mathcal{F} \).

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Definition 1.3 A shift space (or simply shift) over \(A\) is a subset \(X\) of a full shift \(A^\mathbb{Z}\) such that \(X = X_\mathcal{F}\) for some collection \(\mathcal{F}\) of forbidden blocks over \(A\). If \(\mathcal{F}\) is a finite set, then we call \(X = X_\mathcal{F}\) a shift of finite type. When a shift space \(X\) is contained in a shift space \(Y\), we say that \(X\) is a subshift of \(Y\).

Definition 1.4 Let \(X\) be a subset of a full shift, and let \(\mathcal{B}_n(X)\) denote the set of all \(n\)-blocks that occur in points in \(X\). The language of \(X\) is the collection

\[\mathcal{B}(X) = \cup_{m=0}^{\infty} \mathcal{B}_n(X)\]

We consider a map \(\phi\) from a shift space \(X\) over \(A\) to a shift space \(Y\) over another alphabet \(\mathfrak{A}\) described as follows. Fix integer \(m\) and \(n\) with \(-m \leq n\). Let \(\Phi: \mathcal{B}_{m+n+1}(X) \rightarrow \mathfrak{A}\) be a fixed map from \(\mathcal{B}_{m+n+1}(X)\), the set of all \((m+n+1)\)-blocks to the alphabet \(\mathfrak{A}\), which is called an \((m+n+1)\)-block map from allowed \((m+n+1)\)-blocks in \(X\) to symbols in \(\mathfrak{A}\).

Definition 1.5 The map \(\phi: X \rightarrow \mathfrak{A}^\mathbb{Z}\) defined by \(y = \phi(x)\) with \(y_t\) given by

\[y_t = \Phi(x_{i-m} \cdots x_{i+n})\]

is called the sliding block code with memory \(m\) and anticipation \(n\) induced by \(\Phi\). We will denote the formation of \(\phi\) from \(\Phi\) by \(\phi = \Phi_{m+n}\), or simply by \(\phi = \Phi\) if the memory and anticipation of \(\phi\) are understood. If not specified, the memory is taken to be 0. If \(Y\) is a shift space contained in \(\mathfrak{A}^\mathbb{Z}\) and \(\phi(X) \subseteq Y\), we write \(\phi: X \rightarrow Y\).

Obviously, if \(\phi: X \rightarrow Y\) is a sliding block code between shift spaces, then \(\phi \circ \sigma_X = \sigma_Y \circ \phi\), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma_X} & X \\
\phi \downarrow & & \phi \downarrow \\
Y & \xrightarrow{\sigma_Y} & Y
\end{array}
\]

where \(\sigma_X\) and \(\sigma_Y\) are the shifts maps of \(X\) and \(Y\), respectively. If a sliding block code \(\phi: X \rightarrow Y\) has an inverse, i.e., a sliding block code \(\psi: Y \rightarrow X\) such that \(\psi(\phi(x)) = x\) for all \(x \in X\) and \(\phi(\psi(y)) = y\) for all \(y \in Y\), we call \(\phi\) invertible. If \(\phi\) is invertible, its inverse \(\psi\) is unique, so we can write \(\psi = \phi^{-1}\).

Definition 1.6 A sliding block code \(\phi: X \rightarrow Y\) is a conjugacy from \(X\) to \(Y\), if it is invertible. Two shift spaces \(X\) and \(Y\) are conjugate (written \(X \cong Y\)) if there is a conjugacy from \(X\) to \(Y\).

Definition 1.7 A graph \(G\) consists of a finite set \(V = V(G)\) of vertices (or states) together with a finite set \(\mathcal{E} = \mathcal{E}(G)\) of edges. Each edge \(e \in \mathcal{E}(G)\) starts at a vertex denoted by \(i(e) \in V(G)\) and terminates at a vertex \(t(e) \in V(G)\) (which can be the same as \(i(e)\)). Equivalently, the edge has initial state \(i(e)\) and terminale state \(t(e)\).

There may be more than one edge between a given initial state and terminal state; a set of such edges is called a set of multiple edges.

Definition 1.8 A graph \(G\) is irreducible if for every ordered pair of vertices \(i, j \in V\) there is a path in \(G\) starting at \(i\) and terminating at \(j\).

Definition 1.9 Let \(G\) be a graph with vertex set \(V\). For vertices \(i, j \in V\), let \(A_{ij}\) denote the number of edges in \(G\) with initial state \(i\) and terminal state \(j\). Then the adjacency matrix of \(G\) is \(A = [A_{ij}]\), and its formation from \(G\) is denoted by \(A = A(G)\) or \(A = A_G\).

Definition 1.10 Let \(G\) be a graph with edge set \(\mathcal{E}\) and adjacency matrix \(A\). The edge shift \(X_G\) or \(X_A\) is the shift space over the alphabet \(A = \mathcal{E}\) specified by

\[X_G = X_A = \{ \mathbf{X} = \{x_i\}_{i \in \mathbb{Z}} \in \mathcal{E}^\mathbb{Z} \mid t(x_i) = i(x_{i+1}) \text{ for all } i \in \mathbb{Z} \}.\]

The shift map on \(X_G\) or \(X_A\) is called the edge shift map and is denoted by \(\sigma_G\) or \(\sigma_A\).
According to the definition, a bi-infinite sequence of edges is in $X_G$ exactly when the terminal state of each edges is the initial state of the next one; i.e., the sequence describes a bi-infinite walk or bi-infinite trip on $G$.

**Definition 1.11** A labeled graph $\mathcal{G}$ is a pair $(G, L)$, where $G$ is a graph with edge set $E$, and the labeling $L: E \to A$ assigns to each edge $e$ of $E$, and a label $L(e)$ from the finite alphabet $A$. The underlying graph of $\mathcal{G}$ is $G$. A labeled graph is irreducible if its underlying graph is irreducible.

**Definition 1.12** A subset $X$ of a full shift is a sofic shift if $X = X_S$ for some labeled graph $\mathcal{G}$. A presentation of a sofic shift $X$ is a labeled graph $\mathcal{G}$ for which $X_S = X$. The shift map on $X_S$ is denoted by $\sigma_S$.

**Definition 1.13** A shift space $X$ is mixing if, for every ordered pair $u, v \in B(X)$, there is an $N$ such that for each $n \geq N$ there is word $w \in B_n(X)$ such that $uwv \in B(X)$.

For two points $x, y$ of a full shift $A^Z$ over an alphabet $A$, we put

$$\rho(x, y) = \begin{cases} 2^{-k} & \text{if } x \neq y \text{ and } k \text{ is maximal so that } x[-k,k] = y[-k,k] \\ 0 & \text{if } x = y. \end{cases}$$

(1.1)

This $\rho$ satisfies the axioms of a metric, so it makes $A^Z$ a metric space. As we can see easily the topology on $A^Z$ induced by this metric is equivalent to the product topology of discrete topology on $A^Z$. The discrete topology on a finite set $A$ makes it compact. Hence, due to Tikhonov’s theorem, $A^Z$ with the metric $\rho$ is also compact. In what follows we always consider this metric on $A^Z$. The shift map $\sigma$ is a homeomorphism with respect to the topology induced by the metric $\rho$, so $(A^Z, \sigma)$ can be considered as an invertible dynamical system (cf. [1], Definition 6.2.1). A subset $X$ of $A^Z$ is a shift space if and only if it is shift-invariant, i.e., $\sigma(X) = X$, and compact (cf. [1], Theorem 6.1.21). From this fact it follows that any shift space can also be considered as an invertible dynamical system. When we consider a shift space with the metric $\rho$ in (1.1) as a dynamical system, we call it a shift dynamical system. A homeomorphism $\phi$ from a shift $(X, \sigma_X)$ to another one $(Y, \sigma_Y)$ is said to be a topological conjugacy if $\phi \circ \sigma_X = \sigma_Y \circ \phi$. Two shift spaces are said to be topological conjugate if there is a topological conjugacy between them. The fact that two shifts spaces are conjugate as a shift space if and only if they are topologically conjugate as a dynamical system follows from the following theorem ([1], Theorem 6.2.9).

**Theorem 1.14** (Curtis-Lyndon-Hedlund Theorem) Suppose that $(X, \sigma_X)$ and $(Y, \sigma_Y)$ are shift dynamical systems, and that $\theta: X \to Y$ is a (not necessarily continuous) map. Then $\theta$ is a sliding block code if and only if it is continuous and commutes with shift maps, i.e., $\theta \circ \sigma_X = \sigma_Y \circ \theta$.

## 2 RLL(m,n) shifts and their characteristic polynomials

For each pair $(m, n)$ of positive integers with $m < n$, we define $X(m, n)$ to be the set of all binary sequences for all 1’s occur infinitely often in $x$ in both directions, and there are at least $m$ 0’s, but no more than $n$ 0’s, between two 1’s. $X(m, n)$ is called $(m, n)$ run-length limited shift. $X(m, n)$ is the sofic shift associated to the following labeled graph:

![Figure 2.1: X(m,n)](image)

We denote by $G(m, n)$ the underlying graph of the labeled graph in Figure 2.1. We name each vertex and each edge in $G(m, n)$ as follows:
Proposition 2.1  If we put
\[ \mathcal{F} = \left\{ 11, 101, 1001, \ldots, \underbrace{0 \cdots 0}_m, 1 \underbrace{0 \cdots 0}_n \right\}, \]
then \( X(m, n) = X_\mathcal{F} \).

Proof: Let \( x \) be a point in \( A^Z \) where \( A = \{0, 1\} \). Assume \( x \notin X_\mathcal{F} \). Then \( x \) must contain a block in \( \mathcal{F} \), and so \( x \notin X(m, n) \). Hence \( X(m, n) \subset X_\mathcal{F} \). To prove the converse inclusion, assume \( x \in X_\mathcal{F} \). Then, since the number of consecutive 0's which occurs in \( x \) can not exceed \( n + 1 \), \( 1 \) occurs infinitely many times in both directions. If we look at a block \( w \) in \( x \) which has consecutive 0's sandwiched by two 1's like \( 10 \cdots 01 \), then the number of 0's in the block is more than \( m - 1 \) and less than \( n + 1 \). There exists the unique path \( \pi \) in the graph in Figure 2.2 such that \( L(\pi) = w \) where \( L : E(G(m, n)) \rightarrow \{0, 1\} \) is the labeling in Figure 2.1. The path \( \pi \) always ends at the vertex \( v_0 \). Hence, there exists the unique infinite path \( \pi_\infty(x) \) in both direction in the graph in Figure 2.2 such that \( L(\pi_\infty(x)) = x \), that is, \( x \in X(m, n) \). Therefore \( X_\mathcal{F} \subset X(m, n) \). Consequently, \( X_\mathcal{F} = X(m, n) \).

Proposition 2.2 \( X(m, n) \) is conjugate to \( X_{G(m, n)} \) as a shift space.

Proof: We will construct a sliding block code \( \phi : X(m, n) \rightarrow X_{G(m, n)} \) which gives a conjugacy between them. As shown in the proof of Proposition 2.1, for any point \( x \) of \( X(m, n) \) there exists the unique infinite path \( \pi_\infty(x) \) in both directions in the graph in Figure 2.2 such that \( L(\pi_\infty(x)) = x \). We define \( \phi(x) = \pi_\infty(x) \). Obviously, \( \phi \) commutes with the shift maps. Furthermore, \( \phi(x)_0 \) is a function of \( x_{[-n,n]} \). To see this fact, argue as follows: If \( x_0 = 1 \), then all of \( x_{-i} \) are zeros for \( 1 \leq i \leq n \), or there exists \( k \) with \( m \leq k \leq n - 1 \) such that \( x_{-1} = \cdots = x_{-k} = 0 \) and \( x_{-(k+1)} = 1 \). In the former case \( \phi(x)_0 = f_n \), and in the latter case \( \phi(x)_0 = f_k \). If \( x_0 = 0 \), then there exist \( k \) and \( \ell \) with \( 0 \leq k \leq n - 1 \) and \( 0 \leq \ell \leq n - 1 \) and \( m + 1 \leq k + \ell + 1 \leq n \) such that \( x_{-k+1} = x_{\ell+1} = 1 \), \( x_{-1} = \cdots = x_{-k} = 0 \) \( (1 \leq k) \), and \( x_1 = \cdots = x_\ell = 0 \) \( (1 \leq \ell) \). In this case \( \phi(x)_0 = e_k \). In all cases \( \phi(x)_0 \) is decided by \( x_{[-n,n]} \), which means that \( \phi(x)_0 \) is a function of \( x_{[-n,n]} \). Therefore \( \phi \) is a sliding block code \( ([1], \text{Proposition 1.5.8}) \). The map \( \psi : X_{G(m, n)} \rightarrow X(m, n) \) defined by \( \psi(\pi_\infty) = L(\pi_\infty) \) for \( \pi_\infty \in X_{G(m, n)} \) gives the inverse of \( \phi \), so \( \phi \) gives a conjugacy between \( X(m, n) \) and \( X_{G(m, n)} \).

Proposition 2.3 \( X(m, n) \) is mixing.

Proof: We consider \( X_{G(m, n)} \) instead of \( X(m, n) \) due to Proposition 2.2. In \( X_{G(m, n)} \) there are periodic points of period \( m + 1 \) and \( m + 2 \). Since \( m + 1 \) and \( m + 2 \) are co-prime, \( \text{per}(X_{G(m, n)}) = 1 \), where \( \text{per}(X_{G(m, n)}) \) is the greatest common divisor of the periods of \( X_{G(m, n)} \)'s periodic points. Obviously, \( X_{G(m, n)} \) is irreducible. Therefore we can conclude that \( X_{G(m, n)} \) is mixing \( ([1], \text{Proposition 4.5.10}) \).

In what follows we always consider \( X_{G(m, n)} \) instead of \( X(m, n) \). The adjacency matrix of the graph \( G(m, n) \) denoted by \( A(m, n) \) is as follows:
We define the characteristic polynomial of RLL(m,n) shift space is to be that of the adjacency matrix $A(m,n)$ of the graph $G(m,n)$, which is calculated as follows:

$$
\chi_{A(m,n)}(t) = |tE_{n+1} - A(m,n)|
$$

\[
\begin{bmatrix}
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{array}
\end{bmatrix}
\]
\[
\begin{align*}
T^m &=
\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -1 & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix} \\
T^{m+1} &=
\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -1 & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix} \\
t(t^{m+1} - 1) - 1 &=
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -1 & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix} \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 0 \\
p(t) &=
\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & -1 \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\end{align*}
\]

\[
= (-1)^{n+2} \times (-1)^np(t) = p(t)
\]

where \(p(t) = t^{n+1} - \sum_{k=0}^{n-m} t^{n-m-k}\).

3 The numbers of periodic points of RLL(m,n) shifts

First, we give the definitions of a periodic point, a cycle and the zeta function of a topological dynamical system. Let \((M, \phi)\) be a topological dynamical system. We denote \(\phi^k\) the composition of \(\phi\) with itself \(k > 0\) times, and call it \(k\)-th iteration of \(\phi\).

Definition 3.1 A point \(x \in M\) is periodic for \(\phi\) if \(\phi^k(x) = x\) for some \(k \geq 1\), and we say that \(x\) has
period \( k \) under \( \phi \). If \( x \) is periodic, the smallest positive integer \( k \) for which \( \phi^k(x) = x \) is the least period of \( x \). If \( \phi(x) = x \), then \( x \) is called a fixed point for \( \phi \).

**Definition 3.2** For a periodic point \( x_0 \) for \( \phi \) of the least period \( k > 0 \), we call the subset \( C = \{ x_0, \phi(x_0), \ldots, \phi^{k-1}(x_0) \} \) of finite points of \( M \) a cycle of period \( k \) for \( \phi \).

Let \( p_k(\phi) \) denote the number of periodic points of period \( k \), and \( q_k(\phi) \) that of the least period \( k \). For a shift space \( (X, \phi_X) \), we write \( p_k(X) \) for \( p_k(\sigma_X) \) (resp. \( q_k(X) \) for \( q_k(\sigma_X) \)). From the Möbius inversion formula ([2]), \( q_k(\phi) \) and \( p_k(\phi) \) are related as follows:

\[
q_k(\phi) = \sum_{\ell | k} \mu\left(\frac{k}{\ell}\right) p_\ell(\phi)
\]

where \( \mu \) is the Möbius function defined by

\[
\mu(k') = \begin{cases} 
(-1)^r & \text{if } k' \text{ is the product of } r \text{ distinct primes,} \\
0 & \text{if } k' \text{ contains a square factor,} \\
1 & \text{if } k' = 1.
\end{cases}
\]

We are now going to calculate \( p_k(X(m,n)) \) the number of periodic points of period \( k \) in the RLL\((m,n)\) shift \( X(m,n) \) for \( k \geq 1 \). In what follows we write \( p_k(m,n) \) for \( p_k(X(m,n)) \) for simplicity. We denote by \( \mathbb{Z}_{\geq 0} \) the set of all non-negative integers, and by \( \mathbb{Z}_{\geq 0}^{m+1} \) the product of \( n - m + 1 \) copies of \( \mathbb{Z}_{\geq 0} \). For a non-negative integer \( k \), we define the subset \( D_k(m,n) \) of \( \mathbb{Z}_{\geq 0}^{n-m+1} \) by

\[
D_k(m,n) = \left\{ (p_{m+1}, \ldots, p_{n+1}) \in \mathbb{Z}_{\geq 0}^{n-m+1} : \sum_{i=1}^{n-m+1} (m+i)p_{m+i} = k \right\}.
\]

Then, the number \( p_{k+1}(m,n) \) for \( k \geq 0 \) is given as follows:

**Proposition 3.3**

\[
p_{k+1}(m,n) = \sum_{\{p_{m+1}, \ldots, p_{n+1}\} \in D_{k+1}(m,n)} \frac{(k+1)(p_{m+1} + \cdots + p_{n+1} - 1)!}{p_{m+1}! \cdots p_{n+1}!} 
\quad (0 \leq k \leq m - 1),
\]

where we understand that if \( D_{k+1}(m,n) = \emptyset \), then the sum is equal to zero.

**Proof:** First, note that \( p_{k+1}(m,n)/k + 1 \) is the number of cycles of period \( k + 1 \) in the graph \( G(m,n) \), though, if a cycle has period \( k + 1 \) with \( k + 1 = \ell(k' + 1) \) (\( \ell \geq 1 \)), then we should count this cycle as \( 1/\ell \) of a cycle of period \( k + 1 \). There is one-to-one correspondence between closed paths of length \( k + 1 \) in the graph \( G(m,n) \) and cycles of period \( k + 1 \) in the shift \( X(m,n) \) if we do not take account of which vertex and edge closed paths start at. Therefore, it suffices to count the number of cycles starting at the vertex \( v_0 \) of length \( k + 1 \) in the graph \( G(m,n) \). In \( G(m,n) \) there are \( n - m + 1 \) closed paths \( \pi_{m+1}, \ldots, \pi_{n+1} \) starting at \( v_0 \) specified by

\[
\begin{align*}
\pi_{m+1} &= e_0 \circ \cdots \circ e_{m-1} \circ f_m, \\
\pi_{m+2} &= e_0 \circ \cdots \circ e_{m-1} \circ e_m \circ f_{m+1}, \\
\vdots \\
\pi_{n+1} &= e_0 \circ \cdots \circ e_{m-1} \circ e_m \circ \cdots \circ e_{n-1} \circ f_n.
\end{align*}
\]

We call these closed paths "fundamental paths" since every closed path \( \pi \) starting at \( v_0 \) in \( G(m,n) \) can be expressed as

\[
\pi = \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_s},
\]
where each \( \pi_{i_k} \) \((1 \leq k \leq s)\) in this expression is one of \( \pi_{m+1}, \ldots, \pi_{n+1} \). If we denote by \( p_{m+1}(\pi) \) the number of \( \pi_{m+1} \) which occurs in the expression in (3.2) for \( 1 \leq i \leq n-m+1 \), then \( s = \sum_{i=1}^{n-m+1} p_{m+1}(\pi) \) and the length of the closed path \( \pi \) is \( \sum_{i=1}^{n-m+1} (m+i)p_{m+1}(\pi) \). Note that, in the expression in (3.2), all of \( \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_s} \), \( \pi_{i_2} \circ \pi_{i_3} \circ \cdots \circ \pi_{i_s} \), \( \cdots \), \( \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_s} \) represent the same closed path in \( G(m,n) \). To count the number of closed paths \( \pi \)'s which can be expressed as in (3.2) with \( p_{m+1}(\pi) = p_{m+1} \) for \( 1 \leq i \leq n-m+1 \) for a given \( (p_{m+1}, \ldots, p_{n+1}) \in D_{k+1}(m,n) \), we think of each \( p_{m+1} \) as the number of positions where the closed path \( \pi_{m+1} \) occurs in the expression in (3.2). Then the number we seek is

\[
\frac{1}{(p_{m+1} + \cdots + p_{n+1})} \left( \begin{array}{c} p_{m+1} + \cdots + p_{n+1} \\ p_{m+2} \\
 \vdots \\
 p_{n+1} \end{array} \right) = \frac{(p_{m+1} + \cdots + p_{n+1} - 1)!}{p_{m+1}! \cdots p_{n+1}!}
\]

Thus we have the equality in (3.1). \( \blacksquare \)

4 Zeta functions of RLL(m,n) shifts

First, we give the definition of zeta function of a topological dynamical system.

**Definition 4.1** Let \((M, \phi)\) be a topological dynamical system for which \( p_k(\phi) < \infty \) for all \( k \geq 1 \). The zeta function \( \zeta_\phi(t) \) of \((M, \phi)\) is defined by

\[
\zeta_\phi(t) = \exp \left( \sum_{k=1}^{\infty} \frac{p_k(\phi)}{k} t^k \right).
\]

We know that if \( X \) is a shift of finite type, then there is an \( r \times r \) non-negative integer matrix \( A \) with \( X = X_A \) ([1], Theorem 2.3.2) and the zeta function of \( X_A \) is given by

\[
\zeta_{\sigma_A}(t) = \frac{1}{t^{\chi_A(t^{-1})}} = \frac{1}{\det(E_r - tA)},
\]

where \( \sigma_A \) denotes the shift map of \( X_A \), and \( \chi_A \) the characteristic polynomial of \( A \) ([1], Theorem 6.4.6). The equality in (4.2) follows from the definition of \( \zeta_\phi(t) \) and the fact that

\[ p_k(\sigma_A) = \text{tr} A^k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_r^k \]

where \( \lambda_1, \ldots, \lambda_r \) the root of \( \chi_A(t) \) listed with multiplicity ([1], Proposition 2.2.12).

Now, we calculate \( p_{k+1}(m,n) \) for \( k \geq 0 \) by use of the zeta function of the shift dynamical system \( X_{G(m,n)} \). Since the characteristic polynomial \( \chi_{A(m,n)}(t) \) of the adjacency matrix \( A(m,n) \) of the graph \( G(m,n) \) is specified by

\[ X_{A(m,n)}(t) = t^{n+1} - (t^{m-1} + t^{n-m-1} + \cdots + t + 1), \]

the zeta function \( \zeta_{\sigma_{A(m,n)}}(t) \) of \( X_{A(m,n)} \) is \( X_{G(m,n)} \) is given by

\[
\zeta_{\sigma_{A(m,n)}}(t) = \frac{1}{t^{\chi_{A(m,n)}(t^{-1})}} = \frac{1}{1 - (t^{m+1} + \cdots + t^{n+1})}.
\]

Therefore, by (4.1),

\[
\sum_{k=0}^{\infty} \frac{p_{k+1}(m,n)}{k+1} t^{k+1} = \log \zeta_{\sigma_{A(m,n)}}(t) = -\log \left[ 1 - (t^{m+1} + \cdots + t^{n+1}) \right].
\]

\[
\sum_{k=0}^{\infty} \frac{p_{k+1}(m,n)}{k+1} t^{k+1} = \log \zeta_{\sigma_{A(m,n)}}(t) = -\log \left[ 1 - (t^{m+1} + \cdots + t^{n+1}) \right].
\]
We will calculate the Taylor's expansion of the R.H.S. of (4.3) at $t = 0$. Put
\[
\varphi(t) = -\log \{1 - (t^{m+1} + \cdots + t^{n+1})\},
\]
\[
g(t) = (m+1)t^m + \cdots + (n+1)t^n, \quad \text{and}
\]
\[
f(t) = 1 - (t^{m+1} + \cdots + t^{n+1}).
\]
Then
\[
\varphi'(t) = \frac{g(t)}{f(t)}, \quad \text{and}
\]
\[
\varphi^{(k+1)}(t) = \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{1}{\ell+1}\right)^{\ell} \varphi^{(\ell)}(t)\varphi^{(0)}(t) \quad (k \geq 0),
\]
and so
\[
\varphi^{(k+1)}(0) = \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{1}{\ell+1}\right)^{\ell} \varphi^{(\ell)}(0) \varphi^{(0)}(0) \quad (k \geq 0).
\]
Since
\[
g^{(t)}(0) = \begin{cases} (\ell + 1)! & m \leq \ell \leq n, \\ 0 & \text{otherwise}, \end{cases}
\]
we have
\[
\varphi^{(k+1)}(0) = \begin{cases} 0 & (0 \leq k \leq m-1), \\ \sum_{\ell=m}^{n} \binom{n}{\ell} \left(\frac{1}{\ell+1}\right)^{\ell} \varphi^{(\ell)}(0) \varphi^{(0)}(0) & (m \leq k \leq n-1) \\ \sum_{\ell=m}^{n} \binom{n}{\ell} \left(\frac{1}{\ell+1}\right)^{\ell} \varphi^{(\ell)}(0) \varphi^{(0)}(0) & (n \leq k). \end{cases}
\]
(4.4)

To know $(1/f)^{(k-\ell)}(0)$, we calculate the Taylor expansion of $(1/f)(t)$ at $t = 0$, which is given by
\[
\left(\frac{1}{f}(t) = \sum_{\alpha_0=0}^{\infty} (t^{n+1} + \cdots + t^{m+1})^{\alpha_0}
\right)
\]
\[
= \sum_{\alpha_0=0}^{\infty} \sum_{\alpha_1=0}^{\alpha_0-1} \cdots \sum_{\alpha_n-m=0}^{\alpha_n-m-1} \binom{\alpha_0}{\alpha_1} \binom{\alpha_0-\alpha_1}{\alpha_2} \cdots \binom{\alpha_0-(\alpha_1+\cdots+\alpha_n-m)}{\alpha_n-m} 
\times \left(\frac{1}{\sum_{i=0}^{m} (m+i) \alpha_i + (n+1)(\alpha_n - \sum_{i=1}^{n-m} \alpha_i)}\right).
\]
(4.5)

Hence, if we put
\[
D_k(m, n) = \left\{ (\alpha_1, \cdots, \alpha_{n-m}, \alpha_{n-m+1}) \in \mathbb{Z}_{\geq 0}^{n-m+1} \mid \alpha_1 \geq 0, \cdots, \alpha_{n-m+1} \geq 0, \sum_{i=1}^{n-m+1} (m+i) \alpha_i = k \right\},
\]
We will generalize the formula in (4.7). Let

\[ \left( \frac{1}{T} \right)^{(k)}(t) = \sum_{k=0}^{\infty} \left\{ \sum_{(a_1, \ldots, a_{n-m}, a_{n-m+1}) \in D_k(m,n)} \frac{(\alpha_1 + \cdots + \alpha_{n-m} + a_{n-m+1})!}{\alpha_1! \cdots \alpha_{n-m+1}!} \right\} t^k, \]

and so

\[ \left( \frac{1}{T} \right)^{(k-\ell)}(0) = (k-\ell)! \sum_{(a_1, \ldots, a_{n-m}, a_{n-m+1}) \in D_{k-\ell}(m,n)} \frac{(\alpha_1 + \cdots + \alpha_{n-m} + a_{n-m+1})!}{\alpha_1! \cdots \alpha_{n-m+1}!}. \]

Hence, by (4.4),

\[ (4.6) \]

\[ \phi^{(k+1)}(0) = \begin{cases} 0 & (0 \leq k \leq m - 1) \\ \frac{1}{k+1} \sum_{\ell=m}^{n} \sum_{(a_1, \ldots, a_{n-m}, a_{n-m+1}) \in D_{k-\ell}(m,n)} \frac{(\ell+1)(\alpha_1 + \cdots + \alpha_{n-m+1})!}{\alpha_1! \cdots \alpha_{n-m+1}!} & (m \leq k). \end{cases} \]

By the definition of \( \phi(t) \),

\[ \frac{\phi^{(k+1)}(0)}{(k+1)!} = \frac{p_{k+1}(m,n)}{k+1}. \]

Therefore, by (4.6) and (3.1) we have the following equality:

\[ (4.7) \]

\[ \sum_{(p_{m+1}, \ldots, p_{n+1}) \in D_{k+1}(m,n)} \frac{(k+1)(p_{m+1} + \cdots + p_{n+1} - 1)!}{p_{m+1}! \cdots p_{n+1}!} = \sum_{\ell=m}^{n} \sum_{(p_{m+1}, \ldots, p_{n+1}) \in D_{k-\ell}(m,n)} \frac{(\ell+1)(p_{m+1} + \cdots + p_{n+1})!}{p_{m+1}! \cdots p_{n+1}!} \quad (k \geq m) \]

5 A formula in combinatorics

We will generalize the formula in (4.7). Let \( a_1, \ldots, a_s \) \((s \geq 1)\) be positive integers with \( a_1 < a_2 < \cdots < a_s \), and \( k \) a non-negative integer. We define \( D_k(a_1, \ldots, a_s) \) by

\[ D_k(a_1, \ldots, a_s) = \left\{ (p_1, \ldots, p_s) \in \mathbb{Z}_{\geq 0}^s \mid \sum_{i=1}^{s} (a_i + 1)p_i = k \right\}. \]

When \( s = n - m + 1 \) and \( a_1 = m, a_2 = m + 1, \ldots, a_{n-m+1} = n, D_k(a_1, \ldots, a_s) \) is nothing but \( D_k(m,n) \) defined before. With this notation, we have:

**Theorem 5.1**

\[ (5.1) \]

\[ \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(k+1)(p_1 + \cdots + p_s - 1)!}{p_1! \cdots p_s!} = \sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k-a_i}(a_1, \ldots, a_s)} \frac{(a_i + 1)(p_1 + \cdots + p_s)!}{p_1! \cdots p_s!} \quad (k \geq a_1) \]
Proof: We use the induction on $s$.

(I) In the case $s = 1$: $D_{k+1}(a_1)$ ($k \geq a_1$) is not empty if and only if $k + 1$ is a multiple of $a_1 + 1$. If this is the case, we have

$$\frac{k + 1}{p_1} = a_1 + 1$$

for $p_1 \in D_{k+1}(a_1)$ and the map which assigns $p_1$ to $p_1 - 1$ gives a one-to-one correspondence from $D_{k+1}(a_1)$ to $D_{k-a_1}(a_1)$. Hence the equality in (5.1) holds.

(II) In the case $s \geq 2$: We assume that the equality in (5.1) holds for $s - 1$. We put

$$S = \sum_{(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s)} \frac{(k + 1)(p_1 + \cdots + p_s - 1)!}{p_1! \cdots p_s!}$$

For $i$ with $1 \leq i \leq s$,

$$S = \sum_{(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s) \atop p_i \geq 1} \frac{(k + 1)(p_1 + \cdots + p_s - 1)!}{p_1! \cdots p_s!} + \sum_{(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s) \atop p_i = 0} \frac{(k + 1)(p_1 + \cdots + \bar{p_i} + \cdots + p_s - 1)!}{p_1! \cdots \bar{p_i}! \cdots p_s!}$$

where $\bar{p_i}$ and $\bar{p_i}!$ denotes deleting the symbols $p_i$ and $p_i!$. Hence, we have

$$S = \frac{1}{s} \left\{ \sum_{i=1}^{s} \sum_{(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s) \atop p_i \geq 1} \frac{(k + 1)(p_1 + \cdots + p_s - 1)!}{p_1! \cdots p_s!} + \sum_{i=1}^{s} \sum_{(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s) \atop p_i = 0} \frac{(k + 1)(p_1 + \cdots + \bar{p_i} + \cdots + p_s - 1)!}{p_1! \cdots \bar{p_i}! \cdots p_s!} \right\}$$

(5.2)

Since each $(p_1, \cdots, p_s) \in D_{k+1}(a_1, \cdots, a_s)$ satisfies

$$\sum_{j=1}^{s} (a_j + 1)p_j = k + 1$$

and so,

$$\frac{k + 1}{p_i} = (a_i + 1) + \sum_{j=1 \atop j \neq i}^{s} \frac{a_j + 1}{p_j} \frac{p_j}{p_i}$$
for every i with 1 ≤ i ≤ s, the first term in the braces in (5.2) is transformed as follows:

\[
\sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(k+1)[p_1 + \cdots + p_s - 1]!}{p_1 \cdots p_s} = \sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{k + 1 \cdot (p_1 + \cdots + (p_i - 1) + \cdots + p_s)!}{p_i \cdot (p_i - 1)! \cdots p_s!} + \sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(a_i + 1) \cdot (p_1 + \cdots + (p_i - 1) + \cdots + p_s)!}{p_1 \cdots (p_i - 1)! \cdots p_s!}
\]

(5.3)

On the other hand, by the induction hypothesis the second term in the braces in (5.2) is transformed as follows:

\[
\sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s, \tilde{p}_1, \ldots, \tilde{p}_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(k+1)[p_1 + \cdots + \tilde{p}_1 + \cdots + p_s - 1]!}{p_1 \cdots \tilde{p}_i \cdots p_s} = \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(a_j + 1) \cdot [p_1 + \cdots + \tilde{p}_i + \cdots + p_s)!}{p_1 \cdots \tilde{p}_i \cdots p_s!}
\]

(5.4)

We claim that

\[
\sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \sum_{p_i \geq 1} \frac{(a_j + 1) \cdot p_i \cdot [p_1 + \cdots + (p_i - 1) + \cdots + p_s)!}{p_1 \cdots (p_i - 1)! \cdots p_s!} + \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(a_j + 1) \cdot [p_1 + \cdots + \tilde{p}_i + \cdots + p_s)!}{p_1 \cdots \tilde{p}_i \cdots p_s!}
\]

(5.5)

\[
= (s - 1) \sum_{i=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}(a_1, \ldots, a_s)} \frac{(a_i + 1) \cdot [p_1 + \cdots + p_s)!}{p_1 \cdots p_s!}
\]
Indeed, this can be proved as follows:

The first term on the L.H.S. in (5.5)

\[
\sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots (p_i - 1)! \cdots p_s!}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots (p_i - 1)! \cdots p_s!}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots (p_i - 1)! \cdots p_s!}
\]

Hence:

The L.H.S in (5.5)

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots p_s!}
\]

\[
+ \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots (p_i - 1)! \cdots p_s!}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots p_s!}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots p_s!}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots p_s!}
\]

\[
= (s - 1) \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{(p_1, \ldots, p_s) \in D_{k+1}} \frac{(a_j + 1)! (p_1 + \cdots + p_i)!}{p_1! \cdots p_s!}
\]

Thus, the equality in (5.5) holds. Consequently, by (5.2), (5.3), (5.4) and (5.5), we obtain the equality in (5.1).

6 RLL\((a_1, \ldots, a_s)\) shifts

The number on the L.H.S. in (5.1) relates to the number of \(k+1\) cycles of a certain shift dynamical system. For positive integers \(a_1, \ldots, a_s\) \((s \geq 1)\) with \(a_1 < a_2 < \cdots < a_s\), we define the \((a_1, \ldots, a_s)\) run-length limited shift, denoted by \(RLL(a_1, \ldots, a_s)\), or \(X(a_1, \ldots, a_s)\), to be the shift space associated to the labeled graph in Figure 6.1. We denote by \(G(a_1, \ldots, a_s)\) the underlying graph of the labeled one in Figure 6.1, by \(A(a_1, \ldots, a_s)\) the adjacency matrix of the graph \(G(a_1, \ldots, a_s)\) and by \(X_G(a_1, \ldots, a_s)\), or \(X_{\Lambda(a_1, \ldots, a_s)}\), the edge shift associated to \(G(a_1, \ldots, a_s)\). As in the case of RLL\((m, n)\) shift, \(X(a_1, \ldots, a_s)\) is conjugate to \(X_{G(a_1, \ldots, a_s)}\) as a shift space. The number on the L.H.S. in (5.1) is equal to \(p_{k+1}(X(a_1, \ldots, a_s))\), the number of periodic points of period \(k+1\) in \(X(a_1, \ldots, a_s)\). The adjacency matrix \(A(a_1, \ldots, a_s)\) is given by

\[\begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 1 & a_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_s & 1 \\ 0 & \cdots & 0 & 1 & a_1 \end{bmatrix}\]
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As in the case of (4.7) to prove it. to generalize it to that of the form in (5.1), since we cannot apply the induction argument to the equality in (6.1).

Figure 6.1: $X(a_1, \ldots, a_s)$

$$A(a_1, \ldots, a_s) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}$$

and so the characteristic polynomial $X_{A(a_1, \ldots, a_s)}(t)$ of $A(a_1, \ldots, a_s)$ is given by

$$X_{A(a_1, \ldots, a_s)}(t) = |tE_{a_s+1} - A(a_1, \ldots, a_s)| = t^{a_s+1} - (t^{a_s-a_1} + t^{a_s-a_2} + \cdots + t^{a_s-a_{s-1}} + 1).$$

Hence the zeta function $\zeta_{A(a_1, \ldots, a_s)}(t)$ of $X_{A(a_1, \ldots, a_s)} = X_{G(a_1, \ldots, a_s)}$ is given by

$$\zeta_{A(a_1, \ldots, a_s)}(t) = \frac{1}{\{t^{a+1}X_{A(a_1, \ldots, a_s)}(t^{-1}) - 1 - (t^{a+1} + \cdots + t^{a+1})\}}.$$

As in the case of RLL(m, n) shift, we can prove the equality in (5.1) by use of the identity

$$(6.1) \quad \sum_{k=0}^{\infty} \frac{p_{k+1}(X(a_1, \ldots, a_s))}{k+1} = \log \zeta_{A(a_1, \ldots, a_s)}(t) = \log [1 - (t^{a+1} + \cdots + t^{a+1})].$$

Notice that if we want to prove the equality in (4.7) directly, not using the zeta function of $X_{G(m,n)}$, we need to generalize it to that of the form in (5.1), since we cannot apply the induction argument to the equality in (4.7) to prove it.

References


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