A Generalization of the Classical Brachistochrone Curve A Preliminary Study

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A Generalization of the Classical Brachistochrone Curve A Preliminary Study

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Abstract

The classical Brachistochrone curve is one of fastest descent on a vertical plane. In this paper curves on fastest descent on surfaces that can be represented by elliptic integrals are studied. In the first part it is shown that such curves have to lie on spheres or cones. In the second part curves on cones are studied in detail and it is found that these curves approach to cycloids as cones become cylinders gradually.

1 Introduction

A Brachistochrone curve, or curve of fastest descent, is the curve between two points that is covered in the least time by a body that starts at the first point with zero speed and is constrained to move along the curve to the second point, under the action of constant gravity and assuming no friction.

The problem of Brachistochrone curve is first posed by Johann Bernoulli, and is solved simultaneously by himself and several his contemporaries ([1]). It is very old, but has continously atrracted many people until now ([2], [3]).

In this paper we study brachistochrone curves on surfaces of rotation. Consider a surface of rotation defined by

$$\mathbf{r} = \begin{pmatrix} r(z)\cos\varphi \\ r(z)\sin\varphi \\ z \end{pmatrix}.$$

Then we have

$$\frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -r(z)\sin\varphi \\ r(z)\cos\varphi \\ 0 \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial z} = \begin{pmatrix} r_1(z)\cos\varphi \\ r_1(z)\sin\varphi \\ 1 \end{pmatrix},$$

where $r_1 = \dfrac{dr}{dz}$. Acordingly

$$E = \left(\frac{\partial \mathbf{r}}{\partial \varphi}\right)^2 = r^2,\ F = \frac{\partial \mathbf{r}}{\partial \varphi}\frac{\partial \mathbf{r}}{\partial z} = 0,\ G = \left(\frac{\partial \mathbf{r}}{\partial z}\right)^2 = r_1^2 + 1.$$

We consider a curve that lies on the above surface, $(\varphi(u), z(u))$, where u is a parameter. Then an infinitesimal length of the curve is given by

$$ds = \sqrt{E\varphi'^2 + 2F\varphi' z' + Gz'^2}du = \sqrt{r^2\varphi'^2 + (r_1^2+1)z'^2}du,$$

where $\varphi' = \dfrac{d\varphi}{du}$ and $z' = \dfrac{dz}{du}$.

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Suppose that the desired curve starts at a fixed point $\varphi = O, z = z_0$ and reaches a fixed point $\varphi = \varphi_1, z = z_1$. When a body passes through a plane of height z, then its speed becomes $\sqrt{2g(z_0 - z)}$, where g denotes the gravity accelleration. Therefore our problem is to find a curve that minimizes

$$\int J(\varphi, z, \varphi', z')du,$$

where

$$J(\varphi, z, \varphi', z') = \frac{\sqrt{r^2\varphi'^2 + (r_1^2 + 1)z'^2}}{\sqrt{z_0 - z}}.$$

Since J does not contain φ explicitly, the Euler equation of our variational problem reduces to

$$\frac{d}{du}\left(\frac{\partial J}{\partial \varphi'}\right) = \frac{\partial J}{\partial \varphi} = 0.$$

Consequently we can deduce

$$\frac{\partial J}{\partial \varphi'} = \frac{1}{\sqrt{z_0 - z}} \cdot \frac{r^2\varphi'}{\sqrt{r^2\varphi'^2 + (r_1^2 + 1)z'^2}} = C_1,$$

where C_1 is a constant. Hence the differential equation follows

$$d\varphi = \frac{C_1\sqrt{z_0 - z}\sqrt{r_1(z)^2 + 1}}{r(z)\sqrt{r(z)^2 - C_1^2(z_0 - z)}}dz.$$

Therefore its general solution can be formally expressed as

$$\varphi = C_1 \int \frac{\sqrt{z_0 - z}\sqrt{r_1(z)^2 + 1}}{r(z)\sqrt{r(z)^2 - C_1^2(z_0 - z)}}dz + C_2, \tag{1.1}$$

where C_1, C_2 are constants which can be determined by z_0, φ_1, z_1.

We seek a surface such that the integral that appears in (1.1) can be reduced to an elliptic integral. We write

$$r(z) = \sqrt{P(z)}.$$

Then we have

$$\int \frac{\sqrt{z_0 - z}\sqrt{r_1(z)^2 + 1}}{r(z)\sqrt{r(z)^2 - C_1^2(z_0 - z)}}dz = \int \frac{\sqrt{z_0 - z}\sqrt{P'(z)^2 + 4P(z)}}{2P(z)\sqrt{P(z) - C_1^2(z_0 - z)}}dz.$$

Accordingly it is necessary that $P(z)$ is a quardratic polynomial and $\sqrt{P'(z)^2 + 4P(z)}$ is a square of a polynomial. Thus we assume that $P(z) = az^2 + bz + c$. Then we have

$$P'(z)^2 + 4P(z) = 4a(a+1)z^2 + 4(a+1)bz + (b^2 + 4c).$$

Since it is a square of a polynomial of degree one, its determinant $(a+1)(b^2 - 4ac)$ must vanish. Hene it follows that $a = -1$ or $b^2 - 4ac = 0$. In the former case we have $r(z) = \sqrt{-z^2 + bz + c}$, or

$$r(z)^2 + \left(z - \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c,$$

which is an equation of a sphere. In the latter case we have $r(z) = \sqrt{a}z + \frac{b}{2\sqrt{a}}$, which is an equation of a cone. In this preliminary study we treat only a cone. Without loss of generality we may assume that a cone is specified by $r(z) = a + bz$.

2 Brachistochrone on a cone

Suppose that a surface of rotation is a cone $r(z) = a + bz$, where b is supposed to be positive. Then the equation (1.1) becomes

$$\frac{d\varphi}{dz} = \frac{C_1\sqrt{b^2+1}\sqrt{z_0 - z}}{(a+bz)\sqrt{(a+bz)^2 - C_1^2(z_0 - z)}}.$$

Hence, changing variable by $x = \sqrt{z_0 - z}$, we have

$$\frac{d\varphi}{dx} = \frac{Cx^2}{(x^2-\lambda^2)\sqrt{(x^2-\lambda^2)^2 - (2\mu x)^2}}, \tag{2.1}$$

where we put

$$C = \frac{2C_1\sqrt{b^2+1}}{b^2}, \ \lambda = \sqrt{\frac{a+bz_0}{b}}, \ \mu = \frac{C_1}{2b}.$$

To solve the differential equation (2.1), we need to evaluate an elliptic integral

$$I = \int \frac{x^2 dx}{(x^2-\lambda^2)\sqrt{f(x)}}, \tag{2.2}$$

where

$$f(x) = (x^2-\lambda^2)^2 - (2\mu x)^2.$$

In the next section we will evaluate the integral I and find the following result

$$I = \frac{1}{4\mu}\left[\theta + \arctan\left(\sqrt{1-k^2}\tan\theta\right) - 2(1-k^2)^{\frac{1}{4}}\cdot \mathrm{sn}^{-1}(\sin\theta)\right],$$

where a variable θ is related to x as

$$x = \lambda\cdot\frac{\sqrt{1-k^2\sin^2\theta} - \nu}{\sqrt{1-k^2\sin^2\theta} + \nu}$$

with parameters ν and k defined by (3.1) and (3.2) respectively. Therefore we obtain the following theorem.

Theorem *The Brachistochrone on a cone is a curve defined by*

$$\varphi = \frac{\sqrt{b^2+1}}{b}\left[\theta + \arctan\left(\sqrt{1-k^2}\tan\theta\right) - 2(1-k^2)^{\frac{1}{4}}\cdot \mathrm{sn}^{-1}(\sin\theta)\right] + C \tag{2.3}$$

and

$$z = z_0 - \left(\frac{a}{b} + z_0\right)\cdot\left[\frac{\sqrt{1-k^2\sin^2\theta} - (1-k^2)^{\frac{1}{4}}}{\sqrt{1-k^2\sin^2\theta} + (1-k^2)^{\frac{1}{4}}}\right]^2 \tag{2.4}$$

where k and C are constants determined by z_0, z_1, φ_1.

3 Evaluation of an elliptic integral

In this section we evaluate an elliptic integral

$$I = \int \frac{x^2 dx}{(x^2-\lambda^2)\sqrt{f(x)}},$$

where

$$f(x) = (x^2 - \lambda^2)^2 - (2\mu x)^2.$$

To evaluate the integral I, we change variable as

$$x = \lambda \cdot \frac{\mathrm{dn}\,(u,k) - \nu}{\mathrm{dn}\,(u,k) + \nu},$$

where we define a positive parameter ν and k by

$$\nu^2 = \frac{\sqrt{\mu^2 + \lambda^2} - \lambda}{\sqrt{\mu^2 + \lambda^2} + \lambda} \tag{3.1}$$

and

$$k^2 = 1 - \nu^4. \tag{3.2}$$

Since (3.1) implies

$$\mu = \frac{2\lambda\nu}{1 - \nu^2}, \tag{3.3}$$

we have

$$\begin{aligned} f(x) &= \lambda^4 \left\{ \left(\frac{\mathrm{dn}\,u - \nu}{\mathrm{dn}\,u + \nu} \right)^2 - 1 \right\}^2 - \left\{ 4\lambda^2 \cdot \frac{\nu}{1 - \nu^2} \cdot \frac{\mathrm{dn}\,u - \nu}{\mathrm{dn}\,u + \nu} \right\}^2 \\ &= \frac{16\lambda^4\nu^2}{(1 - \nu^2)^2(\mathrm{dn}\,u + \nu)^4} \left[(1 - \nu^2)^2 \mathrm{dn}^2\,u - (\mathrm{dn}^2\,u - \nu^2)^2 \right]. \end{aligned}$$

Using (3.2) we see

$$\begin{aligned} (1 - \nu^2)^2 \mathrm{dn}^2\,u - (\mathrm{dn}^2\,u - \nu^2)^2 &= (2 - k^2)\mathrm{dn}^2\,u - \mathrm{dn}^4\,u - (1 - k^2) \\ &= (1 - \mathrm{dn}^2\,u)(\mathrm{dn}^2\,u - (1 - k^2)) \\ &= k^2 \mathrm{sn}^2\,u \cdot k^2 \mathrm{cn}^2\,u. \end{aligned}$$

Accordingly we get

$$\sqrt{f(x)} = \frac{4\lambda^2\nu}{1 - \nu^2} \cdot \frac{k^2 \mathrm{sn}\,u\,\mathrm{cn}\,u}{(\mathrm{dn}\ + \nu)^2}. \tag{3.4}$$

On the other hand, since

$$\frac{d}{du}\,\mathrm{dn}\,u = -k^2 \mathrm{sn}\,u\,\mathrm{cn}\,u,$$

we have

$$\frac{dx}{du} = -2\lambda\nu \cdot \frac{k^2 \mathrm{sn}\,u\,\mathrm{cn}\,u}{(\mathrm{dn}\,u + \nu)^2}. \tag{3.5}$$

Consequently, from (3.4) and (3.5), it follows that

$$\frac{dx}{\sqrt{f(x)}} = -\frac{1 - \nu^2}{2\lambda}\,du.$$

That is, by (3.3), we obtain

$$\frac{dx}{\sqrt{f(x)}} = -\frac{\nu}{\mu}\,du \tag{3.6}$$

Now, since

$$\frac{x^2}{x^2 - \lambda^2} = -\frac{1}{4\nu} \cdot \frac{(\mathrm{dn}\ - \nu)^2}{\mathrm{dn}\,u} = -\frac{1}{4\nu} \left[\mathrm{dn}\,u - 2\nu + \frac{\nu^2}{\mathrm{dn}\,u} \right],$$

we have

$$I = \frac{1}{4\mu}\left[\int \operatorname{dn} u\, du + \nu^2 \int \frac{du}{\operatorname{dn} u} - 2\nu u\right]$$

Then, by change of variable as $\operatorname{sn} u = \sin\theta$, we have $\operatorname{cn} u \operatorname{dn} u\, du = \cos\theta$. Hence

$$\operatorname{dn} u\, du = \frac{\cos\theta d\theta}{\operatorname{cn} u} = \frac{\cos\theta d\theta}{\sqrt{1-\operatorname{sn}^2 u}} = \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = d\theta.$$

Accordingly we see

$$\int \operatorname{dn} u\, du = \theta.$$

Similarly we can see that

$$\int \frac{du}{\operatorname{dn} u} = \int \frac{\operatorname{dn} u\, du}{\operatorname{dn}^2 u} = \int \frac{d\theta}{1-k^2\sin^2\theta} = \frac{1}{\sqrt{1-k^2}} \arctan(\sqrt{1-k^2}\tan\theta).$$

Therefore we obtain (2.2).

4 Asymptotic curve as k tends to zero

As b tends to zero with both a and C_1 remaining unchaged, we have

$$\frac{\lambda}{\mu} \approx \frac{2\sqrt{a}}{\sqrt{C_1}}\sqrt{b}.$$

Then

$$\nu^2 = \frac{\sqrt{1+\left(\frac{\lambda}{\mu}\right)^2} - \frac{\lambda}{\mu}}{\sqrt{1+\left(\frac{\lambda}{\mu}\right)^2} + \frac{\lambda}{\mu}} \approx 1 - \frac{\lambda}{\mu}$$

and

$$k^2 = 1-\nu^4 \approx 2\frac{\lambda}{\mu} \approx \frac{4\sqrt{a}}{\sqrt{C_1}}\sqrt{b}.$$

Hence we reasonably conjecture that the asymptotic Brachistochrone curve as k vanishes is the same as that on a cylinder. In this section we will confirm this conjecture.

Let us start with the Taylor expansion of arctan function:

$$\arctan z \approx \arctan z_0 + (z-z_0)\cdot\frac{1}{1+z_0^2} + \frac{(z-z_0)^2}{2}\cdot\frac{-2z_0}{(1+z_0^2)^2}$$

as $z-z_0$ vanishes. Since

$$\sqrt{1-k^2} \approx 1 - \frac{k^2}{2} - \frac{k^4}{8},$$

we set

$$z_0 = \tan\theta,\ z - z_0 = -\left(\frac{k^2}{2} + \frac{k^4}{8}\right)$$

in the above Taylor expansion. Then we get

$$\arctan(\sqrt{1-k^2}\tan\theta) \approx \theta - k^2\frac{1}{2}\sin\theta\cos\theta - k^4\frac{1}{8}\sin\theta\cos\theta(1+2\sin^2\theta). \tag{4.1}$$

Next we approximate the first kind of elliptic integral as follows

$$\begin{aligned}\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} &= \int_0^{\phi}\left[1+\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2\cdot 4}\sin^4\theta+\cdots\right]d\theta \\ &= \phi+\frac{k^2}{2}\int_0^{\phi}\sin^2\theta\,d\theta+\frac{3k^4}{8}\int_0^{\phi}\sin^4\theta\,d\theta+\cdots \\ &\approx \phi+\frac{k^2}{4}(\phi-\sin\phi\cos\phi)+\frac{9k^4}{64}\left(\phi-\sin\phi\cos\phi-\frac{2}{3}\sin^3\phi\cos\phi\right).\end{aligned}$$

Hence we can derive an asymptotic formula for the inverse of sn

$$\begin{aligned}\mathrm{sn}^{-1}\,z = \int_0^{z}\frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} &= \int_0^{\arcsin z}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &\approx \arcsin z+\frac{k^2}{4}(\arcsin z-z\sqrt{1-z^2}) \\ &\quad +\frac{9k^4}{64}\left(\arcsin z-z\sqrt{1-z^2}-\frac{2}{3}z^3\sqrt{1-z^2}\right).\end{aligned}$$

In this asymptotic formula, setting $z=\sin\theta$, we have

$$\begin{aligned}\mathrm{sn}^{-1}(\sin\theta) &\approx \theta+k^2\,\frac{1}{4}\,(\theta-\sin\theta\cos\theta) \\ &\quad +k^4\,\frac{9}{64}\left(\theta-\sin\theta\cos\theta-\frac{2}{3}\sin^3\theta\cos\theta\right)\end{aligned}$$

Accordingly we get

$$\begin{aligned}(1-k^2)^{\frac{1}{4}}\,\mathrm{sn}^{-1}(\sin\theta) &\approx \theta-k^2\,\frac{1}{4}\,(\theta-\sin\theta\cos\theta) \\ &\quad -k^4\,\frac{1}{64}\,(\theta+5\sin\theta\cos\theta 6\sin^3\theta\cos\theta)\end{aligned}\tag{4.2}$$

By (4.1) and (4.2) we can deduce

$$\begin{aligned}&\theta+\arctan(\sqrt{1-k^2}\tan\theta)-2(1-k^2)^{\frac{1}{4}}\,\mathrm{sn}^{-1}(\sin\theta) \\ &\approx \frac{k^4}{32}(\theta+\sin\theta\cos\theta(1-2\sin^2\theta)) \\ &= \frac{k^4}{128}(4\theta+\sin 4\theta)\end{aligned}$$

Therefore we obtain

$$\varphi\approx\frac{a}{8C_1^2}\,(4\theta+\sin 4\theta)\tag{4.3}$$

On the other hand, since

$$\sqrt{1-k^2\sin^2\theta}-\nu\approx\frac{k^2}{4}(1-2\sin^2\theta)=\frac{k^2}{4}\cos 2\theta$$

and

$$\sqrt{1-k^2\sin^2\theta}+\nu\approx 2,$$

we have

$$x\approx\frac{\lambda k^2}{8}\cos 2\theta\approx\frac{a}{2C_1}\cos 2\theta.$$

Therefore we obtain

$$z_0 - z \approx \frac{a^2}{8C_1^2}(1 + \cos 4\theta) \tag{4.4}$$

By (4.3) and (4.4) we obtain the following result.

Corollary *The Brachistochrone curve on a cone approaches to a cycloid on a cylinder.*

References

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