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ASYMPTOTIC PROPERTIES OF LINEAR COMBINATIONS OF U-STATISTICS WITH DEGENERATE KERNELS

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Abstract

As an estimator of a real estimable parameter, we consider a linear combination of U-statistics which include V-statistic and limit of Bayes estimate (Toda and Yamato [9]). In the case that the kernel of the estimable parameter is degenerate, we show functional limit theorems (invariance principles) for the linear combination of U-statistics. As their applications we give the asymptotic distribution of a linear combination of U-statistics.

Key Words: Degenerate kernel; Estimable parameter; functional limit theorem; invariance principle; linear combination of U-statistics; V-statistic

1 Introduction

Let $\theta = \theta(F)$ be a real estimable parameter or a real regular functional of a distribution F and $g(x_1, ..., x_k)$ be its kernel of degree k. Let $X_1, ..., X_n$ be a random sample of size n from the distribution F. U-statistic U_n is well-known as estimators of $\theta(F)$, which is given by

$$U_n = \binom{n}{k}^{-1} \sum_{1 \le j_1 < \cdots < j_k \le n} g(X_{j_1}, \ldots, X_{j_k}), \tag{1.1}$$

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where $\sum_{1\le j_1<\cdots<j_k\le n}$ denotes the summation over all integers $j_1,\ldots,j_k$ satisfying $1\le j_1<\cdots<j_k\le n$. We put

$$\psi_j(x_1,\ldots,x_j)=E[g(X_1,\ldots,X_k)\mid X_1=x_1,\ldots,X_j=x_j],\quad j=1,...,k$$
$$\sigma_j^2=Var[\psi_j(X_1,\ldots,X_j)],\quad j=1,...,k.$$

In this paper we assume that

$$\sigma_1^2=\cdots=\sigma_{d-1}^2=0\quad\text{and}\quad\sigma_d^2>0\quad(d\le k).$$

That is, the U-statistic and/or the kernel g is degenerate of order $d-1$. So $E\psi_d(X_1,\ldots,X_d)=\theta$ and almost surely (a.s.) $\psi_1(X_1)=\theta,\ldots,\psi_{d-1}(X_1,\ldots,X_{d-1})=\theta$. We put

$$g^{(1)}(x_1)=\psi_1(x_1)-\theta,$$
$$g^{(c)}(x_1,\ldots,x_c)=\psi_c(x_1,\ldots,x_c)-\sum_{j=1}^{c-1}\sum_{(c,j)}g^{(j-1)}(x_{i_1},\ldots,x_{i_j})-\theta$$

for $c=2,3,...,k$, where the sum $\sum_{(c,j)}$ is taken over all integers such that $1\le i_1<\cdots<i_j\le c$. By the degeneracy, $g^{(1)}=\cdots=g^{(d-1)}=0$, $g^{(d)}=\psi_d-\theta$. (See, for example, Lee [6], and Koroljuk and Borovskich [4].)

Let $D[0,1]$ be the space of all real functions on $[0,1]$ which are right continuous and have left-hand limits. We consider a random process given by $\{U_{[nt]}:0\le t\le 1\}$ which belongs in $D[0,1]$, where $[x]$ is the greatest integer not greater than x. The functional limit theorem based on $\{U_{[nt]}:0\le t\le 1\}$ is given by Theorem 5.5.4 of Koroljuk and Borovskikh [4]. We take the sub-space $D[0,1]$ instead of $D[0,\infty)$ of Theorem 5.5.4 of Koroljuk and Borovskikh [4]. Under the condition $Eg(X_1,\ldots,X_k)^2<\infty$, $n^{d/2}t^d(U_{[nt]}-\theta)$ converges in distribution to $\binom{k}{d}J_d^t(g^{(d)})$ as $n\to\infty$ in the space $D[0,1]$. That is,

$$n^{d/2}t^d(U_{[nt]}-\theta)\xrightarrow{\mathcal{D}}\binom{k}{d}J_d^t(g^{(d)}),\tag{1.2}$$

where

$$J_d^t(f)=\sum_{i_1=1}^{\infty}\cdots\sum_{i_d=1}^{\infty}(f,e_{i_1}\cdots e_{i_d})\prod_{l=1}^{\infty}H_{r_l(\mathbf{i})}(\mathbf{w}_j(t),t),$$

$e_1,e_2,...$ is an orthonormal basis of $L_2(F)$, $\mathbf{w}_1(t)$,$\mathbf{w}_2(t)$,... are independent standard Brownian motion on $[0,1]$ such that $E\mathbf{w}_j(t)=0$, $\mathbf{w}_j^2(t)=t$ $(j=1,2,\ldots)$, and $r_l(\mathbf{i})=\sum_{j=1}^d I(i_j=l)$ is the number of indices among $\mathbf{i}=(i_1,...,i_d)$ equal to l. H is the Hermite polynomial of two variables whose generating function is given by

$$exp[z\tau_{n1}-z^2\tau_{n2}/2]=\sum_{m=0}^{\infty}\frac{z^m}{m!}H_m(\tau_{n1},\tau_{n2})$$

(Borovskikh and Korolyuk [2]). $J_d^t(f)$ can be also expressed by the multiple stochastic integral (see Borovskikh [1], pp.178–179). The other functional limit theorem for U_n is as follows (Borovskich [1], pp.176). For the U-statistic U_n, we consider a random process $\{U_n(t) : 0 \le t \le 1\}$ given by

$$U_n(t) = \sum_{c=d}^{k} \binom{k}{c} \binom{n}{c}^{-1} S_{[nt],c}, \quad 0 \le t \le 1, \tag{1.3}$$

where $S_{[nt],c} = \sum_{1 \le i_1 < \cdots < i_c \le [nt]} g^{(c)}(X_{i_1}, \ldots, X_{i_c})$. The random process $\{U_n(t) : 0 \le t \le 1\}$ belongs in $D[0,1]$. Under the condition $E \mid g^{(c)} \mid^{2c/(2c-d)} < \infty$ for $c = d, \ldots, k$,

$$n^{d/2} U_n(t) \xrightarrow{\mathcal{D}} \binom{k}{d} J_d^t(g^{(d)}). \tag{1.4}$$

As an estimator of $\theta(F)$, Toda and Yamato [9] introduce a linear combination Y_n of U-statistics as follows: Let $w(r_1, \ldots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$, where k is the fixed degree of the kernel g. We assume that at least one of $w(r_1, \ldots, r_j; k)$'s is positive. We put

$$d(k,j) = \sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k)$$

for $j = 1, 2, ..., k$, where the summation $\sum^{+}_{r_1+\cdots+r_j=k}$ is taken over all positive integers $r_1, ..., r_j$ satisfying $r_1 + \cdots + r_j = k$ with j and k fixed. For $j = 1, \ldots, k$, let $g_{(j)}(x_1, ..., x_j)$ be the kernel given by

$$g_{(j)}(x_1, \ldots, x_j) = \frac{1}{d(k,j)} \sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k) g(x_1^{r_1}, \ldots, x_j^{r_j}), \tag{1.5}$$

where

$$g(x_1^{r_1}, \ldots, x_j^{r_j}) = g(\underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_j, \ldots, x_j}_{r_j}).$$

Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1, \ldots, x_j)$ for $j = 1, \ldots, k$. The kernel $g_{(j)}(x_1, \ldots, x_j)$ is symmetric because of the symmetry of $w(r_1, \ldots, r_j; k)$. If $d(k,j)$ is equal to zero for some j, then the associated $w(r_1, \ldots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero.
Then the linear combination Y_n of U-statistics is given by

$$Y_n = \frac{1}{D(n,k)} \sum_{j=1}^{k} d(k,j) \binom{n}{j} U_n^{(j)}, \tag{1.6}$$

where $D(n,k)=\sum_{j=1}^{k} d(k,j)\binom{n}{j}$. Since the w's are nonnegative and at least one of them is positive, $D(n,k)$ is positive.

If $w(1,1,\ldots,1;k)=1$ and $w(r_1,\ldots,r_j;k)=0$ for positive integers $r_1,\ldots,r_j$ such that $j=1,\ldots,k-1$ and $r_1+\cdots+r_j=k$, then $d(k,k)=1$, $d(k,j)=0$ $(j=1,\ldots,k-1)$ and $D(n,k)=\binom{n}{k}$. The corresponding statistic Y_n is equal to U-statistic U_n given by (1.1).

If w is the function given by $w(r_1,\ldots,r_j;k)=k!/(r_1!\cdots r_j!)$ for positive integers $r_1,\ldots,r_j$ such that $j=1,\ldots,k$ and $r_1+\cdots+r_j=k$, then $d(k,j)=j!\mathcal{S}(k,j)$ $(j=1,\ldots,k)$ and $D(n,k)=n^k$, where $\mathcal{S}(k,j)$ are the Stirling number of the second kind. The corresponding statistic Y_n is equal to V-statistic V_n given by

$$V_n=\frac{1}{n^k}\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n} g(X_{j_1},\ldots,X_{j_k}).$$

If w is the function given by $w(r_1,\ldots,r_j;k)=1$ for positive integers $r_1,\ldots,r_j$ such that $j=1,\ldots,k$ and $r_1+\cdots+r_j=k$, then $d(k,j)=\binom{k-1}{j-1}$ $(j=1,\ldots,k)$ and $D(n,k)=\binom{n+k-1}{k}$. The corresponding statistic Y_n is equal to the limit of Bayes estimate (LB-statistic) B_n which is given by

$$B_n=\binom{n+k-1}{k}^{-1}\sum_{r_1+\cdots+r_n=k} g(X_1^{r_1},\ldots,X_n^{r_n}),$$

where $\sum_{r_1+\cdots+r_n=k}$ denotes the summation over all non-negative integers $r_1,\ldots,r_n$ satisfying $r_1+\cdots+r_n=k$ (see Toda and Yamato [9] and for the limit of Bayes estimate see Yamato [10]).

In the following, for $D(n,k)$ we suppose that

$$\frac{n^k}{D(n,k)} \text{ is nondecreasing and converges to } \frac{k!}{d(k,k)} \text{ as } n\to\infty. \tag{1.7}$$

For the V-statistic V_n, relation (1.7) is satisfied because of $d(k,k)=k!$ and $n^k/D(n,k)=1$. For the LB-statistic B_n, relation (1.7) is satisfied because of $d(k,k)=1$ and $n^k/D(n,k)=n^k/\binom{n+k-1}{k}$. On the other hand, the U-statistic U_n does not satisfy relation (1.7) because of $n^k/D(n,k)=n^k/\binom{n}{k}$ and so the U-statistic is not included in the following discussion.

For the U-statistic and the V-statistic with the non-degenerate kernel, their functional limit theorems are discussed by Miller and Sen [7], Sen [8], Denker [3] and Koroljuk and Borovskich [4]) and others. Kondo and Yamato [5] discusses the functional limit theorems for a linear combination Y_n of U-statistics with a non-degenerate kernel.

For the U-statistic with a degenerate kernel, the functional limit theorems are given by Koroljuk and Borovskich [4] and Borovskikh [1], as stated in the previous paragraph. In Section 2, we show functional limit theorems for a linear combination Y_n of U-statistics with a degenerate kernel.

As an application of these functional limit theorems, in Section 3 we give the asymptotic distribution of Y_n. In Section 4, we give the proofs of lemmas and theorems in the previous sections.

2 Functional Limit Theorems for Y-statistic

For the kernel $g_{(j)}(x_1,\ldots,x_j)$ given by (1.3), we put for $c=1,\ldots,j$ and $j=1,...,k$

$$\psi_{(j),c}(x_1,...,x_c)=E[g_{(j)}(X_1,...,X_j)\mid X_1=x_1,...,X_c=x_c]$$

$$=\frac{1}{d(k,j)}\sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1,\ldots,r_j;k)Eg(x_1^{r_1},...,x_c^{r_c},X_{c+1}^{r_{c+1}},...,X_j^{r_j}).$$

The U-statistics $U_n^{(j)}$, $j=1,\ldots,k$ corresponding to to the kernel $g_{(j)}$ have the following properties.

Lemma 2.1

$$E[U_n^{(j)}]=\theta,\quad k-\frac{d-1}{2}\le j\le k$$

or

$$E[U_n^{(k-j)}]=\theta,\quad 0\le j\le\frac{d-1}{2}.$$

Lemma 2.2 *The order of degeneracy of $U_n^{(k-j)}$ is at least $d-2j-1$ for $1\le j\le (d-1)/2$ and*

$$\psi_{(k-j),d-2j}(x_1,...,x_{d-2j})=\theta+ \tag{2.1}$$

$$\frac{1}{d(k,k-j)}\binom{k-d+j}{j}w(1^{k-2j},2^j;k)[\varphi_{d,d-2j}(x_1,...,x_{d-2j})-\theta],$$

where for $1\le j\le (d-1)/2$

$$\varphi_{d,d-2j}(x_1,...,x_{d-2j})=E[\psi_d(x_1,...,x_{d-2j},X_{d-2j+1},X_{d-2j+1},...,X_{d-j},X_{d-j})] \tag{2.2}$$

and

$$w(1^r, 2^s; k) = w(\underbrace{1, 1, \cdots, 1}_{r}, \underbrace{2, 2, \cdots, 2}_{s}; k).$$

We note that for $1 \le j \le (d-1)/2$,

$$E\varphi_{d,d-2j}(X_1, ..., X_{d-2j}) = \theta. \tag{2.3}$$

From Lemmas 2.1 and 2.2, we have the following results: If $d = 2l+1$ and l is a positive integer, then $EU_n^{(k)} = EU_n^{(k-1)} = \cdots = EU_n^{(k-l+1)} = EU_n^{(k-l)} = \theta$. The orders of degeneracy of $U_n^{(k-1)}$,..., $U_n^{(k-l+1)}$, $U_n^{(k-l)}$ are at least $2(l-1), ..., 2, 0$, respectively. If $d = 2l$ and l is a positive integer, then $EU_n^{(k)} = EU_n^{(k-1)} = \cdots = EU_n^{(k-l+2)} = EU_n^{(k-l+1)} = \theta$. The orders of degeneracy of $U_n^{(k-1)}$,...,$U_n^{(k-l+2)}$,$U_n^{(k-l+1)}$ are at least $2l-3, ..., 3, 1$, respectively.

Lemma 2.3 *In case of $d = 2l$,*

$$EU_n^{(k-l)} - \theta = \frac{1}{d(k, k-l)} \binom{k-l}{k-d} w(1^{k-d}, 2^l; k)[E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta]. \tag{2.4}$$

For the statistic Y_n, we consider a random process given by $\{Y_{[nt]} : 0 \le t \le 1\}$ which belongs in $D[0,1]$. We shall show the functional limit theorem for Y_n based on $\{Y_{[nt]} : 0 \le t \le 1\}$. For this purpose, we must consider at the same time the functional limit theorems of the U-statistics $U_n^{(k)}$, $U_n^{(k-1)}$,...,$U_n^{(k-l)}$ ($d = 2l+1$) and $U_n^{(k)}$, $U_n^{(k-1)}$,...,$U_n^{(k-l+1)}$ ($d = 2l$). So we suppose that $E[g(X_{j_1}, X_{j_2}, ..., X_{j_k})^2] < \infty$ for all $j_1, j_2, ..., j_k$ such that $1 \le j_1 \le j_2 \le \cdots \le j_k \le k$.

Theorem 2.4 *We suppose that*

$$E[g(X_{j_1}, X_{j_2}, ..., X_{j_k})^2] < \infty$$

for all $j_1, j_2, ..., j_k$ such that $1 \le j_1 \le j_2 \le \cdots \le j_k \le k$. We assume (1.7).
Then in case of $d = 2l+1$ ($l = 1, 2, \ldots$), in the space $D[0,1]$ we have

$$n^{d/2} t^d (Y_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!} \sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!} \cdot \frac{w(1^{k-2j}, 2^j; k)}{w(1^k; k)} J_{d-2j}^t(\xi_{d,d-2j}), \tag{2.5}$$

where

$$\xi_{d,d-2j}(x_1, ..., x_{d-2j}) = \varphi_{d,d-2j}(x_1, ..., x_{d-2j}) - \theta \quad (0 \le j \le \frac{d-1}{2}).$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2}t^d(Y_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!} \cdot \frac{w(1^{k-2j}, 2^j; k)}{w(1^k; k)} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{1}{l!}\frac{w(1^{k-d}, 2^l; k)}{w(1^k; k)}[E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta]\}. \qquad (2.6)$$

We note that $w(1^{k-2j}, 2^j; k)/w(1^k; k) = 1/2^j$ for the V-statistic and $w(1^{k-2j}, 2^j; k)$ $/w(1^k; k) = 1$ for the LB-statistic, respectively.

Corollary 2.5 *In case of* $d = 2l + 1$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2}t^d(V_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!2^j} J^t_{d-2j}(\xi_{d,d-2j}).$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2}t^d(V_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!2^j} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{1}{l!2^l}[E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta]\}.$$

Corollary 2.6 *In case of* $d = 2l + 1$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2}t^d(B_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!} J^t_{d-2j}(\xi_{d,d-2j}).$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2}t^d(V_{[nt]} - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{1}{l!}[E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta]\}.$$

We consider another functional limit theorem for Y_n. Let $K_n^{(c)}$ be the Y-statistic given by (1.6), based on the kernel $g^{(c)}$. We consider a random process $\{Y_n(t) : 0 \leq t \leq 1\}$ which belongs in $D[0,1]$:

$$Y_n(t) = \sum_{c=d}^{k} \binom{k}{c} \frac{D([nt], c)}{D(n, c)} K^{(c)}_{[nt]}.$$

Theorem 2.7 *We suppose that*

$$E[| g(X_{j_1}, X_{j_2}, ..., X_{j_k}) |^2] < \infty$$

for all $j_1, j_2, ..., j_k$ *such that* $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq k$. *We assume* (1.7). *Then in case that* $d = 2l + 1$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2} Y_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!} \sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!} \cdot \frac{w(1^{d-2j}, 2^j; d)}{w(1^d; d)} J^t_{d-2j}(\xi_{d,d-2j}). \tag{2.7}$$

In case that $d = 2l$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2} Y_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!} \{ \sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!} \cdot \frac{w(1^{d-2j}, 2^j; d)}{w(1^d; d)} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+ \frac{1}{l!} \frac{w(2^l; d)}{w(1^d; d)} [E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta] \}. \tag{2.8}$$

We put

$$V_n(t) = \sum_{c=d}^{k} \binom{k}{c} \frac{[nt]^c}{n^c} K^{(c)}_{V,[nt]}, \quad B_n(t) = \sum_{c=d}^{k} \binom{k}{c} \frac{\binom{[nt]+c-1}{c}}{\binom{n+c-1}{c}} K^{(c)}_{B,[nt]},$$

where for $c = d, \cdots, k$, $K^{(c)}_{V,n}$ and $K^{(c)}_{B,n}$ are the V-statistic and the LB-statistic based on the kernel $g^{(c)}$, respectively. We note that $w(1^{d-2j}, 2^j; d)/w(1^d; d) = 1/2^j$ for the V-statistic and $w(1^{d-2j}, 2^j; d)\ /w(1^d; d) = 1$ for the LB-statistic, respectively. By Theorem 2.7 we get the asymptotic distributions of $V_n(t)$ and $B_(t)$, respectively.

Corollary 2.8 *In case of* $d = 2l + 1$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2} V_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!} \sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!2^j} J^t_{d-2j}(\xi_{d,d-2j}).$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$, *in the space* $D[0,1]$ *we have*

$$n^{d/2} V_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!} \{ \sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!2^j} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+ \frac{1}{l!2^l} [E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta] \}.$$

Corollary 2.9 *In case of $d = 2l + 1$ $(l = 1, 2, \ldots)$, in the space $D[0, 1]$ we have*

$$n^{d/2}B_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\sum_{j=0}^{l} t^j \frac{1}{(d-2j)!j!} J^t_{d-2j}(\xi_{d,d-2j}).$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$, in the space $D[0, 1]$ we have

$$n^{d/2}B_n(t) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1} t^j \frac{1}{(d-2j)!j!} J^t_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{1}{l!}[E\psi_d(X_1, X_1, ..., X_l, X_l) - \theta]\}.$$

Thus, we know that for the V-statistic and the LB-statistic the asymptotic distributions of $n^{d/2}t^d(V_{[nt]} - \theta)$ and $n^{d/2}t^d(V_{[nt]} - \theta)$ are equal to the ones of $n^{d/2}V_n(t)$ and $n^{d/2}B_n(t)$, respectively.

3 Asymptotic distribution of Y-statistic

The Hermite polynomial $H_m(\tau_1, \tau_2)$ of two variables with $\tau_2 = 1$ is equal to the Hermite polynomial $H_m(\tau_1)$ of one variable. We use the notation $J_d(f)$ given by

$$J_d(f) = \sum_{i_1=1}^{\infty}\cdots\sum_{i_d=1}^{\infty}(f, e_{i_1}\cdots e_{i_d})\prod_{l=1}^{\infty} H_{r_l(\mathbf{i})}(Z_j),$$

where Z_1,Z_2,... are independent standard normal random variables. The asymptotic distribution of $n^{d/2}(Y_n - \theta)$ can be obtained from Theorem 2.4 by putting $t = 1$, as follows.

Theorem 3.1 *We suppose that*

$$E[g(X_{j_1}, X_{j_2}, ..., X_{j_k})^2] < \infty \tag{3.1}$$

for all $j_1, j_2, ..., j_k$ such that $1 \le j_1 \le j_2 \le \cdots \le j_k \le k$. We assume (1.7)*. Then in case of $d = 2l + 1$ $(l = 1, 2, \ldots)$, we have*

$$n^{d/2}(Y_n - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\sum_{j=0}^{l}\frac{1}{(d-2j)!j!}\cdot\frac{w(1^{k-2j}, 2^j; k)}{w(1^k; k)} J_{d-2j}(\xi_{d,d-2j}). \tag{3.2}$$

In case of $d = 2l$ $(l = 1, 2, \ldots)$ we have

$$n^{d/2}(Y_n - \theta) \xrightarrow{\mathcal{D}} \frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1}\frac{1}{(d-2j)!j!}\cdot\frac{w(1^{k-2j}, 2^j; k)}{w(1^k; k)} J_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{w(1^{k-d},2^l;k)}{l!w(1^k;k)}[E\psi_d(X_1,X_1,...,X_l,X_l)-\theta]\}. \quad (3.3)$$

We have that $w(1^{k-2j},2^j;k)/w(1^k;k)=1/2^j$ for the V-statistic V_n and $=1$ for the LB-statistic B_n, respectively. Therefore, we get the asymptotic distributions of the V-statistic V_n and the LB-statistic B_n, respectively (Yamato and Toda [11]):
In case of $d=2l+1$ $(l=1,2,\dots)$ we have

$$n^{d/2}(V_n-\theta)\xrightarrow{\mathcal{D}}\frac{k!}{(k-d)!}\sum_{j=0}^{l}\frac{1}{(d-2j)!j!2^j}J_{d-2j}(\xi_{d,d-2j}). \quad (3.4)$$

In case of $d=2l$ $(l=1,2,\dots)$ we have

$$n^{d/2}(V_n-\theta)\xrightarrow{\mathcal{D}}\frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1}\frac{1}{(d-2j)!j!2^j}J_{d-2j}(\xi_{d,d-2j}) \quad (3.5)$$

$$+\frac{1}{l!2^l}[E\psi_d(X_1,X_1,...,X_l,X_l)-\theta]\}.$$

In case of $d=2l+1$ $(l=1,2,\dots)$ we have

$$n^{d/2}(B_n-\theta)\xrightarrow{\mathcal{D}}\frac{k!}{(k-d)!}\sum_{j=0}^{l}\frac{1}{(d-2j)!j!}J_{d-2j}(\xi_{d,d-2j}).$$

In case of $d=2l$ $(l=1,2,\dots)$ we have

$$n^{d/2}(B_n-\theta)\xrightarrow{\mathcal{D}}\frac{k!}{(k-d)!}\{\sum_{j=0}^{l-1}\frac{1}{(d-2j)!j!}J_{d-2j}(\xi_{d,d-2j})$$

$$+\frac{1}{l!}[E\psi_d(X_1,X_1,...,X_l,X_l)-\theta]\}.$$

The asymptotic distribution of Y_n can also be derived directly by same methods as the asymptotic distribution of B_n by Yamato and Toda [11].

If we put $t=1$ in $V_n(t)$, we have

$$V_n(1)=\sum_{c=d}^{k}\binom{k}{c}K_{V,n}^{(c)},$$

which is the expression of V-statistic corresponding to the H-decomposition of U-statistic, where $K_n^{(c)}$ is the V-statistic based on the kernel $g^{(c)}$ which contains θ for

$c = d, \ldots, k$. Thus Corollary 2.8 also yields the asymptotic distribution of V_n given by (3.4) and (3.5). The asymptotic distribution of V-statistic is also given by

$$\binom{k}{d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{\psi_d - \theta\} Q(dx_1) \cdots Q(dx_d),$$

where Q is a centered Gaussian random measure with covariance function $EQ(A)\, Q(B) = F(A \cap B) - F(A)F(B)$ for any Borel sets A, B (see, for example, Borovskikh [1], p.113). From (3.4) and (3.5), we know that the asymptotic distribution of V_n is not affected by the U-statistics $U_n^{(\cdot)}$ with lower degrees in the expression of (1.6) for V-statistic (Yamato and Toda [11]).

4 Appendix

We can prove Lemmas 2.1, 2.2 and 2.3 by methods similar to those in the proofs of Lemmas 2.1, 2.2 and 2.4 of Yamato and Toda (2001). We state here the idea of the proofs of these Lemmas. We consider the solution of $r_1, \ldots, r_j$ satisfying $r_1 + \cdots + r_j = k$ with k and $j (= 1, \ldots, k)$ fixed. If j is large, then the solution contains necessarily $r.$ equal to 1. If $j \geq k - (d-1)/2$, then the solution contains $r.$ equal to 1 whose number is at least $2j - k (\geq k - d + 1)$. For example, in case of $r_j = \cdots = r_{j-k+d} = 1$, we have $Eg(X_1^{r_1}, \cdots, X_j^{r_j}) = E\psi_{d-1}(X_1^{r_1}, \cdots, X_{j-1}^{r_{j-1}}) = \theta$, because of $\psi_{d-1} = \theta$.

Proof of Theorem 2.4: We shall show the case of $d = 2l$ given by (2.6). By a similar method we can show the case of $d = 2l + 1$ given by (2.5). We put

$$Y_{[nt]} - \theta = \sum_{r=0}^{l-1} Y^*_{[nt],r} + Y^*_{[nt],l} + \sum_{r=1}^{k-l-1} Y^{**}_{[nt],r}, \tag{4.1}$$

where

$$Y^*_{[nt],r} = d(k, k-r) \frac{\binom{[nt]}{k-r}}{D([nt], k)} (U_{[nt]}^{(k-r)} - \theta), \quad r = 0, 1, \ldots, l,$$

$$Y^{**}_{[nt],r} = d(k, r) \frac{\binom{[nt]}{r}}{D([nt], k)} (U_{[nt]}^{(r)} - \theta), \quad r = 1, \ldots, k - l - 1.$$

For $Y^*_{[nt],r}$, $r = 0, 1, \ldots, l-1$, we have

$$I_{1n} = \sup_{0 \leq t \leq 1} \mid n^{d/2} t^d Y^*_{[nt],r} - \frac{k! d(k, k-r)}{(k-r)! d(k,k)} t^r n^{\frac{d-2r}{2}} t^{d-2r} (U_{[nt]}^{(k-r)} - \theta) \mid$$

$$\leq \max_{k\leq j\leq n} \frac{d(k,k-r)}{(k-r)!} \mid \frac{j^{(k-r)}j^r}{D(j,k)} - \frac{k!}{d(k,k)} \mid \cdot (\frac{j}{n})^r n^{\frac{d-2r}{2}} (\frac{j}{n})^{d-2r} \cdot \mid U_j^{(k-r)} - \theta \mid .$$

From (1.7), for any $\varepsilon > 0$ there exists a positive integer N such that

$$\mid \frac{j^{(k-r)}j^r}{D(j,k)} - \frac{k!}{d(k,k)} \mid < \varepsilon \text{ for } j \geq N.$$

Thus we have

$$I_{1n} \leq \max\{\frac{1}{n^{d/2}} \max_{k\leq j\leq N} \frac{d(k,k-r)}{(k-r)!} \cdot \mid \frac{j^{(k-r)}j^r}{D(j,k)} - \frac{k!}{d(k,k)} \mid \cdot j^{d-r} \cdot \mid U_j^{(k-r)} - \theta \mid, \quad (4.2)$$

$$\varepsilon \frac{d(k,k-r)}{(k-r)!} \sup_{0\leq t\leq 1} n^{\frac{d-2r}{2}} t^{d-2r} \mid U_{[nt]}^{(k-r)} - \theta \mid \}.$$

By Lemma 2.2, for $r = 1, \cdots, l-1$, the order of degeneracy of $U_n^{(k-r)}$ is at least $d-2r-1$. Among the last term, $\sup_{0\leq t\leq 1} n^{d/2-r} t^{d-2r} \mid U_{[nt]}^{(k-r)} - \theta \mid$ converges to $\binom{k-r}{d-2r} \sup_{0\leq t\leq 1} \mid J_{d-2r}^t(\psi_{(k-r),d-2r} - \theta) \mid$ in distribution as $n \to \infty$ by (1.2) and Lemma 2.1 if the order of degeneracy of $U_n^{(k-r)}$ is $d-2r-1$ exactly. Hence by letting n tend to ∞ and then ε tend to zero, we get that the right-hand side of (4.2) converges to zero in probability and therefore I_{1n} converges to zero in probability.

Since $U_j^{(k-l)}$ converges to $\theta_{k-l} (= EU_j^{(k-l)})$ almost surely (a.s.) as $j \to \infty$, for any $\varepsilon > 0$ there exists a positive integer N such that $\mid U_j^{(k-l)}(\omega) - \theta_{k-l} \mid < \varepsilon$, $j > N$ where ω belongs to the event E such that $P(E) = 1$. Then, using (1.7), we have

$$\sup_{0\leq t\leq 1} \mid n^{d/2} t^d \frac{\binom{[nt]}{k-l}}{D([nt],k)} (U_{[nt]}^{(k-l)}(\omega) - \theta_{k-l}) \mid$$

$$\leq \max\{\frac{1}{n^l (k-l)!} \max_{k\leq j\leq N} \frac{j^{(k-l)}}{D(j,k)} \mid U_{[nt]}^{(k-l)}(\omega) - \theta_{k-l} \mid, \varepsilon \frac{k!}{(k-l)! d(k,k)}\},$$

for any $\omega \in E$. By letting n tend to ∞ and then ε tend to zero, we get

$$\sup_{0\leq t\leq 1} \mid n^{d/2} t^d \frac{\binom{[nt]}{k-l}}{D([nt],k)} (U_{[nt]}^{(k-l)} - \theta_{k-l}) \mid \to 0 \ \ a.s. \quad (4.3)$$

Since $j^{(k-l)} j^l / D(j,k)$ increases as j increases by (1.7), we have

$$\sup_{0\leq t\leq 1} n^{d/2} t^d \frac{\binom{[nt]}{k-l}}{D([nt],k)} = \frac{n^{(k-l)} n^l}{(k-l)! D(n,k)} \to \frac{k!}{(k-l)! d(k,k)}, \quad n \to \infty. \quad (4.4)$$

Therefore by (4.3) and (4.4), we get

$$\sup_{0\le t\le 1} n^{d/2}t^d Y^*_{[nt],l} \to \frac{k!d(k,k-l)}{(k-l)!d(k,k)}(\theta_{k-l}-\theta) \ \ a.s. \tag{4.5}$$

For $Y^{**}_{[nt],r}$, $r=1,\ldots,k-l-1$, we have

$$\sup_{0\le t\le 1} n^{d/2}t^d \mid Y^{**}_{[nt],r} \mid \le \frac{d(k,r)}{r!} \max_{k\le j\le n} \frac{j^{(r)}j^{k-r}}{D(j,k)} \mid U_j^{(r)}-\theta \mid \frac{1}{n^{k-r-l}}, \ \ k-r-l\ge 1.$$

Since $j^{(r)}j^{k-r}/D(j,k)$ converges to $k!/d(k,k)$ and $U_j^{(r)}$ converges $\theta_r(=EU_j^{(r)})$ a.s. as $j\to\infty$, $[j^{(r)}j^{k-r}/D(j,k)] \mid U_j^{(r)}-\theta \mid$ is bounded almost surely. Thus

$$\sup_{0\le t\le 1} n^{d/2}t^d \mid Y^{**}_{[nt],r} \mid \to 0 \ \ a.s.$$

By these convergence of $Y^*_{[nt],r}, r=0,1,\ldots,l$ and $Y^{**}_{[nt],r}$ $(r=1,\ldots,k-l-1)$, we know that

$$\sup_{0\le t\le 1} n^{d/2}t^d(Y_{[nt]}-\theta)$$

has the same asymptotic distribution as that of

$$\sum_{r=0}^{l-1} \sup_{0\le t\le 1} \frac{k!d(k,k-r)}{(k-r)!d(k,k)} t^r n^{\frac{d-2r}{2}} t^{d-2r}(U^{(k-r)}_{[nt]}-\theta) + \frac{k!d(k,k-l)}{(k-l)!d(k,k)}(\theta_{k-l}-\theta), \tag{4.6}$$

where $\theta_{k-l}-\theta$ is given by the right-hand side of (2.4). By applying (1.2) to each terms of the first summation and $d(k,k)=w(1^k;k)$, we get the asymptotic distribution (2.6).

Even if the order of degeneracy of $U_n^{(k-r)}$ is larger than $d-2r-1$ for some $r(=1,\ldots,l-1)$, relation (2.6) is still valid because $\sup_{0\le t\le 1} n^{d/2}t^d Y^*_{[nt],r}$ converges to zero in probability and $J^t_{d-2r}(\xi_{d,d-2r})=0$ by $\xi_{d,d-2r}=0$: For example, we suppose that the order of degeneracy of $U_n^{(k-r)}$ is $d-2r$. Then we have

$$I_{1n} \le \max_{k\le j\le n} \frac{d(k,k-r)}{(k-r)!} \mid \frac{j^{(k-r)}j^r}{D(j,k)} - \frac{k!}{d(k,k)} \mid \cdot \frac{1}{\sqrt{n}}(\frac{j}{n})^{r-1} n^{\frac{d-2r+1}{2}}(\frac{j}{n})^{d-2r+1} \cdot \mid U_j^{(k-r)}-\theta \mid .$$

Therefore the second term of the right-hand side of (4.2) is replaced by

$$\varepsilon\frac{d(k,k-r)}{(k-r)!}\cdot\frac{1}{\sqrt{n}}\cdot \sup_{0\le t\le 1} n^{\frac{d-2r+1}{2}} t^{d-2r+1} \mid U^{(k-r)}_{[nt]}-\theta \mid,$$

which converges to zero in probability as $n\to\infty$ since the order of degeneracy of $U_n^{(k-r)}$ is $d-2r$ and $\sup_{0\le t\le 1} n^{d/2-r+1/2}t^{d-2r+1}(U^{(k-r)}_{[nt]}-\theta)$ converges in distribution to $\binom{k-r}{d-2r+1}\sup_{0\le t\le 1}|J^t_{d-2r+1}(\psi_{(k-r),d-2r+1}-\theta)|$ by (1.2). Therefore I_{1n} converges to zero in probability as $n\to\infty$. On the other hand, by the same reasoning

$$|\sup_{0\le t\le 1} t^r n^{\frac{d-2r}{2}} t^{d-2r}(U^{(k-r)}_{[nt]}-\theta)|\le \frac{1}{\sqrt{n}}|\sup_{0\le t\le 1} n^{\frac{d-2r+1}{2}} t^{d-2r+1}(U^{(k-r)}_{[nt]}-\theta)|$$

converges to zero in probability. Therefore, $\sup_{0\le t\le 1} n^{d/2}t^d Y^*_{[nt],r}$ converges to zero in probability. □

We note that in the above proof, the convergence of $Y^*_{[nt],l}$ and $Y^{**}_{[nt],r}$ $(r=1,\dots,k-l-1)$ can be derived under the condition that $E\mid g_{(j)}(X_1,\dots,X_j)\mid<\infty$ for $j=1,\dots,l$.

Proof of Theorem 2.5: We note that

$$n^{d/2}Y_n(t)=\binom{k}{d}\frac{D([nt],d)}{D(n,d)}n^{d/2}K^{(d)}_{[nt]}+\sum_{c=d+1}^{k}\binom{k}{c}\frac{D([nt],c)}{D(n,c)}n^{d/2}K^{(c)}_{[nt]}. \tag{4.7}$$

By the assumption (1.7), for any $\varepsilon>0$ there exists a positive integer N such that

$$|\frac{n^d}{D(n,d)}\cdot\frac{D(j,d)}{j^d}-1|<\varepsilon \ \text{ for } \ n\ge j\ge N.$$

Thus we have

$$\begin{aligned}\sup_{0\le t\le 1}|\frac{D([nt],d)}{D(n,d)}n^{d/2}K^{(d)}_{[nt]}-n^{d/2}t^dK^{(d)}_{[nt]}|&\le\max_{d\le j\le n}|\frac{D(j,d)}{D(n,d)}-(\frac{j}{n})^d|\,n^{d/2}K^{(d)}_{[nt]}\\ &\le\max\{\max_{d\le j\le N}|D(j,d)\frac{n^d}{D(n,d)}-j^d|\frac{K^{(d)}_j}{n^{d/2}},\varepsilon\sup_{0\le t\le 1}n^{d/2}t^dK^{(d)}_{[nt]}\}.\end{aligned} \tag{4.8}$$

Since $\sup_{0\le t\le 1}n^{d/2}t^dK^{(d)}_{[nt]}$ converges in distribution by Theorem 2.4, by letting n tend to ∞ and then ε tend to zero we have that (4.8) converges to zero in probability as $n\to\infty$.

By a discussion similar to the one above, for $d+1\le c\le k$ we have that

$$\sup_{0\le t\le 1}|\frac{D([nt],c)}{D(n,c)}n^{c/2}K^{(c)}_{[nt]}-n^{c/2}t^cK^{(c)}_{[nt]}|$$

converges to zero in probability as $n\to\infty$. Since $n^{c/2}t^cK^{(c)}_{[nt]}$ converges in distribution by Theorem 2.4, $(D([nt],c)/D(n,c))n^{c/2}K^{(c)}_{[nt]}$ converges in distribution for $d+1\le c\le k$. Thus for $d+1\le c\le k$ we get that

$$\sup_{0\le t\le 1}|\frac{D([nt],c)}{D(n,c)}n^{d/2}K^{(c)}_{[nt]}|=\sup_{0\le t\le 1}|\frac{D([nt],c)}{D(n,c)}n^{c/2}K^{(c)}_{[nt]}|/n^{\frac{c-d}{2}}$$

converges to zero in probability as $n \to \infty$.

From this convergence of $[D([nt], c)/\ D(n, c)]n^{d/2}K^{(c)}_{[nt]}$, $c = d, d+1, \ldots, k$, and the convergence in (4.8), we know that the asymptotic distribution of $n^{d/2}Y_n(t)$ given by (4.7) is equal to the one of $\binom{k}{d}n^{d/2}t^d K^{(d)}_{[nt]}$. The kernel of $K^{(d)}_n$ is $g^{(d)} = \psi_d - \theta$. Since $g^{(d)}$ is completely degenerate, its degree and its order of degeneracy are equal to d. By applying (2.4) and (2.5) to this Y-statistic $K^{(d)}_n$, we get the asymptotic distributions of $n^{d/2}Y_n(t)$ given by (2.7) and (2.8). □

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