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Rational integrals of the second kind on a complex projective manifold and its primitive cohomology *

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Abstract: Let $X$ be a complex algebraic manifold of dimension $n + 1$ embedded in a sufficiently higher dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$, and $Y$ a generic hyperplane section of $X$. By sheaf cohomological method, we prove the well-known facts that the primitive cohomology group $H^p(X, \mathbb{C})_0$ ($1 \leq p \leq n + 1$) is isomorphic to the De Rham cohomology group $I^p(X, (p + 1)Y)_0$ of closed rational $p$-forms of the 2nd kind on $X$, having poles of order $p + 1$ (at most) along $Y$ only, and that the Hodge filtration of $H^p(X, \mathbb{C})_0$ is isomorphic to the one of $I^p(X, (p + 1)Y)_0$ defined by the order of poles along $Y$. On the other hand, we have a long exact sequence of cohomology

$$
\rightarrow H^p(X, \mathbb{C}) \xrightarrow{\iota^p} H^p(X - Y, \mathbb{C}) \xrightarrow{\rho^p} H^{p-1}(Y, \mathbb{C}) \xrightarrow{G^{p-1}} H^{p+1}(X, \mathbb{C}) \rightarrow \cdots,
$$

which is dual to

$$
\rightarrow H_p(X, \mathbb{C}) \xleftarrow{\iota_p} H_p^c(X - Y, \mathbb{C}) \xleftarrow{\rho_{p-1}} H_{p-1}(Y, \mathbb{C}) \xleftarrow{G_{p+1}} H_{p+1}(X, \mathbb{C}) \rightarrow \cdots,
$$

where $H_p^c$ denotes compact support homology group (cf. (1.2)). Using these exact sequences, we describe the mixed Hodge structure on $H^p(X - Y, \mathbb{C})$ and the Hodge filtration of the middle primitive cohomology group $H^n(Y, \mathbb{C})_0$ of $Y$ in terms of rational integrals on $X$.

Key words: Primitive cohomology, Rational integral of the 2nd kind, Generalized Poincaré résidué map, Hodge filtration, Mixed Hodge structure

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Summary

Let $X$ be a non-singular irreducible algebraic variety of dimension $n + 1$ embedded in a sufficiently higher dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$, and $Y$ a generic hyperplane section of $X$. We shall use the following notation:

- $\Omega^n_X$: the sheaf of germs of holomorphic $q$-forms on $X$,
- $\Omega^p_X(kY)$: the sheaf of germs of meromorphic $q$-forms having poles of order $k$ (at most) along $Y$ as their only singularities on $X$,
- $\Omega^p_X(*Y)$: the sheaf of germs of meromorphic $q$-forms having poles of arbitrary order along $Y$ as their only singularities on $X$,
- $\Omega^q_X(\log Y)$: the sheaf of germs of meromorphic $q$-forms having logarithmic poles (at most) along $Y$ as their only singularities on $X$.

We denote by $\Phi^p_X$, $\Phi^q_X(kY)$, e.t.c., the subsheaves consisting of closed forms of each one. On the complex $\Omega_X$ we define a decreasing filtration $F = \{F^k\}_{0 \leq k \leq n+1}$ (the Hodge filtration) by the subcomplexes

$$F^k(\Omega_X)^q = \begin{cases} 0 & q < k \\ \Omega^q_X & k \leq q. \end{cases}$$

On the complex $\Omega_X(\log Y)$ we define the Hodge filtration similarly, and another increasing filtration $W = \{W_0 \subset W_1\}$ (the weight filtration) by

$$W_0(\Omega_X(\log Y)) = \Omega_X, \quad W_1(\Omega_X(\log Y)) = \Omega_X(\log Y).$$

Then $(\Omega_X, F)$ becomes the cohomological Hodge complex, and $(\Omega_X(\log Y), W, F)$ the cohomological mixed Hodge complex (cf. §3). They induce the Hodge structure on the cohomology $H^p(X, \mathbb{C})$, and the mixed Hodge structure on the cohomology $H^p(X, \log Y)$.

We define

$$I^p_k(X, (p+1)Y) := \frac{\Gamma(X, \Phi^p_X((p-k+1)Y))}{d\Gamma(X, \Omega^{q-1}_X((p-k))Y)} \quad (0 \leq k \leq p)$$

and denote by $I^p_k(X, (p+1)Y)$ the subspace of $I^p_k(X, (p+1)Y)$ generated by closed meromorphic $p$-forms of the second kind (cf. Definition 2.2). Assume that

$$H^p(X, \Omega^q_X(kY)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1.$$

Then we have

$$F^k H^p(X - Y, \mathbb{C}) \simeq I^p_k(X, (p+1)Y) \quad 0 \leq k \leq p,$$

$$F^k H^p(X, \mathbb{C})_0 \simeq I^p_k(X, (p+1)Y)_0 \quad 0 \leq k \leq p,$$

$$G^W_1 H^q(X - Y, \mathbb{C}) = W[q]_0 H^q(X - Y, \mathbb{C}) = I^q(X, *Y)_0,$$

$$G^W_1 H^q(X - Y, \mathbb{C}) = I^q(X, *Y)/I^q(X, *Y)_0,$$

$$F^k G^W_1 H^q(X - Y, \mathbb{C}) \simeq F^k H^q(X, \mathbb{C})_0, \quad \text{and}$$

$$F^k G^W_1 H^q(X - Y, \mathbb{C}) \simeq \text{Ker}(F[-1]^k H^{q-1}(Y, \mathbb{C})_0 \xrightarrow{\sigma} F^k H^{q+2}(Y, \mathbb{C})),$$

where $H^p(X, \mathbb{C})_0$ denotes the $p$-th primitive cohomology of $X$, $F^k$ the $k$-th Hodge filtration of cohomology, and $W[q]$ the shift to the right on the degree of $W$ by $q$. (Theorem 3.1, Theorem 3.3 and Proposition 2.3). Furthermore, let $Y'$ be a generic hypersurface of $\mathbb{P}^N(\mathbb{C})$ of sufficiently higher degree so that

$$H^p(Y, \Omega^q_X(kZ)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1,$$

where $Z = Y \cdot Y'$. Then we can define the generalized Poincaré résidues map

$$\text{Ré} : I^{n+1}(X, (n+2)Y) \to I^n(Y, (n+1)Z)_0$$
and prove that

\[ F^k H^n(Y, \mathbb{C})_0 \simeq I^n_k(Y, (n + 1)Z)_0 \]
\[ \simeq \text{Res}(I^{n+1}_{k+1}(X, (n + 2)Y)) \oplus r^n(I^n_k(X, (n + 1)Y)_0) \],

where \( r^n \) denotes the map induced by the natural map \( H^n(X, \mathbb{C})_0 \to H^n(Y, \mathbb{C})_0 \) (Theorem 4.1). These results might be considered as a generalization of those by P. A. Griffith in the case of a hypersurface in a complex projective space (cf. [9]).

### 1 Some remarks on primitive cohomology and homology of algebraic manifolds

Let \( X \) be a non-singular irreducible algebraic variety of dimension \( n + 1 \) embedded in a higher dimensional complex projective space \( \mathbb{P}^N(\mathbb{C}) \) and \( Y \) a generic hyperplane section of \( X \). In what follows we call such \( Y \) a prime section of \( X \). We denote by \( \Omega \) the restriction to \( X \) of the fundamental form of the Fubini-Study metric on \( \mathbb{P}^N(\mathbb{C}) \). \( \Omega \) is a closed 2-form whose cohomology class \([\Omega] \in H^2(X, \mathbb{C})\) is the Poincaré dual of the homology class \([Y] \in H_{2n}(X, \mathbb{C})\) associated to the prime section \( Y \). We define \( L(\omega) := \Omega \wedge \omega \) for a (\( \mathbb{C} \)-valued) \( C^\infty \) differential form \( \omega \) on \( X \). If \( \omega \) is a closed form (resp. derived form), then \( L(\omega) \) is also a closed form (resp. derived form) for \( \Omega \) is a closed form. Hence \( L \) define a homomorphism \( H^q(X, \mathbb{C}) \to H^{q+2}(X, \mathbb{C}) \) \((0 \leq q \leq 2n)\).

Throughout this paper we always identify the ordinary cohomology with the De Rham cohomology. We call this cohomology operator Hodge operator and denote it by the same letter \( L \).

**Definition 1.1.** A \( C^\infty \) differential form \((0 \leq q \leq n + 1)\) \( \omega \) is said to be primitive if \( L^{n-q+2}(\omega) = 0 \) \((L^{n-q+2} = \underbrace{L \circ \cdots \circ L}_{n-q+2 \text{ times}})\). A (De Rham) cohomology class containing a closed, primitive \( C^\infty \) differential form is said to be a primitive cohomology class.

We call the subgroup of \( H^q(X, \mathbb{C}) \) which consists of all primitive cohomology classes the \( q \)-th primitive cohomology group of \( X \), which we denote by \( H^q(X, \mathbb{C})_0 \).

**Remark 1.1.** Originally, a \( C^\infty \) differential form \((0 \leq q \leq n + 1)\) \( \omega \) on \( X \) is defined to be primitive if \( \Lambda \omega = 0 \), \( \Lambda \) is the adjoint operator of \( L \) with respect to the Hodge metric on \( X \) which is the restriction of the Fubini-Study metric on \( \mathbb{P}^N(\mathbb{C}) \). The above definition of primitive forms is equivalent to the original one (cf. [11]).

The following facts are fundamental for the Hodge operator \( L \).

**Theorem 1.1.** (Hard Lefshets Theorem)

\[ L^k : H^{n+1-k}(X, \mathbb{C}) \simeq H^{n+1+k}(X, \mathbb{C}) \quad (1 \leq k \leq n + 1) \]

**Theorem 1.2.** (Lefshets decomposition)

(i) \( L : H^{q-2}(X, \mathbb{C}) \to H^q(X, \mathbb{C}) \) is injective and

\[ H^q(X, \mathbb{C}) \simeq LH^{q-2}(X, \mathbb{C}) \oplus H^q(X, \mathbb{C})_0 \quad (2 \leq q \leq n + 1) \].

(ii) \( H^{n+1+k}(X, \mathbb{C}) \simeq L^k H^{n+1+k}(X, \mathbb{C})_0 \oplus L^{k+1} H^{n-1-k}(X, \mathbb{C}) \)

By restriction \( C^\infty \) differential \( q \)-forms on \( X \) to \( Y \), we obtain a cohomology map \( r^q : H^q(X, \mathbb{C}) \to H^q(Y, \mathbb{C}) \), for which the following holds.

**Theorem 1.3.** (Weak Lefshetz Theorem)

(i) \( r^q : H^q(X, \mathbb{C}) \simeq H^q(Y, \mathbb{C}) \quad (0 \leq q \leq n - 1) \).

(ii) \( r^n : H^n(X, \mathbb{C}) \to H^n(Y, \mathbb{C}) \) is injective.

For the proofs of the theorems above we refer to [11].
Corollary 1.4.

\[ 0 \to H^{n+1}(X, \mathbb{C})_0 \to H^{n+1}(X, \mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(Y, \mathbb{C}) \to 0. \quad \text{(exact)} \]

Proof. By (1.2), (i) and (1.1), we have

\[
\begin{align*}
0 & \to H^{n-1}(X, \mathbb{C}) \xrightarrow{r} H^{n+1}(X, \mathbb{C}) \\
& \xrightarrow{\approx} \xrightarrow{r^{n+1}} \xrightarrow{\approx} H^{n+1}(Y, \mathbb{C})
\end{align*}
\]

and,

\[ H^{n+1}(X, \mathbb{C}) = H^{n+1}(X, \mathbb{C})_0 \oplus LH^{n-1}(X, \mathbb{C}). \]

Therefore,

\[ \text{Ker } r^{n+1} = H^{n+1}(X, \mathbb{C})_0 \]

\[ \square \]

Corollary 1.5.

\[ 0 \to H^{n+1}(X, \mathbb{C})_0 \to H^{n+1}(X, \mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(Y, \mathbb{C}) \to 0 \quad \text{(exact)} \]

In what follows, homology and cohomology are with coefficient in the complex number field if otherwise explicitly mentioned. Taking a topological tublar neighborhood \( U \) of \( Y \) in \( X \), we consider the homology exact sequence concerning a pair of the topological spaces \((X, X - U)\), which is written as follows:

\[ \ldots \to H^q (U, X - U) \xrightarrow{\iota_q} H_q (X) \xrightarrow{j_q} H_q (X, X - U) \xrightarrow{\partial_q} H^{q-1}_c (X - U) \to \ldots, \]

where \( H^*_c \) denotes compact support homology groups. Since \( X - U \) is a deformation retract of \( X - U \), \( H^*_c (X - U) \simeq H^*_c (X - Y) \). By the excision axiom, \( H^*_c (X, X - U) \simeq H^*_c (U, \partial U) \). By the Thom isomorphism, \( H^*_c (U, \partial U) \simeq H^{q-2}_c (Y) \) for \( q \geq 2 \). We obviously have \( H_q (U, X - U) = 0 \) for \( 0 \leq q \leq 1 \). Therefore the homology exact sequence (1.1) is rewritten as follows:

\[ \ldots \to H^q (X - Y) \xrightarrow{\iota_q} H_q (X) \xrightarrow{G_q} H_q - 2 (Y) \xrightarrow{r^{q-2}} H^{q-1}_c (X - Y) \to \ldots, \]

where

(i) the map \( \iota_q : H^*_c (X - U) \to H^*_c (X) \) is the one induced by the natural inclusion map \( \iota : X - Y \to X \),

(ii) the map \( G_q : H^*_c (X) \to H^*_c - 2 (Y) \) is the one which assigns each \( q \)-cycle on \( M \) to its intersection cycle with \( Y \), and

(iii) the map \( r^{q-2} : H^*_c - 2 (Y) \to H^*_c - 1 (X - U) \) is the one which assigns each \( (q - 2) \)-cycle on \( Y \), say \( \gamma \), to the cycle \( \partial U_{\gamma} \) on \( X \), the restriction of \( \partial U \) over \( \gamma \).

In the subsequence we denote the cycle \( \partial U_{\gamma} \) in (iii) above by \( \tau (\gamma) \). Taking the cohomology exact sequence dual to (1.2), we have

\[ \ldots \to H^q (X - Y) \xrightarrow{\iota^*} H^q (X) \xrightarrow{G^q - 2} H^{q+2} - 2 (Y) \xrightarrow{r^{q-1}} H^{q+1}_c (X - Y) \to \ldots, \]

Here the map \( G^q - 2 : H^{q+2 - 2} (Y) \to H^q (X) \) is the so-called Gysin map. We are now going to describe the Gysin map \( G^q - 2 \) by use of differential forms. We take a sufficiently fine, finite open covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( X \) such that, in each open subset \( U_i, Y \) is defined by a holomorphic equation \( \sigma_i = 0 \). We put \( t_{ij} = \sigma_i / \sigma_j \) for each pair of indexes \((i, j)\) with \( U_i \cap U_j \neq \emptyset \). Then the system of transition functions, with respect to the covering \( \mathcal{U} \), of the line bundle \([Y]\) associated to \( Y \) are given by \( \{ t_{ij} \} \), and \( \sigma = \{ \sigma_i \} \) give rise to a cross-section...
of \([Y]\) whose zero locus is \(Y\). We take a system \(\{a_i\}\) of real positive functions \(a_i\) of class \(C^\infty\) defined in \(U_i\), respectively, satisfying
\[
\frac{a_i}{a_j} = |t_{ij}|^2, \quad \text{in } U_i \cap U_j \neq \emptyset.
\]
The system \(\{a_i\}\) defines a fiber metric on the line bundle \([Y]\). The length function \(|\sigma|\) of the cross-section \(\sigma = \{\sigma_i\}\) of \([Y]\) with respect to this fiber metric is given by
\[
|\sigma| = \sqrt{\sigma_i \overline{a_i}} = |\sigma_i| \sqrt{a_i}
\]
in each \(U_i\). Note that \(|\sigma|^2\) is a globally defined real non-negative function of class \(C^\infty\). We define
\[
\eta := \frac{1}{2\pi i} \partial \log |\sigma|^2,
\]
\[
\omega = \bar{\partial} \eta = \frac{1}{2\pi i} \bar{\partial} \log |\sigma|^2.
\]
On each \(U_i\), \(\eta\) and \(\omega\) are written as
\[
\eta := \frac{1}{2\pi i} (d \log a_i + \partial \log a_i),
\]
\[
\omega = \frac{1}{2\pi i} \bar{\partial} \log a_i.
\]
Note that \(\omega\) is a globally defined closed \(C^\infty\) form of type \((1,1)\) on \(X\), representing the first Chern class \(c_1([Y])\) of the line bundle \([Y]\). We denote by \(A^* (X)\), \(A^* (X - Y)\) and \(A^* (Y)\) the De Rham complexes of \(\mathbb{C}\)-valued, \(C^\infty\) differential forms on \(X, X - Y\) and \(Y\), respectively.

**Definition 1.2.** \(A^* (\log Y)\) is defined to be the sub-complex of \(A^* (X - Y)\) generated by \(A^* (X)\) and \(\eta\).

A form \(\varphi \in A^* (\log Y)\) may be (non-uniquely) written as
\[
(1.4) \quad \varphi = \alpha \wedge \eta + \beta
\]
where \(\alpha, \beta \in A^* (V)\). The restriction \(\alpha_Y \in A^* (Y)\) is, however, not ambiguous. Hence we may define \(R^* : A^* (\log Y) \to A^{n - 1} (Y)\) by
\[
(1.5) \quad R^* (\varphi) := 2\pi \sqrt{-1} \alpha_Y,
\]
which we call \(\text{Résidue map}\). Let \(W^* \subset A^* (\log Y)\) be the kernel of \(R^*\). There is an obvious inclusion
\[
\iota \quad A^* (X) \subset W^*
\]

**Proposition 1.6.** The inclusion \(\iota\) induces isomorphisms on \(d\) and \(\partial\) cohomologies.

For the proof we refer to ([9]), p.49~p.50.

**Proposition 1.7.** The Gysin map \(G^{q - 2} : H^{q - 2} (Y, \mathbb{C}) \to H^q (X, \mathbb{C})\) is described using differential forms as follows: For \(\alpha \in A^{q - 2} (Y)\), choose \(\tilde{\alpha} \in A^{q - 2} (X)\) with \(\tilde{\alpha}_Y = \alpha\) and set
\[
\gamma (\alpha) = d (\tilde{\alpha} \wedge \eta) = d \tilde{\alpha} \wedge \eta \wedge \eta + (-1)^{q-2} \tilde{\alpha} \wedge \omega.
\]
If \(\alpha\) is a closed form (resp. derived form), then \(\gamma (\alpha)\) is a closed form (resp. derived form) in \(W^q\). Furthermore \(\gamma (\alpha)\) is independent of the choice of \(\tilde{\alpha}\) modulo derived form in \(W^q\). Hence, by virtue of Proposition 1.7, the correspondence \([\alpha] \mapsto [\gamma (\alpha)]\) defines a map
\[
H^{q - 2} (Y, \mathbb{C}) \simeq H^{q - 2} (A^* (X)) \to H^q (X, \mathbb{C}) \simeq H^q (W^*),
\]
which coincides, up to a factor of \(\pm 1\), with the Gysin map \(G\).
Proof. By the definition of \( W^a \text{st} \), \( \gamma(\alpha) \in W^a \text{st} \). It is obvious that if \( \alpha \) is a closed form, then \( \gamma(\alpha) \) is also closed in \( W^q \). Assume \( \alpha \) is written as \( d\beta = \alpha \) for \( \beta \in A^{q-3}(Y) \). We choose \( \tilde{\beta} \in A^{q-3}(X) \) with \( \tilde{\beta}|_Y = \beta \) and set

\[
\xi = (\tilde{\alpha} - d\tilde{\beta}) \wedge \eta + (-1)^{q-2}\tilde{\beta} \wedge d\eta.
\]

Then \( \xi \in W^{q-1} \) and

\[
d\xi = d\tilde{\alpha} \wedge \eta + (-1)^{q-2}(\tilde{\alpha} - d\tilde{\beta}) \wedge d\eta + (-1)^{q-2}d\tilde{\beta} \wedge d\eta
\]

\[
= d\tilde{\alpha} \wedge \eta + (-1)^{q-2}\tilde{\alpha} \wedge d\eta
\]

\[
= \gamma(\alpha)
\]

Thus \( \gamma(\alpha) \) is a derived form in \( W^* \).

The fact that \( \gamma(\alpha) \) is independent of the choice of \( \tilde{\alpha} \) modulo derived forms in \( W^* \) is almost trivial. In fact, if \( \tilde{\alpha}' \) is another form in \( A^{q-2}(X) \) with \( \tilde{\alpha}'|_Y = \alpha \), then \( (\tilde{\alpha} - \tilde{\alpha}') \wedge \eta \in W^{q-1}(X) \) and \( d((\tilde{\alpha} - \tilde{\alpha}') \wedge \eta) = d\tilde{\alpha} \wedge \eta - d\tilde{\alpha}' \wedge \eta \), which shows \( \gamma(\alpha) \) is uniquely determined up to derived forms in \( W^* \). We are now going to show that the correspondence \([\alpha] \mapsto [\gamma(\alpha)]\) coincides with the Gysin map \( G \). To do this it suffices to show that for any \( q \)-cycle \( c_q \) on \( X \), the integral \( \int_{\Gamma} \gamma(\alpha) \) converges and

\[
\int_{c_q} \gamma(\alpha) = \pm \int_{c_q \cap Y} \alpha
\]

holds, where \( \Gamma \cdot Y \) denotes the intersection cycle of \( \Gamma \) with \( Y \). We may assume that \( c_q \) intersects \( Y \) normally in a \( (q - 2) \)-cycle \( c_{q-2} \) with respect to some given hermitian metric on \( X \). For a sufficiently small positive \( \varepsilon \), we take a \( \varepsilon \)-tube with axis \( c_{q-2} \), and lying in \( c_q \), normally,

\[
T_\varepsilon(c_{q-2}) := \{ p \in c_q \mid d_X(p, c_{q-2}) \leq \varepsilon \}
\]

where \( d_X(\cdot, \cdot) \) denotes the distance function on \( X \) defined by the given hermitian metric. We give natural orientation to \( T_\varepsilon(c_{q-2}) \). Then,

\[
\lim_{\varepsilon \to 0} \int_{c_q - T_\varepsilon(c_{q-2})} \gamma(\alpha) = \lim_{\varepsilon \to 0} \int_{c_q - T_\varepsilon(c_{q-2})} d(\tilde{\alpha} \wedge \eta)
\]

\[
= \lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta \quad \text{(by Stokes's Theorem)}
\]

Using local coordinates \( (z_1, \cdots, z_n, z_{n+1}) \) on \( X \) such that \( Y \) is defined by \( z_{n+1} = 0 \), \( \tilde{\alpha} \wedge \eta \) and \( \partial T_\varepsilon(c_{q-2}) \) are locally written as

\[
\tilde{\alpha} \wedge \eta = \frac{1}{2\pi i} \tilde{\alpha} \wedge \frac{dz_{n+1}}{z_{n+1}} + \text{(regular form)}
\]

\[
\pm \partial T_\varepsilon(c_{q-2}) = c_{q-2} \times \{ z_{n+1} \in \mathbb{C} \mid |z_{n+1}| = \varepsilon \}
\]

(with natural orientation)

Hence,

\[
\int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta = \pm \int_{c_{q-2}} \tilde{\alpha} + \int_{\partial T_\varepsilon(c_{q-2})} \text{(regular form),}
\]

and since \( \lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(c_{q-2})} \text{(regular form)} = 0 \),

\[
\int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta = \pm \int_{c_{q-2}} \tilde{\alpha}
\]

From (1.7) and (1.8) it follows that the integral \( \int_{c_q} \gamma(\alpha) \) converges and the equality in (1.6) holds as required.
Proposition 1.8. We have the following commutative diagram:

\[
\begin{array}{ccc}
H^q(X, \mathbb{C}) & \xrightarrow{r^q} & H^q(Y, \mathbb{C}) \\
& L & \\
H^q(Y, \mathbb{C}) & \xrightarrow{G^q} & H^{q+2}(X, \mathbb{C}) \\
& L' & \\
& H^{q+2}(Y, \mathbb{C}) & \xrightarrow{r^{q+2}} \\
\end{array}
\]

where \( L' \) denotes the Hodge operator on \( H^*(Y, \mathbb{C}) \) associated to the fundamental form on \( Y \), the restriction \( \Omega_Y \) of the fundamental form \( \Omega \) to \( Y \).

Proof. We first show that the commutativity of the upper triangle. Let \( \alpha \) be a closed \( C^\infty \) \( q \)-form on \( X \). We denote by \([\alpha] \in H^q(X, \mathbb{C})\) its cohomology class. Then,

\[
(G^q \circ r^q)([\alpha]) = [d(\alpha \wedge \eta)] = [d\alpha \wedge \eta + (-1)^q \alpha \wedge d\eta] = [d\eta \wedge \alpha].
\]

Now, we recall that \( \omega := d\eta \) is a closed \((1,1)\)-form which represents the first Chern class of the line bundle \([Y]\). Hence, \( \omega \) is cohomologous to \( \Omega \) in \( H^2(X, \mathbb{C}) \). From this it follows that

\[
[d\eta \wedge \alpha] = [\Omega \wedge \alpha] = L([\alpha]).
\]

Thus \( (G \circ r^*)([\alpha]) = L([\alpha]) \) as required. Similarly, the commutativity of the lower triangle can be proved. \( \square \)

We now return to the long exact sequence of cohomology (1.3). By Theorem 1.1, Theorem 1.2, Theorem 1.3, Proposition 1.8 and Grothendieck’s theorem in [12] which tells us (among other things) that \( H^q(X - Y, \mathbb{C}) = 0 \) for \( q \geq n + 2 \), we can easily see that the long exact sequence of cohomology (1.3) breaks down into the short exact sequences as follows:

(1.10) \[
0 \rightarrow H^q(X, \mathbb{C}) \xrightarrow{r^q} H^q(X - Y, \mathbb{C}) \rightarrow 0 \quad \text{for } 0 \leq q \leq 1,
\]

(1.11) \[
0 \rightarrow H^{q-2}(Y, \mathbb{C}) \xrightarrow{G^{q-2}} H^q(X, \mathbb{C}) \xrightarrow{r^q} H^q(X - Y, \mathbb{C}) \rightarrow 0 \quad \text{for } 2 \leq q \leq n,
\]

(1.12) \[
0 \rightarrow H^{n-1}(Y, \mathbb{C}) \xrightarrow{G^{n-1}} H^{n+1}(X, \mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(X - Y, \mathbb{C}) \xrightarrow{R^{n+1}} H^n(Y, \mathbb{C}) \xrightarrow{G^n} H^{n+2}(X, \mathbb{C}) \rightarrow 0,
\]

(1.13) \[
0 \rightarrow H^q(Y, \mathbb{C}) \xrightarrow{G^q} H^{q+2}(X, \mathbb{C}) \rightarrow 0 \quad \text{for } n + 1 \leq q \leq 2n.
\]

We now define the notions of primitive cycles and finite cycles on \( X \) with respect to the prime section \( Y \).

Definition 1.3. A \( q \)-cycle \( c_q \) on \( X \) is defined to be primitive if its intersection cycle \( c_q \cdot Y \) with \( Y \) is zero in \( H_{q-2}(Y, \mathbb{C}) \). A \( q \)-cycle \( c_q \) on \( X \) is defined to be finite if its support is contained in \( X - Y \).

We call homology classes of primitive cycles primitive homology classes and those of finite cycles finite homology classes. We denote the subspace of primitive (resp. finite) \( q \)-homology classes by \( H_q(X, \mathbb{C})_0 \) (resp. \( H_q(X, \mathbb{C})_f \)) and call it the primitive \( q \)-homology group of (resp. finite \( q \)-homology groups of \( X \). Then by the definitions,

\[
H_q(X, \mathbb{C})_0 := \text{Ker} \{ H_q(X, \mathbb{C}) \xrightarrow{[Y]} H_{q-2}(Y, \mathbb{C}) \} \\
H_q(X, \mathbb{C})_f := \text{Im} \{ H_q(X - Y, \mathbb{C}) \xrightarrow{\cdot Y} H_q(X, \mathbb{C}) \}.
\]
Proposition 1.9. Primitive $q$-cycles possibly exist on $X$ only for $q$ with $0 \leq q \leq n + 1$, and
\[ H_q(X, \mathbb{C})_0 = H_q(X, \mathbb{C})_f \quad \text{for} \quad 0 \leq q \leq n + 1. \]

Proof. From the homology sequences dual to the cohomology sequences in (1.10) through (1.12) the assertion easily follows. \hfill \Box

To state about the relation between primitive cohomology and homology groups, we introduce the notation for a subspace $S$ of $H^q(X, \mathbb{C})$ (resp. $H_q(X, \mathbb{C})$) as follows:
\[ \text{Ann}(S) := \{ [\alpha] \in H_q(X, \mathbb{C}) \mid < [\omega], [\alpha] >= 0 \quad \text{for any} \quad [\omega] \in S \}, \]
where $<,>$ denotes the pairing between cohomology and homology. We call this the annihilator subspace of $H_q(X, \mathbb{C})$ by the subspace $S$.

Proposition 1.10.
(i) $H_q(X, \mathbb{C}) \simeq H_q(X, \mathbb{C})_0 \quad (0 \leq q \leq 1)$
(ii) $H_q(X, \mathbb{C}) \simeq H_q(X, \mathbb{C})_0 \oplus \text{Ann}(H_q(X, \mathbb{C})_0) \quad (2 \leq q \leq n + 1)$

Proof. The assertion (i) follows from the definition of primitive homology. We will now prove the assertion (ii). By (i) of 1.3 and Proposition 1.9, $G^{q-2}H^{q-2}(X, \mathbb{C}) = LH^{q-2}(X, \mathbb{C})$. Hence, by (ii) of Theorem 1.3,
\[ (1.14) \quad H^{q-2}(X, \mathbb{C}) \simeq G^{q-2}H^{q-2}(Y, \mathbb{C}) \oplus H^q(X, \mathbb{C})_0 \]
Therefore, by duality
\[ (1.15) \quad H_q(X, \mathbb{C}) \simeq \text{Ann}(G^{q-2}H^{q-2}(Y, \mathbb{C})) \oplus \text{Ann}(H^{q-2}(X, \mathbb{C})_0). \]
By considering the pairing between the exact sequences of cohomology (1.10), (1.11) and their dual exact sequences of homology,
\[ (1.16) \quad \text{Ann}(G^{q-2}H^{q-2}(Y, \mathbb{C}) \simeq \iota_* H_q(X - Y, \mathbb{C}) = H_q(X, \mathbb{C})_f \]
From (1.15), (1.16) and Proposition 1.11 follows the assertion (ii). \hfill \Box

Proposition 1.11. For $0 \leq q \leq n + 1$, $r^q : H^q(X, \mathbb{C}) \to H^q(X - Y, \mathbb{C})$ is injective on the subspace $H^q(X, \mathbb{C})_0$ and
\[ H^q(X, \mathbb{C})_0 \simeq r^q H^q(X, \mathbb{C}) \hookrightarrow H^q(X - Y, \mathbb{C}). \]

Proof. By the exactness of the cohomology sequences (1.11) and (??), $\text{Im} \, G = \text{ker} \, r^*$. Hence the assertion follows from (1.14). \hfill \Box

Definition 1.4. Cycles with compact support in $X - Y$ is defined to be résidue cycle if they bounds in $X$. We call their homology classes résidue homology classes.

We denote the subspace of $H^q_\circ(X - Y, \mathbb{C})$ comprising résidue homology classes by $H^q_\circ(X - Y, \mathbb{C})_{\text{rés}}$. By the definition,
\[ H^q_\circ(X - Y, \mathbb{C})_{\text{rés}} = \text{Ker} \{ H^q_\circ(X - Y, \mathbb{C}) \to H^q_\circ(X, \mathbb{C}) \}. \]
Actually, $H^q_\circ(X - Y, \mathbb{C})_{\text{rés}} \neq 0$ only for $q = n + 1$ and
\[ (1.17) \quad H^q_{n+1}(X - Y, \mathbb{C})_{\text{rés}} = \tau_n H^q_{n+1}(Y, \mathbb{C}) \]
because of the exact homology sequence (1.2) which is dual to (1.3).

Proposition 1.12.
\[ r^{n+1} H^{n+1}(X, \mathbb{C}) = \text{Ann}(H^q_{n+1}(X - Y, \mathbb{C})_{\text{rés}}). \]
Proof. By considering the paring between the cohomology exact sequence (1.18) and its dual homology sequence, we have
\[ r^{n+1}H^{n+1}(X, \mathbb{C}) = \text{Ann}(\tau_n H_n(Y, \mathbb{C})). \]
Hence the assertion follows from (1.17).

We denote by \( H^n(Y, \mathbb{C}) \) the primitive cohomology group with respect to the Hodge operator \( L' \) on \( Y \) which is associated to \( \Omega_Y \), the restriction of the fundamental form \( \Omega \) on \( X \). We are now going to discuss the primitive cohomology and homology of \( Y \). For use later we wish to make clear the relation between the image of the map \( R^{n+1} : H^{n+1}(X - Y, \mathbb{C}) \to H^n(Y, \mathbb{C}) \) in the exact sequence (1.18) and the primitive cohomology group \( H^n(Y, \mathbb{C}) \). The result is as follows:

**Lemma 1.13.** The restriction map \( r^n : H^n(X, \mathbb{C}) \to H^n(Y, \mathbb{C}) \), which is injective by the Weak Lefshetz Theorem, give rise to an isomorphism from \( H^n(Y, \mathbb{C}) \) into \( H^n(Y, \mathbb{C}) \) and
\[ r^n(H^n(X, \mathbb{C})) \cap H^n(Y, \mathbb{C}) = r^n(H^n(X, \mathbb{C})_0) \]

**Proof.** By the definition of primitive cohomology, the isomorphism in (1.13) for \( n + 2 \) and 1.3, (ii), we have the following commutative diagram of exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^n(X, \mathbb{C})_0 & \longrightarrow & H^n(X, \mathbb{C}) & \longrightarrow & H^{n+2}(X, \mathbb{C}) \\
& & \downarrow & & \downarrow & & \uparrow\cong \\
0 & \longrightarrow & H^n(Y, \mathbb{C})_0 & \longrightarrow & H^n(Y, \mathbb{C}) & \longrightarrow & H^{n+2}(Y, \mathbb{C}) \\
\end{array}
\]

From this we infer that \( r^n(H^n(Y, \mathbb{C})_0) \to H^n(Y, \mathbb{C})_0 \) and \( r^n_0(H^n(Y, \mathbb{C})_0) \to H^n(Y, \mathbb{C})_0 \) is an isomorphism into. To show the latter part, we consider the following Lefshetz decompositions of \( H^n(Y, \mathbb{C}) \) and \( H^n(Y, \mathbb{C})_0 \):

(1.19) \[ H^n(X, \mathbb{C}) = H^n(X, \mathbb{C})_0 \oplus L^nH^{n-2}(X, \mathbb{C}), \]
(1.20) \[ H^n(Y, \mathbb{C}) = H^n(Y, \mathbb{C})_0 \oplus L'\!H^{n-2}(Y, \mathbb{C}). \]

Note that, since \( r^{n-2} : H^{n-2}(X, \mathbb{C}) \to H^{n-2}(Y, \mathbb{C}) \) is an isomorphism by the Weak Lefshetz Theorem, \( r^n : H^n(X, \mathbb{C}) \to H^n(Y, \mathbb{C}) \) maps \( L^nH^{n-2}(X, \mathbb{C}) \) onto \( L'\!H^{n-2}(Y, \mathbb{C}) \) isomorphically. The inclusion

(1.21) \[ r^nH^n(X, \mathbb{C}) \cap H^n(Y, \mathbb{C}) \]

is obvious, since \( r^nH^n(X, \mathbb{C})_0 \to H^n(Y, \mathbb{C})_0 \) as has been proved just above. We will prove the reverse inclusion. Given \( x \in r^nH^n(X, \mathbb{C}) \cap H^n(Y, \mathbb{C})_0 \), there exists a \( y \in H^n(X, \mathbb{C}) \) with \( r^n(y) = x \). We write \( y = y_1 + y_2 \) where \( y_1 \in H^n(X, \mathbb{C})_0 \) and \( y_2 \in L^nH^{n-2}(X, \mathbb{C}) \). Then \( x = r^n(y) = r^n(y_1) + r^n(y_2) \), and \( r^n(y_1) \in H^n(Y, \mathbb{C})_0 \), \( r^n(y_2) \in L'\!H^{n-2}(Y, \mathbb{C})_0 \). Hence
\[ x - r^n(y_1) = r^n(y_2) \in H^n(Y, \mathbb{C})_0 \cap L'\!H^{n-2}(Y, \mathbb{C})_0 = 0. \]
Thus \( r^n(y_2) = 0 \), from which \( y_2 = 0 \) follows since \( r^n \) maps \( L^nH^{n-2}(X, \mathbb{C}) \) onto \( L'\!H^{n-2}(Y, \mathbb{C}) \) isomorphically. Hence \( x = r^n(y_1) \). This shows that

(1.22) \[ r^nH^n(X, \mathbb{C}) \cap H^n(Y, \mathbb{C})_0 \to r^nH^n(X, \mathbb{C})_0 \]

By (1.21) and (1.22), \( r^nH^n(X, \mathbb{C}) \cap H^n(Y, \mathbb{C})_0 = r^nH^n(X, \mathbb{C})_0 \) as required. \( \square \)
**Lemma 1.14.** There is an exact sequence

\[(1.23) \quad 0 \to LH^n(X, \mathbb{C})_0 \to H^{n+2}(X, \mathbb{C}) \xrightarrow{r^{n+2}} H^{n+2}(Y, \mathbb{C}) \to 0.\]

**Proof.** To see the surjectivity of \(r^{n+2}\), we consider the following commutative diagram:

\[
\begin{array}{ccc}
H^{n+2}(X, \mathbb{C}) & \xrightarrow{r^{n+2}} & H^{n+2}(Y, \mathbb{C}) \\
L \downarrow & & \downarrow L' \\
H^n(X, \mathbb{C}) & \xleftarrow{G^{n-2}} & H^{n-2}(Y, \mathbb{C})
\end{array}
\]

the commutativity of which follows from the description of the Gysin map \(G^{n-2}\) using differential forms (1.8). From this diagram the surjectivity of \(r^{n+2}\) follows, since \(L'\) is an isomorphism (Hard Lefshetz for \(Y\)). The injectivity of \(L : H^n(X, \mathbb{C})_0 \to H^{n+2}(X, \mathbb{C})\) follows from the fact that \(L : H^n(X, \mathbb{C})_0 \to H^{n+2}(X, \mathbb{C})\) is an isomorphism (Hard Lefshetz for \(X\)).

To prove the exactness at the term \(H^{n+2}(X, \mathbb{C})\), we consider the following commutative diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
H^n(X, \mathbb{C})_0 & \xrightarrow{r^n} & H^n(X, \mathbb{C})_0 & \xrightarrow{L} \xrightarrow{r^{n+2}} H^{n+2}(X, \mathbb{C}) \\
\downarrow \text{inclusion} & & \downarrow \text{inclusion} & & \\
H^n(Y, \mathbb{C})_0 & \xrightarrow{r^n} & H^n(Y, \mathbb{C}) & \xrightarrow{L'} \xrightarrow{r^{n+2}} H^{n+2}(Y, \mathbb{C}) & \xrightarrow{0} (\text{exact}) \\
\downarrow & & \downarrow & & \\
0 & & & \\
\end{array}
\]

By this diagram we can easily see \(LH^n(X, \mathbb{C})_0 \subset \ker r^n\). We will prove the converse inclusion by casing the diagram (1.24). Given \(x \in \ker r^n\), there exists a \(y \in H^n(X, \mathbb{C})_0\) with \(L(y) = x\). Then \(L'(r^n(y)) = r^{n+2}(L(y)) = r^{n+2}(x) = 0\), hence \(r^n(y) \in r^nH^n(X, \mathbb{C}) \cap H^n(X, \mathbb{C})_0\). We should now recall that \(r^nH^n(X, \mathbb{C}) \cap H^n(X, \mathbb{C})_0 = r^nH^n(X, \mathbb{C})_0\) (Lemma 1.13) and \(r^n\) is injective. This implies \(y \in H^n(X, \mathbb{C})_0\), that is, \(x = L(y) \in LH^n(X, \mathbb{C})_0\), which means \(\ker r^n \subset LH^n(X, \mathbb{C})_0\). Consequently, we conclude \(\ker r^n = LH^n(X, \mathbb{C})_0\) as required. \(\square\)

**Theorem 1.15.**

\[H^n(Y, \mathbb{C})_0 = R^{n+1}(H^{n+1}(X - Y, \mathbb{C}) \oplus r^n(H^n(X, \mathbb{C})_0)\]

**Proof.** Let us consider the Lefshetz decompositions of \(H^n(Y, \mathbb{C})_0\) and \(H^{n+2}(X, \mathbb{C})\):

\[
H^n(Y, \mathbb{C}) = H^n(Y, \mathbb{C})_0 \oplus L'H^{n-2}(Y, \mathbb{C})
\]

\[
H^{n+2}(X, \mathbb{C}) = LH^n(X, \mathbb{C})_0 \oplus L^2H^{n-2}(X, \mathbb{C}).
\]

**Claim:** Concerning the Gysin map \(G^n : H^n(Y, \mathbb{C}) \to H^{n+2}(X, \mathbb{C})\), we have

(a) \(G^n(L'H^{n-2}(Y, \mathbb{C}) \subseteq L^2H^{n-2}(X, \mathbb{C})\), \(G^n\) maps \(L'H^{n-2}(Y, \mathbb{C})\) onto \(L^2H^{n-2}(X, \mathbb{C})\) isomorphically,

(b) \(G^n(H^n(Y, \mathbb{C})_0) = LH^n(X, \mathbb{C})_0\), and

(c) \(\ker G^n \subset H^n(Y, \mathbb{C})_0\).
Proof of (a): By Proposition 1.9, we have the following commutative diagram:

\[
\begin{array}{cccc}
H^{n-2}(X, \mathbb{C}) & \xrightarrow{r^{n-2}} & L & \xrightarrow{G^{n-2}} H^{n}(X, \mathbb{C}) \\
\downarrow L & & \downarrow & \downarrow L \\
H^{n-2}(Y, \mathbb{C}) & \xrightarrow{L'} & H^{n}(Y, \mathbb{C}) & \xrightarrow{G^{n}} H^{n+2}(X, \mathbb{C}) \\
\end{array}
\]

where \( r^{n-2} : H^{n-2}(X, \mathbb{C}) \to H^{n-2}(Y, \mathbb{C}) \) is an isomorphism by the Weak Lefshetz Theorem. From this diagram \( G^{n}(L'H^{n-2}(Y, \mathbb{C}) \subset L^2H^{n-2}(X, \mathbb{C}) \) follows. The fact that \( G^{n} \) maps \( L'H^{n-2}(Y, \mathbb{C}) \) onto \( L^2H^{n-2}(X, \mathbb{C}) \) isomorphically is proved as follows: Since \( L : H^{n-2}(X, \mathbb{C}) \to H^{n}(X, \mathbb{C}) \) is injective, and since \( L : H^{n}(X, \mathbb{C}) \to H^{n+2}(X, \mathbb{C}) \) is an isomorphism (Hard Lefshetz Theorem), \( L^2 : H^{n-2}(X, \mathbb{C}) \to L^2H^{n-2}(X, \mathbb{C}) \) is an isomorphism. Besides, since \( L' : H^{n-2}(Y, \mathbb{C}) \to H^{n}(Y, \mathbb{C}) \) is injective, \( L' : H^{n-2}(Y, \mathbb{C}) \to L'H^{n-2}(Y, \mathbb{C}) \) is also an isomorphism. Therefore, taking into account that \( r^{n-2} : H^{n-2}(X, \mathbb{C}) \to H^{n-2}(Y, \mathbb{C}) \) is an isomorphism, we conclude that the Gysin map \( G^{n} \) maps \( L'H^{n-2}(Y, \mathbb{C}) \) onto \( L^2H^{n-2}(X, \mathbb{C}) \) isomorphically.

Proof of (b): Combining (1.9) for \( q = n \), Proposition 1.9, (1.23) and (1.13), we have the following commutative diagram:

\[
\begin{array}{cccc}
0 & \xrightarrow{0} & H^{n}(Y, \mathbb{C})_0 & \xrightarrow{G^{n}} H^{n+2}(X, \mathbb{C}) \\
\downarrow & & \downarrow & \downarrow \quad \text{(exact)} \\
H^{n}(Y, \mathbb{C}) & \xrightarrow{L'} & H^{n+2}(Y, \mathbb{C}) & \xrightarrow{0} \quad \text{(exact)} \\
\downarrow & & \downarrow & \downarrow \\
LH^{n}(X, \mathbb{C})_0 & \xrightarrow{0} & H^{n}(Y, \mathbb{C})_0 \\
\end{array}
\]

From this it follows
\[
G^{n}(H^{n}(Y, \mathbb{C})_0) \subset \text{Ker } r^{n+2}_* = LH^{n}(X, \mathbb{C})_0
\]

Actually, they coincide with each other, since \( G^{n} \) is surjective and (a) holds.

Proof of (c): Let \( x \in \text{Ker } G^{n} \). We write it as \( x = x_1 + x_2 \), where \( x_1 \in H^{n}(Y, \mathbb{C})_0 \) and \( x_2 \in L'H^{n-2}(Y, \mathbb{C})_0 \). Then \( G^{n}(x) = G^{n}(x_1) + G^{n}(x_2) = 0 \), and by (a) and (b), \( G^{n}(x_1) \in LH^{n}(X, \mathbb{C})_0 \) and \( G^{n}(x_2) \in L^2H^{n-2}(X, \mathbb{C})_0 \). Hence \( G^{n}(x_2) = -G^{n}(x_1) \in LH^{n}(X, \mathbb{C})_0 \cap L^2H^{n-2}(X, \mathbb{C})_0 = 0 \). Thus \( G^{n}(x_2) = 0 \), whence \( x_2 = 0 \). This is because \( G^{n} \) maps \( L'H^{n-2}(Y, \mathbb{C}) \) onto \( L^2H^{n-2}(Y, \mathbb{C}) \) isomorphically. Therefore \( x = x_1 \in H^{n}(Y, \mathbb{C})_0 \), which means \( \text{Ker } G^{n} \subset H^{n}(Y, \mathbb{C})_0 \).

q.e.d. for the Claim.

Now we can easily deduce the Proposition. In fact, by Lemma 1.13 and the claim (a), (b) (c) above, we have the following commutative diagram:
which implies

\[ H^n(Y, \mathbb{C})_0 \simeq Ker G^n \oplus r^n(H^n(X, \mathbb{C})_0). \]

Here, recall that \( Ker G^n = Im R^{n+1} \) by (1.12), then we are done. \( \square \)

We wish to identify the subspace of \( H^n(Y, \mathbb{C})_0 \) which is dual to \( Im R^{n+1} \). For this purpose we need to introduce the following notion.

**Definition 1.5.** Cycles in \( Y \) are defined to be vanishing cycles with respect to \( X \) if they bound in \( X \). We call their homology classes vanishing homology classes.

We denote the subspace \( H_q(Y, \mathbb{C}) \) comprising vanishing homology classes by \( H_q(Y, \mathbb{C})_v \). Note that \( H_q(Y, \mathbb{C})_v \) may not be zero only if \( q = n \).

**Proposition 1.16.** \( H_q(Y, \mathbb{C})_v \) is included in \( H_n(Y, \mathbb{C})_0 \) and

\[ H_n(Y, \mathbb{C}) = H_n(Y, \mathbb{C})_v \oplus Ann(Im R^{n+1}) \]

or, equivalently

\[ H_n(Y, \mathbb{C})_0 = H_n(Y, \mathbb{C})_v \oplus [Ann(Im R^{n+1}) \cap H_n(Y, \mathbb{C})_0] \]

**Proof.** By virtue of Theorem 1.15, it suffices to show that

\[ H_n(Y, \mathbb{C})_0 \cap Ann(r^nH^n(X, \mathbb{C})_0) = H_n(Y, \mathbb{C})_v. \]

The inclusion \( H_n(Y, \mathbb{C})_v \subset Ann(r^nH^n(X, \mathbb{C})_0) \) is trivial. To see that \( H_n(Y, \mathbb{C})_v \subset H_n(Y, \mathbb{C})_0 \), consider the following diagram:

\[ \begin{array}{ccc}
H_n(Y, \mathbb{C}) & \xrightarrow{[Z]} & H_{n-2}(Z, \mathbb{C})_v \\
\downarrow & & \downarrow \\
H_n(X, \mathbb{C}) & \xrightarrow{[\gamma]} & H_{n-2}(Y, \mathbb{C})_v,
\end{array} \]

where \( Z \) is the intersection of a generic member \( |Y| \) (linear system of effective divisors which are linearly equivalent to \( Y \)) with \( Y \), which is a non-singular, irreducible hypersurface of \( Y \) and for which \( c_1([Z]) \sim \Omega_Y \) (cohomologous), where \( v^n \) (resp. \( v^{n-2} \)) is the homomorphism induced by the inclusion map \( \iota : Y \hookrightarrow X \) (resp. \( \iota : Z \hookrightarrow Y \)), and where \( [Z] \) (resp. \( [Y] \)) is the map which assigns each \( n \)-cycle in \( Y \) (resp. \( X \)) to its intersection cycle with \( Z \) (resp. \( Y \)). By the diagram (1.25), \( H_n(Y, \mathbb{C})_v \hookrightarrow Ker ([Z]) \). Meanwhile, \( Ker ([Z]) = H_n(Y, \mathbb{C})_0 \) by definition. Thus we have \( H_n(Y, \mathbb{C})_v \hookrightarrow H_n(Y, \mathbb{C})_0 \). Hence

\[ H_n(Y, \mathbb{C})_v \hookrightarrow H_n(Y, \mathbb{C})_0 \cap Ann(r^nH^n(X, \mathbb{C})_0). \]

Next we will prove the converse inclusion. It suffices to show that if \( [\gamma] \in H_n(Y, \mathbb{C})_0 \cap Ann(r^nH^n(X, \mathbb{C})_0), \) then \( \int_\gamma \omega = 0 \) for any \( [\omega] \in H^n(X, \mathbb{C}) \). To see this, we use the Lefschetz decomposition

\[ H^n(X, \mathbb{C}) = H^n(Y, \mathbb{C})_0 \oplus LH^{n-2}(X, \mathbb{C}). \]
Assume \([\gamma] \in H_n(Y, \mathbb{C})_0 \cap \text{Ann}(r^nH^n(X, \mathbb{C})_0)\). Then \(\int_\gamma \omega = 0\) for any \([\omega] \in H^n(X, \mathbb{C})_0\), and for any \([\Omega \wedge \omega'] \in LH^{n-2}(X, \mathbb{C})\) \((\omega') \in H^{n-2}(X, \mathbb{C})\),

\[
\int_\gamma \Omega \wedge \omega' = \int_{[\gamma \cdot Y]} \omega' = 0
\]
since \([\gamma \cdot Y] = 0\) by the assumption. Thus \(\int_\gamma \omega = 0\) for any \([\omega] \in H^n(X, \mathbb{C})\) if \([\gamma] \in H_n(Y, \mathbb{C})_0 \cap \text{Ann}(r^nH^n(X, \mathbb{C})_0)\). This implies

\[(1.27) \quad H_n(Y, \mathbb{C})_0 \leftrightarrow H_n(Y, \mathbb{C})_0 \cap \text{Ann}(r^nH^n(X, \mathbb{C})_0)\]

By (1.26) and (1.27), \(H_n(Y, \mathbb{C})_0 = H_n(Y, \mathbb{C})_0 \cap \text{Ann}(r^nH^n(X, \mathbb{C})_0)\) as required.

\[\Box\]

2 Rational De Rham groups of an algebraic manifold and Integrals of the second kind on it

As in §1 we let \(X\) be a non-singular irreducible algebraic variety of dimension \(n + 1\) embedded in a higher dimensional complex projective space \(\mathbb{P}^N(\mathbb{C})\) and \(Y\) a generic hyperplane section of \(X\). By a meromorphic \(q\)-form on \(X\) we shall mean an exterior differential form \(\omega\) of degree \(q\), which has the form

\[
\omega = \sum f_{i_1i_2\cdots i_q} dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_q}
\]

where \((z_1, \cdots, z_{n+1})\) is a complex analytic local coordinate system on \(X\) and \(f_{i_1i_2\cdots i_q}\)'s are meromorphic functions of the variables \((z_1, \cdots, z_{n+1})\). We denote by \(\Omega^q_X(kY)\) the sheaf of germs of meromorphic \(q\)-forms having poles of order \(k\) (at most) along \(Y\) as their only singularities. The direct limit of the sheaves \(\Omega^q_X(kY)\) \(k \to \infty\) we denote by \(\Omega^q_X(\ast Y)\). It is just the sheaf of germs of meromorphic \(q\)-forms with poles of arbitrary order along \(Y\). We put \(\Omega^q_k(\ast Y) := \bigoplus \Omega^q_k(\ast Y)\), which forms a complex of sheaves with respect to the exterior derivative \(d\). We define

\[
\Phi^q(kY) := \text{Ker} \{ \Omega^q_X(kY) \xrightarrow{d} \Omega^{q+1}_X((k+1)Y) \}
\]

call it the sheaf of germs of closed meromorphic \(q\)-forms having poles of order \(k\) (at most) along \(Y\) as their only singularities. We define the sheaf \(\Omega^q_X(\log Y)\) to be the subsheaf of \(\Omega^q_X\(\ast Y)\) consisting of the germs of such local meromorphic \(q\)-forms that both of \(f\omega\) and \(df \wedge \omega\) are holomorphic if \(f\) is a local holomorphic defining equation of \(Y\). If \(g = 0\) is another defining equation of \(Y\), then \(g = uf\) where \(u\) is a non-vanishing local holomorphic function and the relation \(g\omega = uf\omega, dg \wedge \omega = udf \wedge \omega + fdu \wedge \omega\) shows that \(\Omega^q_X(\log Y)\) is well-defined. We call the sheaf of germs of meromorphic \(q\)-forms having logarithmic poles (at most) along \(Y\) as their only singularities. The reason for this naming is that a meromorphic \(q\)-form \(\omega\) \((q \geq 1)\) has logarithmic poles (at most) along \(Y\) as its only singularities if and only if \(\omega\) is locally written as

\[
\omega = \phi \wedge \frac{df}{f} + \psi,
\]

where \(\phi, \psi\) are holomorphic forms and \(f = 0\) is a local holomorphic equation of \(Y\). The following lemma is fundamental for calculations in the subseuil.

**Lemma 2.1.**

(i) The following sheaf sequences are exact:

\[
\begin{align*}
&0 \to \Phi^{q-1}((k-1)Y) \to \Omega^{q-1}_X((k-1)Y) \xrightarrow{d} \Phi^q(kY) \to 0 \quad (q \geq 2, k \geq 2) \\
&(b) \quad 0 \to \Phi^{q-1}(Y) \to \Omega^{q-1}_X(\log Y) \xrightarrow{d} \Phi^q(Y) \to 0 \quad (q \geq 2)
\end{align*}
\]

(ii) There exist naturally the following exact sequences of sheaves:

\[
\begin{align*}
&(c) \quad 0 \to \mathcal{O}_X \to \mathcal{O}((k-1)Y) \xrightarrow{d} \Phi^1(kY) \xrightarrow{\alpha} \mathcal{O}_Y \to 0 \quad (k \geq 1) \\
&(d) \quad 0 \to \Omega^q_X(Y) \to \Omega^q_X(\log Y) \xrightarrow{R} \Omega^q_X(Y) \to 0 \quad (q \geq 1) \\
&(e) \quad 0 \to \Phi^q_X \to \Phi^q_X(Y) \xrightarrow{R} \Phi^q_Y \to 0 \quad (q \geq 1)
\end{align*}
\]
Proof. We take a local coordinate system \((z_1, \ldots, z_n, w)\) on \(X\) such that \(Y\) is defined by \(w = 0\). First, we prove for all pairs of integers \((q,k)\) with \(q \geq 1, k \geq 1\) that if \(\varphi\) is a local holomorphic section of the sheaf \(\Phi^q(kY)\), then \(\varphi\) is written as

\[
\varphi = A \wedge \frac{dw}{w^k} + \frac{B}{w^{k-1}} \tag{2.1}
\]

where \(A, B\) are holomorphic and involve only \(dz_1, \ldots, dz_n\). In fact, as to such \(\varphi\), since \(w^k\varphi\) is holomorphic, we may write

\[
\varphi = A \wedge \frac{dw}{w^k} + B'
\]

where \(A, B'\) are holomorphic and do not involve \(dw\). Since \(\varphi\) is closed,

\[
d\varphi = \frac{dA \wedge dw}{w^k} + \frac{dB'}{w^k} + (-k) \frac{dw \wedge B'}{w^{k+1}} = 0
\]

so that \(B := B'/w\) is holomorphic. Hence we have locally the expression in (2.1) as required. Now we prove the exactness of (i)-(a) and (i)-(b). For a local holomorphic section \(\varphi\) of \(\Phi^q(kY)\) (\(q \geq 1, k \geq 2\) and \(q \geq 2, k = 1\)), we take such an expression as in (2.1). If \(k \geq 2\), letting \(\psi_1 = -(1/(k-1))(A/w^{k-1})\), \(\varphi - d\psi_1\) is a local section of \(\Phi^q(k-1)\). Repeating this argument, we may find a local section \(\psi\) of \(\Omega^{q-1}_X((k-1)Y)\) such that \(\varphi - d\psi\) is a section of \(\Phi^q_X(Y)\). Thus

\[
\varphi - d\psi = E \wedge \frac{dw}{w} + F,
\]

where \(E, F\) are holomorphic and involve only \(dz_1, \ldots, dz_n\). We express \(E\) as follows:

\[
E = E_0(z) + wE_1(z, w)
\]

where \(E_0(z)\) does not involve \(w\). Then,

\[
\varphi - d\psi = E_0(z) \wedge \frac{dw}{w} + F_0
\]

where \(F_0 = E_1 + F\). Since \(d(Edw/w + F) = 0\), \(dE_0(z)dw/w + dF_0 = 0\). Hence \(dE_0(z)dw/w + wdF_0 = 0\). From this it follows that \(d_2E_0(z) = 0\), \(dF_0 = 0\). Therefore, there exist \(D(z)\) and \(G\) such that \(d_2D = E_0\) and \(dG = F_0\), and so

\[
d(D \frac{dw}{w} + G) = E_0 \wedge \frac{dw}{w} + F_0.
\]

Hence,

\[
\varphi = d(\psi + D \wedge \frac{dw}{w} + G), \tag{2.2}
\]

namely, \(\varphi\) is a derived form. This shows the exactness of the sequence (i)-(a). If \(k = 1\), then \(\psi\) does not appear in the expression of \(\varphi\) in (2.2). This shows the exactness of (i)-(b).

Next we prove the exactness of the sequence (ii)-(c). If \(\varphi\) is a local section \(\Phi^1(Y)\), then it is written as

\[
\varphi = A \wedge \frac{dw}{w} + B,
\]

where \(A\) is a holomorphic function and \(B\) is a holomorphic 1-form, involving only \(dz_1, \ldots, dz_n\) (cf. (2.1)). Writting \(A\) as

\[
A(z, w) = A_0(z) + wA_1(z, w),
\]

where \(A_0(z)\) is a function of \(z_1, \ldots, z_n\), we have

\[
\varphi = A_0(z) \wedge \frac{dw}{w} + B_0,
\]
where $B_0 = A_1(z, w)dw + B$. Since

$$d\varphi = \frac{dz A_0(z) \wedge dw}{w} + dB_0 = 0,$$

we have

$$dz A_0(z) dw + w dB_0 = 0,$$

Hence $dz A_0(z) = dB_0 = 0$. From these it follows that $A_0(z)$ is constant and $B_0 = dC$ for some holomorphic function $C(z, w)$. Thus $\varphi$ is written as

$$\varphi = A_0 \wedge \frac{dw}{w} + dC,$$

This means $\Phi^1_X(Y)/d\Omega^1_X$ is locally a constant sheaf. At each point $y \in Y$, we take $[(1/2\pi i)dw/w]_y$, the class of $(\Phi^1_X(Y)/d\Omega^1_X)_y$ determined by $(1/2\pi i)dw/w$, as a generator of $(\Phi^1_X(Y)/d\Omega^1_X)_y$. We can easily see that the class $[(1/2\pi i)dw/w]_y$ is uniquely determined, not depending on the choice of a local defining equation of $Y$. We denote by $\alpha_y : (\Phi^1_X(Y)/d\Omega^1_X)_y \to \mathbb{C}_{Y,y}$ defined by

$$\left[ \frac{1}{2\pi i} \frac{dw}{w} \right]_y \mapsto 1_{Y,y}$$

at each point $y \in Y$, which gives rise to a well-defined sheaf homomorphism $\alpha : \Phi^1_X(Y)/d\Omega^1_X \to \mathbb{C}_Y$ as easily seen. The surjectivity of the map $\alpha$ and that the kernel of the homomorphism $d : \Omega^1_X \to \Phi^1_X(Y)$ coincides with $\mathbb{C}_Y$ is obvious. The sheaf homomorphism $R^1 : \Omega^1_X(\log Y) \to \Omega^2_Y$, which we call Résidues map is defined as follows (resp. $R : \Phi^1_X(Y) \to \Phi^1_Y$): A local cross-section $\varphi$ of the sheaf $\Omega^1_X(\log Y)$ (resp. $\Phi^1_X(Y)$) is written as

$$\omega = \varphi \wedge \frac{dw}{w} + \psi,$$

where $\varphi$ is a holomorphic $(q - 1)$-form and $\psi$ is a holomorphic $q$-form, involving only $dz_1, \cdots, dz_n$. For such $\omega$, we define $R(\omega) := \varphi|_Y$. We can easily seen that this map is well-defined and the sequences (c) and (d) are exact. Thus we are done. $\boxdot$

**Notation.** We denote by $\Omega_X((k_0 + \cdot)Y) (k_0$: a non-negative integer), $\Omega_X(\log Y)$ and $L(Y)$ the complexes of sheaves of $\mathbb{C}$-modules described as follows:

$$\Omega_X((k_0 + \cdot)Y) : \Omega^1_X(k_0 Y) \to \cdots \to \Omega^1_X((k_0 + p)Y) \to \cdots \to \Omega^1_X((k_0 + n)Y),$$

$$\Omega_X(\log Y) : \mathcal{O}_X \to \Omega^1_X(\log Y) \to \cdots \to \Omega^p_X(\log Y) \to \cdots \to \Omega^p_X(\log Y),$$

$$L(Y) : \Omega^1_X \to \Phi^1_X(Y).$$

**Proposition 2.2.** The natural homomorphisms of the complexes of sheaves of $\mathbb{C}$-vector spaces

$$L(Y) : \Omega_X(\log Y) \to \Omega_X((k_0 + \cdot)Y) \to \Omega_X(\cdot Y)$$

give rise to quasi-isomorphisms among them, and so all of the hypercohomology of these are isomorphic to $H^p(X, Y, \mathbb{C})$.

**Proof.** The former part of the proposition follows directly from Lemma 2.1. The latter part is proved as follows: What we shall prove is that $H^p(X, \Omega_X(\log Y)) \simeq H^p(X - Y, \mathbb{C})$ ($p \geq 0$). To do this we form a fine resolution of $\Omega_X(\log Y)$, using semi-meromorphic forms which have poles only on $Y$. Here, after J. Leray ([18]), we call a $C^\infty$-differential form $\varphi$ on $X - Y$ semi-meromorphic form on $X$, having poles of order $k$ (at most) along $Y$ if $w^k \varphi$ is locally a $C^\infty$ regular differential form at every point of $Y$, where $w = 0$ is a local defining equation of $Y$. Similarly, as in the case of meromorphic forms, semi-meromorphic forms having logarithmic poles on $Y$ is defined. We denote by $\mathfrak{F}_{X,Y}^p (\log Y)$ the sheaf of germs of semi-meromorphic forms of type $(p, q)$, having logarithmic poles on $Y$. Using these sheaves, we obtain a fine resolution of $\Omega_X(\log Y)$ as follows:
Letting \( A \) \((2.5)\) → From this the following long exact sequence of hypercohomology is derived: By Lemma 2.1,(d), we have the exact sequence of complexes of sheaves of global \( A \)← respectively, we have \( H^p(X, \Omega^q_X) \simeq H^p(A_X(\log Y)) \quad (p \geq 0) \). Then \( (A_X(\log Y), d) \) forms a complex of \( \mathbb{C} \)-vector spaces and
\[
\mathbb{H}^p(X, \Omega^q_X(\log Y)) \simeq H^p(A_X(\log Y)) \quad (p \geq 0).
\]
By Lemma 2.1,(d), we have the exact sequence of complexes of sheaves of \( \mathbb{C} \)-vector spaces:
\[
0 \to \Omega_X \to \Omega_X(\log Y) \xrightarrow{R} \Omega_Y[-1] \to 0.
\]
From this the following long exact sequence of hypercohomology is derived:
\[
\cdots \to \mathbb{H}^p(\Omega_X) \to \mathbb{H}^p(\Omega_X(\log Y)) \to \mathbb{H}^{p-1}(\Omega_Y) \to \mathbb{H}^{p+1}(\Omega_X) \to \cdots
\]
Letting \( A_X \) and \( A_Y \) be the complexes of \( \mathbb{C} \)-vector spaces of global \( C^\infty \) differential forms on \( X \) and \( Y \), respectively, we have \( \mathbb{H}^p(\Omega_X) \simeq H^p(A_X) \) and \( \mathbb{H}^p(\Omega_Y) \simeq H^p(A_Y) \). Hence the sequence (2.5) is rewritten as:
\[
\cdots \to H^p(A_X) \xrightarrow{R} H^p(A_X(\log Y)) \xrightarrow{E} H^{p-1}(A_Y) \xrightarrow{G^{p-1}} H^{p+1}(A_X) \to \cdots
\]
We claim that this is the dual of the homology sequence
\[
\cdots \to H_p(X, \mathbb{C}) \xrightarrow{R} H_p(X - Y, \mathbb{C}) \xrightarrow{R_{p-1}} H_{p-1}(Y, \mathbb{C}) \xrightarrow{G_{p+1}} H_{p+1}(X, \mathbb{C}) \to \cdots
\]
(cf. (1.3). In fact, since \( A_X(\log Y) \) is a subcomplex of \( A_{X - Y} \) which is the complex of \( \mathbb{C} \)-vector spaces of global \( C^\infty \) differential forms on \( X - Y \), we can define pairings by integrations between the terms corresponding to each other in (2.6) and (2.7). Furthermore, these pairings commute with the homomorphisms in (2.6) and (2.7), since we can easily see \( A_X(\log Y) \) is the same one as defined in Definition 1.2 and the map \( R_p : H^p(A_X(\log Y)) \to H^{p-1}(A_Y) \) is the \( \textit{Résiduum map} \) defined just after Definition 1.2, and since \( G^{p-1} : H^{p-1}(A_Y) \to H^{p+1}(A_X) \) is the \( \textit{Gysin map} \) whose description by use of differential forms has been given in Proposition 1.7. Therefore, by \textit{Five Lemma}, we conclude that the paring between \( H^p(A_X(\log Y)) \) and \( H^p(X - Y, \mathbb{C}) \) is non-degenerated. Hence \( H^p(A_X(\log Y)) \simeq H^p(X - Y, \mathbb{C}) \).
Definition 2.1. We define
\[ I_p(X, *Y) := \Gamma(X, \Phi^p_X(*Y))/d\Gamma(X, \Omega_X^{p-1}(*Y)) \]
and
\[ I_p(X, kY) := \Gamma(X, \Phi^p_X(kY))/d\Gamma(X, \Omega_X^{p-1}((k-1)Y)). \]
We call them the \(p\)-th \( *Y \)-rational De Rham group of \( X \) and \( p\)-th \( *Y \)-rational De Rham group of \( X \) with pole order \( k \), respectively.

Then, by Proposition 2.2, we have the following:

Proposition 2.3. Let \( k_0 \) be a positive integer such that
\[ H^p(X, \Omega_X^p((k_0 + q)Y)) = 0 \quad \text{for} \quad p \geq 1, \; q \geq 0, \]
then,
\[ I_p(X, (k_0 + p)Y) \cong I_p(X, *Y) \cong H^p(X - Y, \mathbb{C}) \quad \text{for} \quad p \geq 0. \]

Remark 2.1. The result in the proposition above is a special case of the theorem of Grothendieck (cf. [12]).

Now we are going to explain the notion of closed meromorphic forms of the \textit{second kind}, having poles only along \( Y \). There are the following three different definitions for this:

Definition 2.2. A closed meromorphic \( q \)-form \( \varphi \) is of the \textit{second kind} if
\begin{itemize}
  \item[(A)] (Picard-Lefschetz definition) at any point \( x \) of \( X \), there exists a meromorphic \( q - 1 \)-form on \( X \) such that \( \varphi - dw \) is holomorphic in a neighborhood of \( x \),
  \item[(B)] (Geometric Residue definition) it has no periods on \textit{résidue cycles} (cf. Definition 1.4) of \( X - Y \), if \( Y \) is sufficiently large subvariety (depending on \( \varphi \)),
  \item[(C)] Hodge and Atiyah’s \textit{algebraic definition}, using spectral sequences associated to the complex of sheaves of \( \mathbb{C} \)-vector spaces \( \Omega_X(*Y) \) (or \( \Omega_X((k_0 + *)Y) \)).
\end{itemize}

We shall explain the last Hodge and Atiyah’s definition ([16]) more precisely by use of the fine resolution \( \mathbb{A}^X(*Y) \) of \( \Omega_X(*Y) \), where \( \mathbb{A}^X(*Y) \) denotes the double complex of \( \mathbb{C} \)-vector spaces comprising \( \mathbb{A}^{p,q}_X(*Y) \), the sheaf of germs of semi-meromorphic forms of type \((p, q)\) on \( X \), having poles only along \( Y \). In the same manner as for \( \mathbb{A}^{p,q}_X(\log Y) \), we define \( \mathbb{A}^{p,q}_X(*Y) \) and \( \mathbb{A}^X(*Y) \). We form the complex of \( \mathbb{C} \)-vector spaces \((A^X(*Y), d)\) for \( \mathbb{A}^X(*Y) \). Then we have
\[ I_p(X, *Y) \cong H^p(X, \Omega_X(*Y)) \cong H^p(A^X(*Y)) \quad (p \geq 0). \]
under these isomorphisms, we identify \( I_p(X, *Y) \) with \( H^p(A^X(*Y)) \) in the following. We set
\[ \nu F^k A^X(*Y) := \oplus_{q \geq k} A^{q,p}_X(*Y), \]
then \( \{\nu F^k\}_{k \geq 0} \) give a finite decreasing filtration to \( A^X(*Y) \) and \( A^X(*Y) \) becomes a filtered complex of \( \mathbb{C} \)-vector spaces. We define
\[ I^p_0(X, *Y) := \text{Im} \{ H^p(\nu F^k A^X(*Y)) \to H^p(A^X(*Y)) \cong I^p(X, *Y) \} \]
then we have a filtration on \( I^p(X, *Y) \):
\[ I^p(X, *Y) := I^p_0(X, *Y) \supset I^p_1(X, *Y) \supset \cdots \supset I^p_p(X, *Y) \supset I^p_{p+1}(X, *Y) = \{0\}. \]
Hodge and Atiyah have defined that a closed meromorphic \( p \)-form \( \varphi \), having poles only along \( Y \), is of the \textit{second kind} if its cohomology class \([\varphi] \in I^p(X, *Y)\) belongs to the subspace \( I^p_0(X, *Y) \), i.e., it has the maximum filtration, and they have proved that the definitions (B) and (C) are equivalent in general. They have also proved that the definition (A) is equivalent to other definitions if \( Y \) is a prime section of \( X \).
Replacing (ii) is trivial if we note that (3.6), (iii) is trivial if we note that 

Then we have:

Theorem 2.4.

(i) \( I^p(X, *Y)_0 \simeq r^p H^p(X, \mathbb{C}) \simeq H^p(X, \mathbb{C})_0 \quad (1 \leq p \leq n + 1) \),

where \( r^p : H^p(X, \mathbb{C}) \rightarrow H^p(X - Y, \mathbb{C}) \) is the map induced by restricting closed forms on \( X \) to \( X - Y \),

(ii) \( I^p(X, *Y)_0 = I^p(X, *Y) \quad 1 \leq p \leq n \),

(iii) \( I^n(X, *Y)/I^n(X, *Y)_0 \simeq \ker \{ H^{n-1}(Y, \mathbb{C})_0 \xrightarrow{G^{n-1}} H^{n+2}(X, \mathbb{C}) \} \),

where \( G^{n-1} \) denotes the Gysin map.

Proof. Replacing \( H^*(A_X(\log Y)) \) by \( I^*(X, *Y) \) in the exact sequence (2.6), we obtain the exact sequence

\[
\rightarrow H^p(A_X) \xrightarrow{\rho_p} I^p(X, *Y) \xrightarrow{R^p} H^{p-1}(A_Y) \xrightarrow{Gr^{p-1}} H^{p+1}(A_X) \rightarrow \cdots ,
\]

which is dual to the homology sequence in (2.7). By the Résumé definition of the second kind, we have

\[
I^p(X, *Y)_0 \simeq \text{Ann}(R_{p-1}(H_{p-1}(Y, \mathbb{C}))),
\]

where the right hand side above denotes the annihilator subspace of \( I^p(X, *Y) \) by \( R_{p-1}(H_{p-1}(Y, \mathbb{C})) \) through the pairing defined by integration between \( I^p(X, *Y) \) and \( H^p(Y - X, \mathbb{C}) \). By the duality between (2.8) and (2.7),

\[
\text{Ann}(R_{p-1}(H_{p-1}(Y, \mathbb{C}))) = r^p H^p(A_X) \simeq r^p H^p(X, \mathbb{C}).
\]

By Proposition 1.11, \( r^p H^p(X, \mathbb{C}) \simeq H^p(X, \mathbb{C})_0 \). Thus we have proved (i). By (i), (ii) follows from that \( r^p : H^*(X, \mathbb{C}) \rightarrow H^*(X - Y, \mathbb{C}) \) is surjective for \( 0 \leq q \leq n \) (cf. (1.10) and (1.11)). By the duality between (2.8) and (2.7), (iii) is trivial if we note that \( H^p(I^p(X, *Y)) \subset H^{p-1}(X - Y, \mathbb{C})_0 \) (Theorem 1.15).

Remark 2.2. As in the case of \( A_X(\log Y) \), we define a finite decreasing filtration \( \{ nF_k \}_{k \geq 0} \) on the complex \( A_X(\log Y) \) by

\[
^nF_k A_X(\log Y) := \oplus_{q \geq k} A^q_X(\log Y)
\]

Then, as is well known in the homological algebra, there arises a spectral sequence from the filtered complex \( (A_X(\log Y), F^p) \) as follows:

\[
E_2^{p,q} := H^p(X, \mathcal{H}^q(\Omega_X(\log Y))) \Rightarrow E_\infty^{p,q} = Gr_{F^p, F^{p+q}}(X, \Omega_X(\log Y)) = Gr_{F^p, F^{p+q}}(X, *Y),
\]

where \( \mathcal{H}^q(\Omega_X(\log Y)) \) (\( q \geq 0 \)) are the cohomology sheaves of the complex of sheaves \( \Omega_X(\log Y) \). From Lemma 2.1 it follows

\[
E_2^{p,q} = \begin{cases} H^p(X, \mathbb{C}) & q = 0 \\ H^p(X, \mathbb{C}) & q = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Hence we have

\[
E_2^{p,q} = E_2^{p+1,q} = \cdots = E_2^{p+1,q} = Gr_{F^p, F^{p+q}}(X, *Y) = 0
\]

for \( q \neq p, p - 1, \) and \( r \geq 2 \)

This amounts to

\[
I^p(X, *Y) = I_{00}^p(X, *Y) \Rightarrow I_0^p(X, *Y) = \cdots = I_{p-1}^p(X, *Y),
\]

namely, the filtration of \( I^p(X, *Y) \) induced by \( \{ nF_k \}_{k \geq 0} \) of \( A_X(\log Y) \) is given by a single subspace \( I_k^p(X, *Y) \). From this we can derive the following exact sequence (cf. [7] Chapitre I, Théorème 4.6.2, p.85):

\[
\ldots \longrightarrow E_2^{-2,1} \xrightarrow{d_2^{-1}} E_2^{0,0} \xrightarrow{i^p} \ldots \\
\underline{\Rightarrow} \underline{\Rightarrow} \underline{\Rightarrow} \underline{\Rightarrow} \\
\ldots \longrightarrow H^{p-2}(Y, \mathbb{C}) \longrightarrow H^p(X, \mathbb{C}) \longrightarrow \ldots
\]
We call the attention of the readers to that Ω

mixed Hodge complex (CMHC) in the sense of Deligne and it induces mixed Hodge structures (MHS) on

Mixed Hodge structures on Hodge-Atiyah's algebraic definition

subspace is nothing but out only on a topological space Y,

(iii) A bifiltered complex (K, W, F) ∈ Ob D+ F(X, C) (W increasing and F decreasing) and α : (K, W) ≃ (K, W) ⊗ C in D+ F(X, C), i.e., GrW(K) and GrF(K) are quasi-isomorphic as graded complexes, satisfying the following axioms:
(A) $\mathbb{R}\Gamma(X, Gr^W_k K) \cong \mathbb{R}\Gamma(X, Gr^W_k K) \otimes \mathbb{C}$ is a Hodge complex (HC) of weight $k$,

where HC of weight $k$ is defined as follows: A Hodge complex (HC) $K$ of weight $k$ is given by

(i) A complex $K \in \text{Ob} \ D^+(X, \mathbb{Z})$ such that the cohomology $H^q(K)$ is a $\mathbb{Z}$-module of finite type for each $q$.

(ii) A filtered complex $(K, F) \in \text{Ob} \ D^+ FC$ and an isomorphism $\alpha : K \cong K \otimes \mathbb{C}$ in $D^+ \mathbb{C}$, satisfying the following axioms:

(AI) The differential $d$ of $K$ is strictly compatible to the filtration $F$, i.e., $F^i \cap \text{Im} \ d = \text{Im} \ (d/F^i)$ or equivalently the spectral sequence defined by $(K, F)$ degenerates at $E_1$ ($E_1 = E_\infty$).

(AII) The filtration $F$ induced on $H^q(K) \cong H^q(K) \otimes \mathbb{C}$ defines a HS of weight $q + k$.

In our case, we take $K \in \text{Ob} \ D^+(X, \mathbb{Z})$, $(K, W) \in \text{Ob} \ D^+ F(X, \mathbb{Q})$ and $(K, W, F) \in \text{Ob} \ D^+ F_2(X, \mathbb{C})$ in the definition above as follows:

$K := \mathbb{R} j_* \mathbb{Z}$,

where $j : X - Y \hookrightarrow X$ is the open immersion,

$K_\mathbb{Q} := \mathbb{R} j_* \mathbb{Q}_{X - Y}$,

$W_p(K_\mathbb{Q}) := \tau_{\leq p}(K_\mathbb{Q})$,

where $\tau_{\leq p}(K_\mathbb{Q})$ denotes the subcomplex of $K_\mathbb{Q}$ defined by

$$\tau_{\leq p}(K_\mathbb{Q})^n = \begin{cases} K^n & q = 0 \\ \ker d & q = 1 \\ 0 & n > p \end{cases}$$

(which we call the canonical filtration)

$K : = \Omega_X(\log Y)$,

$W_0(K) = \Omega_X$,

$W_1(K) = \Omega_X(\log Y)$,

$F^q(K) := \sigma_{\geq q}(\Omega_X(\log Y))$,

where $\sigma_{\geq q}(\Omega_X(\log Y))$ denotes the subcomplex of $\Omega_X(\log Y)$ defined by

$$(\sigma_{\geq q}(\Omega_X(\log Y)))^\ell = \begin{cases} 0 & \ell < q \\ \Omega_X^\ell(\log Y) & q \leq \ell, \end{cases}$$

which we call the stupid filtration. Instead of the filtration $W$, we shall use the filtration $W[q]$ defined by

$W[q]_p := W_{-q} - q$,

namely, a shift by $q$ to the right on the degree of $W$. Then $(W[q], F)$ induces a mixed Hodge structure on $H^q(\mathbb{R}\Gamma(X, \Omega_X(\log Y)) : = H^q(X, \Omega_X(\log Y)) \cong I^q(X, *Y)$. We shall calculate $Gr^W_k I^q(X, *Y) (k = q, q + 1)$ by use of spectral sequences. We put

$K := A_X(\log Y)$, and

$W_0(K) = A_X$, $W_1(K) = A_X(\log Y)$.

$\{W_0(K) \subset W_1(K) = A_X(\log Y)\}$ is the filtration induced by the filtration $\{W_0 \subset W_1 = \Omega_X(\log Y)\}$ on $\Omega_X(\log Y)$. We define

$W_p(K) := W_q - q(K) = W_{-q}(K)$ \hspace{1em} (p \leq -q).
Then \( \{W'_p(K')\} \) is a decreasing filtration of \( K' \). Hence we can consider the spectral sequence concerning the filtration complex \((K', W'(K'))\), whose 0-th term and 1-st one are computed as follows:

\[
W'_p E^{r,s}_0 = Gr_r^{W'_p(K^{r+s})} W_0(K^{s-q}) \\
= \begin{cases} 
W_1(K^{s-q-1})/W_0(K^{s-q-1}) & r = q-1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
A_X^{s-1}(\log Y)/A_X^{s-2} \simeq A_Y^{s-2} \\
= \begin{cases} 
A_X^{s-1}(\log Y)/A_X^{s-2} & r = q-1 \\
0 & \text{otherwise}
\end{cases}
\]

where the isomorphism \( A_X^{s-1}(\log Y)/A_X^{s-2} \simeq A_Y^{s-2} \) comes from the exact sequence of sheaves

\[
0 \to W_X \to A_X(\log Y) \xrightarrow{R} A_Y[-1] \to 0,
\]

(cf. Proposition 1.7) which is the \( C^\infty \) version of the exact sequence (2.4);

\[
w'_r E^{r,s}_1 = \frac{\ker \{w'_r E^r_s \xrightarrow{d_1} w'_r E^{r+1,s}_1\}}{\text{im} \{w'_r E^{r-1,s} \xrightarrow{d_1} w'_r E^r_s\}}
\]

\[
= \begin{cases} 
H^{s-q}(A_X) \simeq H^{s-q}(X, \mathbb{C}_X) & r = q, s \geq q \\
H^{s-q-1}(A_X(\log Y)/A_X) \simeq H^{s-q-1}(A_X(\log Y)/W_X) \\
\simeq H^{s-q-2}(A_Y) \simeq H^{s-q-2}(Y, \mathbb{C}_Y) & r = q-1, s \geq q+1, \\
0 & \text{otherwise}
\end{cases}
\]

Hence we have

\[
w'_r E^{r,p-r} = w'_r E^{r,p-r} = \cdots = w'_r E^{r+1,p-r} = Gr_r^{W'_r} \mathcal{P}(X, \ast Y) = 0
\]

for \( r \neq -q, -q-1 \) and \( t \geq 2 \)

This is equivalent to \( \mathcal{P}(X, \ast Y) = W'_r(\mathcal{P}(X, \ast Y)) \supset W'_{r-1}(\mathcal{P}(X, \ast Y)) = E_{\infty}^{r,q+p} \). From these we obtain the following exact sequence:

\[
\cdots \to w'_r E^{q,p}_0 \xrightarrow{i^p} w'_r E^{p,q}_0 \xrightarrow{j^p} w'_r E^{q-1,p+1}_0 \xrightarrow{d_1^{p+1}} w'_r E^{q,p+1}_1 \xrightarrow{i^{p+1}} w'_r E^{p,q+1}_1 \xrightarrow{j^{p+1}} w'_r E^{q+1,p+2} \cdots
\]

(3.2)

where the maps in this diagram are described as follows:

(i) \( d_1^{p+1} \) is the differential at the first term of \( \{E^{q-1,p+1}_1\} \) of the spectral sequence,
Theorem 3.1. Therefore, combining these results with those of Theorem 2.4, we have

\[ W' E_r^{q+p} = \frac{\text{Ker}\{W_r E_r^{q+p} \to W_r E_r^{q+r,p-r+1}\}}{\text{Im}\{W_r E_r^{q-r,p+r-1} \to W_r E_r^{q,p}\}} \]

\[ = \begin{cases} 
W_r E_r^{q,p} / \text{Im}\{W_r E_r^{q,1,p} \to W_r E_r^{q,p}\}, & r = 1 \\
W_r E_r^{q,p}, & r \geq 2,
\end{cases} \]

there is a surjection from \( W'E_1^{q,p} \) onto \( W' E_\infty^{q,p} = \text{Gr}_{W'}^{q} I^{p-q}(X,*Y) = W'_{q+p} I^{p-q}(X,*Y) \). The map \( i^p \) is the composite of this surjection and the natural injection \( W'_{q} I^{p-q}(X,*Y) \to I^{p-q}(X,*Y) = W'E_\infty^{q,p} \).

(iii) Since

\[ W' E_r^{q-1,p+1} = \frac{\text{Ker}\{W_r E_r^{q-1,p+1} \to W_r E_r^{q-1+r,p-r+2}\}}{\text{Im}\{W_r E_r^{q-1-r,p+r} \to W_r E_r^{q-1,p+1}\}} \]

\[ = \begin{cases} 
\text{Ker}\{W_r E_r^{q-1,p+1} \to W_r E_r^{q,p+1}\}, & r = 1 \\
W_r E_r^{q-1,p+1}, & r \geq 2,
\end{cases} \]

there is an injection from \( W'E_\infty^{q-1,p+1} = \text{Gr}_{W'}^{q-1} I^{p-q}(X,*Y) \) into \( W'E_\infty^{q-1,p+1} \). The map \( j^p \) is the composite of the natural surjection \( W'E_\infty^{q-1} = I^{p-q}(X,*Y) \) onto \( W'E_\infty^{q-1,p+1} = \text{Gr}_{W'}^{q-1} I^{p-q}(X,*Y) \) and the injection above from \( W'E_\infty^{q-1,p+1} \) into \( W'E_1^{q-1,p+1} \).

Chasing these maps more precisely by direct calculation, using differential forms, we can conclude that the exact sequence (2.7) is dual to the homology sequence (2.7). By the definition of the map \( i^p \) and \( j^p \), we have

\[ i^p(W'E_1^{q,p}) = W'_{q} I^{p-q}(X,*Y) \] and
\[ j^p(W'E_\infty^{p+1}) = \text{Gr}_{W'}^{q-1} I^{p-q}(X,*Y), \]

which are rewritten as

\[ r^p-q(H^{p-q}(X,C)) = W'_{q} I^{p-q}(X,*Y) \] and
\[ R^p-q(I^{p-q}(X,*Y)) = \text{Gr}_{W'}^{q-1} I^{p-q}(X,*Y), \]

If we put \( p = 2q \), then we have

\[ H^q(X,C)_0 = r^q(H^q(X,C)) = W'_{q} I^q(X,*Y) = W[q]_q I^q(X,*Y) \]

and

\[ \text{Ker}\{G^{q-1} : H^{q-1}(Y,C)_0 \to F^k H^{q+2}(X,C)\} = R^q I^q(X,*Y) \]

\[ \simeq \text{Gr}_{W'}^{q-1} I^q(X,*Y) \]

\[ \simeq = G^q_{q+1} I^q(X,*Y). \]

Therefore, combining these results with those of Theorem 2.4, we have

**Theorem 3.1.**

(i) \( G^q W[q]_q H^q(X - Y,C) = W[q]_q H^q(X - Y,C) = I^q(X,*Y)_0 \),

(ii) \( G^q W[q] H^q(X - Y,C) = I^q(X,*Y) / I^q(X,*Y)_0 \),

(iii) \( F^k G^q W[q] H^q(X - Y,C) \simeq F^k H^q(X,C)_0 \),
We define a decreasing filtration \( F^k \) by
\[
F^k(\Omega_X((1+\cdot)Y)) := \{ \cdots \to 0 \to \Omega^k_X(Y) \to \Omega^{k+1}_X(2Y) \to \cdots \to \Omega^n_X((n-k+2)Y) \},
\]
and an increasingfiltration \( \{ W'_0 \subset W'_1 \} \) by
\[
W'_0(\Omega_X((1+\cdot)Y)) : \quad \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X,
\]
\[
W'_1(\Omega_X((1+\cdot)Y)) : \quad \Omega_X((1+\cdot)).
\]

Then we have

**Proposition 3.2.** The bi-filtered complexes of sheaves of \( \mathbb{C} \)-vector spaces
\[
(\Omega_X(\log Y), W, F) \quad \text{and} \quad (\Omega_X((1+\cdot)Y), W', F')
\]
are quasi-isomorphic, i.e., bi-graded complex of sheaves of \( \mathbb{C} \)-vector spaces
\[
Gr_F Gr^W(\Omega_X(\log Y)) \quad \text{and} \quad Gr_F Gr^{W'}(\Omega_X((1+\cdot)Y))
\]
are quasi-isomorphic where the filtration \( F \) of \( \Omega_X(\log Y) \) is defined by
\[
F^k(\Omega_X(\log Y)) := \{ \cdots \to 0 \to \Omega^k_X(\log Y) \to \Omega^{k+1}_X(\log Y) \to \cdots \to \Omega^n_X(\log Y) \} \quad (0 \leq k \leq n+1).
\]

**Proof.** First, we have
\[
Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y)) = \Omega_X[-k],
\]
\[
Gr^F_{Gr^W_1}(\Omega_X((1+\cdot)Y)) = (\Omega^{k+1}_X((1+\cdot)Y)/\Omega^k_X((1+\cdot)Y))[-k],
\]
\[
Gr^F_{Gr^W_0}(\Omega_X(\log Y)) = \Omega_X[-k], \quad \text{and}
\]
\[
Gr^F_{Gr^W_1}(\Omega_X(\log Y)) = (\Omega^{k+1}_X(\log Y)/\Omega^k_X(\log Y))[-k]
\]

Thus \( Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y)) \) and \( Gr^F_{Gr^W_0}(\Omega_X(\log Y)) \) are quasi-isomorphic and
\[
H^p(Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y))) = \begin{cases} \Omega^p_X(\log Y)/\Omega^0_X & p = k \geq 1 \\
0 & \text{otherwise.}
\end{cases}
\]

We shall calculate \( H^p(Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y))) \). Obviously,
\[
H^p(Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y))) = 0 \quad \text{for} \quad 0 \leq p \leq k-1, \quad 1 \leq k.
\]

Assume \( p \geq k + 1 \). Let \( [\omega] \in \Omega^p_X((p-k+1)Y)/\Omega^p_X((p-k+1)Y) \) be an element with \( d[\omega] = 0 \) in \( \Omega^p_X((p-k+1)Y)/\Omega^p_X(((p-k+1)Y) \) where \( \omega \) is an element of \( \Omega^{p+1}_X((p-k+1)Y) \). Since \( d\omega \) is a closed form, by Lemma 2.1, (i)-(a), there exists \( \varphi \in \Phi^p_X((p-k+1)Y) \) such that \( d\varphi = d\omega \). Since \( \omega - \varphi \in \Phi^{p+1}_X((p-k+1)Y) \), by the same reason, there exists \( \psi \in \Phi^{p+1}_X((p-k+1)Y) \) such that \( d\psi = \omega - \varphi \). This means \( d[\psi] = [\omega] \). Thus \( H^p(Gr^F_{Gr^W_0}(\Omega_X((1+\cdot)Y))) = 0 \) for \( p \geq k + 1 \). Let \( [\omega] \in \Omega^p_X(Y)/\Omega^p_X \) be an element with \( d[\omega] = 0 \) in
Ω^{k+1}(2Y)/Ω^{k+1}(Y). This amounts to \( dw \in \Omega^{k+1}_X(Y) \). If \( k \geq 1 \), we can easily see that this is the case if and only if \( \omega \in \Omega^k_X(\log Y) \). This fact tells us that

\[
H^k(Gr^k_W Gr^1_W (\Omega_X((1+)Y))) \simeq \Omega^k_X(\log Y)/\Omega^k_X for \quad k \geq 1.
\]

If \( k = 0 \), we can easily see that \( \omega \in \mathcal{O}_X \), since \( \omega \in \mathcal{O}_X(Y) \), \( dw \in \Omega^1_X(Y) \). Hence \( H^0(Gr^0_W Gr^1_W (\Omega_X((1+)Y))) = 0 \). This completes the proof. \( \square \)

We define

\[
I_k^p(X, (p+1)Y) := \Gamma(X, \Phi^p((p-k+1)Y)) = 0 \quad (0 \leq k \leq p)
\]

and denote by \( I_k^p(X, (p+1)Y)_0 \) the subspace of \( I_k^p(X, (p+1)Y) \) generated by closed meromorphic \( p \)-forms of the second kind. The CMHC \( (\Omega_X, W, F) \) induces a mixed Hodge structure on \( H^p(X - Y, \mathbb{C}) \) \( \simeq \mathbb{H}^p(X, \Omega_X(\log Y)) \). We denote by \{\( F^kH^p(X - Y, \mathbb{C}) \}_{0 \leq k \leq p} \) the Hodge filtration of \( H^p(X - Y, \mathbb{C}) \) concerning this mixed Hodge structure, and by \{\( F^kH^p(X, \mathbb{C})_0 \)\}_{0 \leq k \leq p} \) the ordinary Hodge filtration of \( H^p(X, \mathbb{C})_0 \), the \( p \)-th primitive cohomology group of \( X \). With this notation we have

**Theorem 3.3.** If \( Y \) is sufficiently ample so that

\[
H^p(X, \Omega^k_X((k)Y)) = 0 \quad for \quad p \geq 1, q \geq 0, k \geq 1,
\]

then we have

\[
F^kH^p(X - Y, \mathbb{C}) \simeq I_k^p(X, (p+1)Y) \quad 0 \leq k \leq p \quad and
\]

\[
F^kH^p(X, \mathbb{C})_0 \simeq I_k^p(X, (p+1)Y)_0 \quad 0 \leq k \leq p
\]

under the isomorphisms \( H^p(X - Y, \mathbb{C}) \simeq I^p(X, (p+1)Y) \) and \( H^p(X, \mathbb{C})_0 \simeq I^p(X, (p+1)Y)_0 \) in Proposition 2.3 and Theorem 2.4, respectively.

**Proof.** Using the sheaves \( \mathfrak{A}^{p,q}_X(Y) \), the sheaves of germs of semi-meromorphic forms of type \( (p, q) \) on \( X \), having poles of order \( \ell \) (at most) along \( Y \), we can form a fine resolution of it by use of more small sheaves. Let \( \mathfrak{B}^{p,q}_X(Y) \) be the subsheaves of \( \mathfrak{A}^{p,q}_X(Y) \) characterized by the following prescription: Letting \( \varphi \) be a local cross-section of \( \mathfrak{B}^{p,q}_X(Y) \) if and only if \( f^{\ell-1}df \wedge \varphi \) is a \( C^\infty \) regular differential form where \( f = 0 \) is a local holomorphic defining equation for \( Y \). Using \( \mathfrak{B}^{p,q}_X(Y) \), we obtain a fine resolution of \( \Omega_X((1+)Y) \) as follows:

\[
\begin{array}{cccccccc}
\mathfrak{B}^{0,1}_X(Y) & \xrightarrow{\partial^{0,1}} & \mathfrak{B}^{1,1}_X(2Y) & \xrightarrow{\partial^{1,1}} & \mathfrak{B}^{2,1}_X(3Y) & \xrightarrow{\partial^{2,1}} & \cdots & \xrightarrow{\partial^{n-1}} & \mathfrak{B}^{n+1,1}_X((n+2)Y) \\
\mathfrak{B}^{0,0}_X(Y) & \xrightarrow{\partial^{0,0}} & \mathfrak{B}^{1,0}_X(2Y) & \xrightarrow{\partial^{1,0}} & \mathfrak{B}^{2,0}_X(3Y) & \xrightarrow{\partial^{2,0}} & \cdots & \xrightarrow{\partial^{n,0}} & \mathfrak{B}^{n+1,0}_X((n+2)Y) \\
\Omega^1_X(Y) & \xrightarrow{d} & \Omega^1_X(2Y) & \xrightarrow{d} & \Omega^1_X(3Y) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n+1}_X((n+2)Y) \\
0 & \downarrow & 0 & \downarrow & 0 & \downarrow & \cdots & \downarrow & 0
\end{array}
\]

We put

\[
B^{p,q}_X((p+1)Y) := \Gamma(X, \mathfrak{B}^{p,q}_X((p+1)Y)) \quad (p \geq 0, q \geq 0),
\]

\[
B^{k}_X((k+1)Y) := \oplus_{p+q=k} B^{p,q}_X((p+1)Y) \quad d^{p,q} = \partial^{p,q} + (-1)^p \partial^{q,p} \quad and
\]

\[
B_X((1+)Y) := \oplus_{k} \oplus_{p+q=k} B^{p,q}_X((p+1)Y)
\]
Here we should recall that the filtration \( F_{\Omega} \) of \( \Omega_X \) defined in (3.3) induces a filtration on \( B_X((1+\cdot)Y) \), which we denote by \( \{ F^k B_X((1+\cdot)Y) \} \), i.e.,

\[
F^k B_X((1+\cdot)Y) := \bigoplus_{p} \bigoplus_{p \geq q \geq k} B^p_X(p-q)((q+1)Y)
\]

Since \( \Omega_X((1+\cdot)Y), W, F' \) is a CMHC by Proposition 3.2, the spectral sequence, associated to the filtration \( \{ F^k B_X((1+\cdot)Y) \} \) and whose final terms are

\[
F^p E^{p,q}_\infty = Gr^p_{F'} = Gr^p_{F'} H^{p+q}(B_X((1+\cdot)Y)),
\]

is degenerated at the 1-st term (cf. [2], Théorème 3.2.5, [4], Théorème 3.2.1). Therefore, we have

\[
F^k H^p(B) := \text{Im}\{ H^p(F^k(B)) \to H^p(B) \} \quad \text{and}

\[
Gr^k_{F'} H^p(B) = F^k H^p(B)/F^{k+1} H^p(B)
\]

From this and (3.8) it follows that the natural map

\[
H^p(F^k(B)) \to H^p(F^k(B)/F^{k+1}(B))
\]

is surjective. Hence the long exact sequence of cohomology associated to the exact sequence of complex

\[
0 \to F^{k+1}(B) \to F^k(B) \to F^k(B)/F^{k+1}(B) \to 0
\]

breaks up into the following short exact sequences

\[
0 \to H^p(F^{k+1}(B)) \to H^p(F^k(B)) \to H^p(F^k(B)/F^{k+1}(B)) \to 0
\]

\((0 \leq p \leq n, 0 \leq k \leq p)\)

Here \( H^p(F^k(B)/F^{k+1}(B)) \simeq Gr^k_{F'} H^p(B) \). Hence

\[
H^p(F^k(B)) \simeq F^k H^p(B) \simeq F^k H^p(X - Y, \mathbb{C}) \quad (0 \leq k \leq p, 0 \leq p \leq n + 1).
\]

On the other hand, by the assumption 3.4, we have

\[
H^p(F^k(B)) \simeq H^p(F^k(\Omega_X((1+\cdot)Y)))
\]

\[
= H^p(\Omega^k_X((1+\cdot)Y) - k])
\]

\[
= H^{p-k}(\Omega^{k+1}_X((1+\cdot)Y))
\]

\[
\simeq \frac{\Gamma(X, \Phi^p_X((p-k+1)Y))}{d\Gamma(X, \Omega^{p-1}_X((p-k)Y))}
\]

\[
= I^p_k(X, (p+1)Y).
\]

By Proposition 3.2, the ordinary Hodge filrtation \( F^p H^p(X - Y, \mathbb{C}) \) of the cohomology \( H^p(X - Y, \mathbb{C}) \) coincides with \( F^p H^p(X - Y, \mathbb{C}) \). Therefore, by (3.9) and (3.10), we conclude that (3.5) certainly holds. Noticing that \( I^p(X, (p+1)Y)_0 \simeq I^p(X, *Y)_0 \), we obtain (3.6) from (3.5) and Theorem 3.1. \( \square \)
4 Generalized Poincaré résidue map

The setting under which we shall work in this section is as follows: Let $X$ be a non-singular irreducible algebraic variety of dimension $n+1$ embedded in a sufficiently higher complex projective space $\mathbb{P}^N$, $Y$ a generic hyperplane section of $X$ which satisfies the condition (3.4) in Theorem 3.3, and $Y'$ a non-singular, irreducible hypersurface section of sufficiently higher degree such that if we set $Z = Y \cdot Y'$, then

\[(4.1) \quad H^p(Y, \Omega^k_Y(kZ)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1.\]

When we refer to primitive cohomology, we always means the one concerning the Hodge metric whose fundamental forms is dual to the homology class $[Y]$ (resp. $[Z]$). Under this setting and with the same notation as in the previous sections, the purpose of this section is to define the so-called generalized Poincaré residue map

\[
\text{Rés} : I^{n+1}(X, (n+2)Y) \to I^n(Y, (n+1)Z)_0
\]

and prove the following theorem:

Theorem 4.1. Under the setting above, we have

\[
F^k H^n(Y, \mathbb{C})_0 \simeq I^n_k(Y, (n+1)Z)_0 \\
\simeq \text{Rés}(I^{n+1}_k(X, (n+2)Y)) \oplus r^n(I^n_k(X, (n+1)Y')_0),
\]

where $r^n$ denote the map induced by the natural map $H^n(X, \mathbb{C})_0 \to H^n(Y, \mathbb{C})_0$.

We shall prove the theorem after several lemmas and Propositions. We denote by $\Omega^q(kY + *Y')$ the sheaf of germs of meromorphic $q$-forms having poles of order $k$ (at most) along $Y$ and poles of arbitrary order along $Y'$ as their only singularities. We denote by $\Omega^q(\log Y + kY')$ the sheaf of germs of meromorphic $q$-forms having logarithmic poles along $Y$ and poles of order $k$ at most as their only singularities. We consider the following homomorphisms of complexes of sheaves of $\mathbb{C}$-vector spaces:

\[
\begin{array}{ccccccc}
\Omega_X((1+\cdot)Y) : & \mathcal{O}_X(Y) & \longrightarrow & \Omega^1_X(2Y) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\Omega_X((1+\cdot)Y + *Y') : & \mathcal{O}_X(Y + *Y') & \longrightarrow & \Omega^1_X(2Y + *Y') & \longrightarrow & \cdots \\
\mathcal{O}_X(\log Y + (1+\cdot)Y') : & \mathcal{O}_X(Y') & \longrightarrow & \Omega^1_X(\log Y + 2Y') & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\Omega^p_X((p+1)Y) & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}_X((n+2)Y) \\
\downarrow & & \downarrow & & \downarrow & \\
\Omega^p_X((p+1)Y + *Y') & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}_X((n+2)Y + *Y') \\
\downarrow & & \downarrow & & \downarrow & \\
\Omega^p_X(\log Y + (p+1)Y') & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}_X(\log Y + (n+2)Y')
\end{array}
\]

Proposition 4.2. The homomorphism of complexes of sheaves

\[
\Omega_X(\log Y + (1+\cdot)Y') \to \Omega_X((1+\cdot)Y + *Y')
\]

in the diagram (4.2) is a quasi-isomorphism.

Proof. By virtue of Proposition 2.2 it suffices to show that the stalks of the cohomology sheaves $\mathcal{H}^p(\Omega_X(\log Y + (1+\cdot)Y'))$ and $\mathcal{H}^p(\Omega_X((1+\cdot)Y + *Y'))$ are isomorphic at a point $x_0 \in Y \cap Y'$. Let $(z_1, \ldots, z_{n+1})$ be a
holomorphic local coordinate system at \( x_0 \) such that \( z_1 = 0 \) and \( z_2 = 0 \) are local defining equations \( Y \) and \( Y' \), respectively. We are going to show that

\[
\mathcal{H}^p(\Omega_X((1+\cdot)Y + *Y'))
\]

(4.3)

\[
\simeq \mathcal{H}^p(\Omega_X(\log Y + (1+\cdot)Y')) = \begin{cases} 
\mathbb{C} \quad & p = 0 \\
\mathbb{C}\{\frac{dz_1}{z_1}, \frac{dz_2}{z_2}\} \quad & p = 1 \\
\mathbb{C}\{\frac{dz_1dz_2}{z_1z_2}\} \quad & p = 2 \\
0 \quad & \text{otherwise}
\end{cases}
\]

Now let \( \varphi = dz_1 \land \alpha + \beta \) be a local cross-section of \( \Phi^p_X((p + 1)Y + *Y') \) \( (p \geq 1) \) in a neighborhood of \( x_0 \), where \( \alpha, \beta \) are local meromorphic forms, having poles of order \( p + 1 \) (at most) along \( Y \) and poles of arbitrary order along \( Y' \) as their only singularities, and not involving \( dz_1 \). Then we may write

\[
\alpha = \alpha_0 + \frac{\alpha_1}{z_1} + \frac{\alpha_2}{z_1^2} + \cdots + \frac{\alpha_{p+1}}{z_1^{p+1}} \\
\beta = \beta_0 + \frac{\beta_1}{z_1} + \frac{\beta_2}{z_1^2} + \cdots + \frac{\beta_{p+1}}{z_1^{p+1}}
\]

where \( \alpha_i, \beta_i \ (i \geq 1) \) do not involve \( z_1 \) and \( dz_1 \), and \( \alpha_i, \beta_i \ (i \geq 0) \) have poles of arbitrary order (at most) along \( Y' \) as their only singularities. Since \( d\varphi = 0 \), we have

\[
d\varphi = -dz_1 \land d\alpha_0 + d\beta_0 - \frac{dz_1 \land d\alpha_0 + d\beta_1}{z_1} - \frac{dz_1 \land (d\alpha_2 + \beta_1) - d\beta_2}{z_1^2}
\]

\[
\cdots - \frac{dz_1 \land (d\alpha_{p+1} + \beta p) - d\beta_{p+1}}{z_1^{p+1}} - (p + 1) \frac{dz_1 \land \beta_{p+1}}{z_1^{p+2}}
\]

\[
= 0.
\]

Hence,

\[
d\alpha_1 = d\alpha_2 + \beta_1 = d\alpha_3 + 2\beta_2 = \cdots = d\alpha_{p+1} + p\beta_p = 0, \\
(p + 1)\beta_{p+1} = 0, \\
d\beta_1 = d\beta_2 = \cdots = d\beta_{p+1} = 0, \\
d\varphi_0 = 0, \quad \text{where} \quad \varphi_0 = dz_1 \land \alpha_0 + \beta_0.
\]

Put

\[
\theta = -\frac{\alpha_2}{z_1} - \frac{\alpha_3}{2z_1^2} \cdots - \frac{\alpha_{p+1}}{pz_1^p},
\]

then

\[
\varphi = d\theta + \frac{dz_1}{z_1} \land \alpha_1 + \varphi_0, \quad \text{and} \quad d\varphi_0 = 0.
\]

Hence if \( p \geq 3 \), since \( d\alpha_1 = d\varphi_0 = 0 \), there exist local cross-sections \( \gamma \) of \( \Omega^{p-2}(*Y') \) and \( \varphi_1 \) of \( \Omega^{p-1}(*Y') \) with \( d\gamma = \alpha_1 \) and \( d\varphi_1 = \varphi_0 \) in a neighborhood of \( x_0 \). Put

\[
\theta_1 = \frac{dz_1}{z_1} \land \gamma + \varphi_1,
\]

then \( \theta + \theta_1 \) is a local cross-section of \( \Omega^{p-1}(pY + *Y') \) and \( \varphi = d(\theta + \theta_1) \). This shows that \( \mathcal{H}^p(\Omega_X((1+\cdot)Y + *Y')) = 0 \) for \( p \geq 3 \). If \( p = 2 \), \( \alpha_1 \) in the expression (4.5) of \( \varphi \) is a local cross-section of \( \Phi^1(*Y') \).
Hence, as shown in the proof of Lemma 2.1 (ii)-(c), there exists a constant $\lambda \in \mathbb{C}$ and a local cross-section $\gamma$ of $\Omega^0_X(\ast Y')$ with
\[
\alpha_1 = \lambda \frac{dz_2}{z_2} + d\gamma.
\]
Furthermore, since $\varphi_0$ is a local cross-section of $\Phi^2(\ast Y')$, by Lemma 2.1 (i)-(a), there exists a local cross-section $\varphi_1$ of $\Omega^1(\ast Y')$ with $d\varphi_1 = \varphi_0$. Put
\[
\theta_1 = \frac{dz_1}{z_1} \wedge \gamma + \varphi_1
\]
then $\theta + \theta_1$ is a local cross-section of $\Omega^1(2Y + \ast Y')$ at $x_0$ and
\[
\varphi = d\theta + \lambda \frac{dz_1 \wedge dz_2}{z_1 z_2} + \frac{dz_1}{z_1} \wedge d\gamma + \varphi_0
\]
\[
= \lambda \frac{dz_1 \wedge dz_2}{z_1 z_2} + d(\theta + \theta_1)
\]
This shows that
\[
\mathcal{H}^2(\Omega^1_X((1 + \cdot)Y + \ast Y'))_{x_0} \approx \mathbb{C}\{ \frac{dz_1 \wedge dz_2}{z_1 z_2} \}.
\]
If $p_1 = 1$, $\alpha_1$ is a meromorphic function, hence $d\alpha_1 = 0$ implies that $\alpha_1 = \lambda$, a constant. Since $\varphi_0$ is a local cross-section of $\Phi^1(\ast Y')$, by Lemma 2.1 (ii)-(c), there exists $\varphi_1 \in \Omega^0(\ast Y')$ at $x_0$ such that
\[
\varphi_0 = \mu \frac{dz_2}{z_2} + d\varphi_1.
\]
Hence the expression of $\varphi$ in (4.5) becomes
\[
\varphi = \lambda \frac{dz_1}{z_1} + \mu \frac{dz_2}{z_2} + d(\varphi_1 + \theta).
\]
Since $\varphi_1 + \theta \in \Omega^1(Y + \ast Y')$, this shows
\[
\mathcal{H}^1(\Omega^1_X((1 + \cdot)Y + \ast Y')) \approx \mathbb{C}\{ \frac{dz_1}{z_1}, \frac{dz_2}{z_2} \}.
\]
\[
\mathcal{H}^0(\Omega^1_X((1 + \cdot)Y + \ast Y')) \approx \mathbb{C}_X \text{ is obvious. To prove the same for } \mathcal{H}^p(\Omega^1_X(\log Y + (1 + \cdot)Y')) \text{ is rather easy.}
\]
If $\varphi$ is a local cross-section of $\Phi^p(\log Y + (p + 1)Y')$ in a neighborhood of $x_0$, then $\varphi$ is written as
\[
\varphi = \frac{dz_1}{z_1} \wedge \alpha + \beta,
\]
where $\alpha \in \Omega^{p-1}((p + 1)Y')$, $\beta \in \Omega^p((p + 1)Y')$ do not involve $z_1$. Furthermore, we may assume that $\alpha$ does not involve $z_1$. Then $d\varphi$ implies $d\alpha = d\beta = 0$, and by the same arguments as in the case of $\Omega^1_X((p + 1)Y + \ast Y')$, we can show that (4.3) for $\mathcal{H}^p(\Omega^1_X(\log Y + (p + 1)Y'))$. 

**Lemma 4.3.** Assume we are under the setting at the beginning of this section. Particularly, we assume that the following conditions are satisfied:
\[
H^p(X, \Omega^3_X(kY)) = 0,
\]
\[
H^p(Y, \Omega^3_Y(kZ)) = 0 \quad \text{for } p \geq 1, q \geq 0, k \geq 1.
\]
Then we have
\[
H^p(X, \Omega^3_X((q + 1)Y')) = 0 \quad \text{for } p \geq 1, q \geq 0.
\]
Proof. We consider the following exact sequence
\[ 0 \to \Omega^q_X \to \Omega^q_X((\log Y)^{(p+1)}) \xrightarrow{R} \Omega^{p-1}_Y \to 0 \quad (q \geq 1), \]
where \( R \) is the residue map (cf. Lemma 2.1 (ii)-(c)). Tensoring \( \mathcal{O}_X((q+1)Y') \) to this exact sequence, we have
\[ 0 \to \Omega^q_X((q+1)Y') \to \Omega^q_X((\log Y + (q+1)Y') \to \Omega^{p-1}_Y((q+1)Z) \to 0. \]
From the long exact sequence of cohomology associated to this sequence, the assertion of the lemma follows. 

We define
\[
I^p(X, \log Y + (p+1)Y') := \frac{\Gamma(X, \Phi^p_X((\log Y + (p+1)Y'))}{d\Gamma(X, \Omega^{p-1}_X((\log Y + pY'))},
\]
\[
I^p(X, (p+1)Y + *Y') := \frac{\Gamma(X, \Phi^p_X((p+1)Y + *Y'))}{d\Gamma(X, \Omega^{p-1}_X(pY + *Y'))}.
\]
Combining Proposition 4.2 with Lemma 4.3 implies the following:

**Proposition 4.4.** Assume that we are under the setting at the beginning of this section. Then
\[
I^p(X, \log Y + (p+1)Y') \simeq I^p(X, (p+1)Y + *Y') \quad \text{for} \quad p \geq 0.
\]

We are now ready to define the Résidue map
\[ R\ell s : I^p(X, (p+1)Y) \to I^{p-1}(Y, pZ)_0 \]
Let \( \omega \in \Gamma(X, \Phi^p_X((p+1)Y)) \) be given. We think of \( \omega \) as an element of \( \Gamma(X, \Phi^p_X((p+1)Y + *Y')) \). Then, by Proposition 4.4, there exists a \( \varphi \in \Gamma(X, \Omega^{p-1}_X((pY + *Y')) \) such that \( \omega - d\varphi \in \Gamma(\Phi^p_X((\log Y + (p+1)Y'))) \). We take an open covering \( \{U_i\}_{i \in I} \) of \( X \) such that there is a local coordinate system \((z_1, \cdots, z_{n+1})\) on each \( U_i \), satisfying the following conditions:

(a) If \( U_i \cap Y \neq \emptyset, z_1 = 0 \) is a defining equation of \( Y \) in \( U_i \).

(b) If \( U_i \cap (Y \cap Y') \neq \emptyset, z_1 = 0 \) and \( z_2 = 0 \) are defining equations of \( Y \) and \( Y' \) in \( U_i \), respectively.

In each \( U_i \) with \( U_i \cap Y \neq \emptyset \), we can write \( \omega - d\varphi \) as

\[
\omega - d\varphi = \frac{dz_1}{z_1} \land \alpha_i + \beta_i,
\]

where \( \alpha_i \in \Gamma(U_i, \Phi^p_X((p+1)Y')) \), \( \beta_i \in \Gamma(U_i, \Phi^p_X((p+1)Y')) \), \( \alpha_i \) and \( \beta_i \) does not involve \( dz_1 \). We can easily see \( \alpha_{ijY} = \alpha_{ijY'} \) if \( U_i \cap U_j \cap Y \neq \emptyset \), hence \( \{\alpha_{ijY}\} \) defines an element of \( \Gamma(Y, \Phi^p_X((p+1)Z)) \). We claim that \( \{2\pi \sqrt{-1} \alpha_{ijY}\} \) determine a unique element of \( I^{p-1}(Y, (p+1)Z) \), not depending on the choice of \( \varphi \). In fact, if \( \varphi' \) is another element of \( \Gamma(X, \Omega^{p-1}_X((p+1)(Y + Y')) \) with \( \omega - d\varphi' \in \Gamma(X, \Phi^p_X((\log Y + (p+1)Y'))) \) and

\[
\omega - d\varphi' = \frac{dz_1}{z_1} \land \alpha_i' + \beta_i',
\]

is the expression of \( \omega - d\varphi' \) as in (4.7), then

\[
d(\varphi' - \varphi) = \frac{dz_1}{z_1} \land (\alpha_i - \alpha_i') + (\beta_i - \beta_i') \in \Gamma(X, \Phi^p_X((\log Y + (p+1)Y')) \]

is zero in $I^p(X, \log Y + (p+1)Y')$. Hence, by Proposition 4.4, there exists an element $\psi \in \Omega^{p-1}_X(\log Y + pY')$ such that $d\psi = d(\varphi' - \varphi)$. Let

$$\psi = \frac{dz}{z_1} \wedge \gamma_i + \delta_i$$

be the expression of $\psi$ as in (4.7). Then, since $d\psi = d(\varphi' - \varphi)$, we have

$$d\gamma_i|Y = d\gamma_i|Y = \alpha_{i|Y} - \alpha_{i|Y}$$

(4.8)

for each $i$ with $U_i \cap Y \neq \emptyset$, where $d\gamma_i$ denotes the exterior derivative on $Y$. Since $\{\gamma_i|Y\}$ is a global cross-section of $\Gamma(Y, \Omega^{p-1}_X(pZ))$, (4.8) shows that $\{\alpha_{i|Y} \}$ is in $I^p(Y, (p+1)Z)$. Furthermore, the arguments above also show that if $\omega$ is a derived form, then so is $\{\alpha_{i|Y} \}$. Therefore, we conclude that the correspondence

$$\omega \mapsto \{\alpha_{i|Y} \}$$

determine a map $I^p(X, (p+1)Y) \rightarrow I^{p-1}(Y, (p+1)Z)$. Since $I^{p-1}(Y, (p+1)Z) \simeq I^{p-1}(Y, pZ)$ by Proposition 2.3, this map is thought of as a map from $I^p(X, (p+1)Y)$ to $I^{p-1}(Y, pZ)$, which we define to be the generalized Poncaré résidue map and denote it $R\dot{\varepsilon}s$. We denote $\{\alpha_{i|Y} \}$ by $r\dot{\varepsilon}s[\omega]$ (determined up to derived forms) and call résidue form of $\omega$.

**Proposition 4.5.**

$$R\dot{\varepsilon}s(I^p(X, (p+1)Y)) \subset I^{p-1}(Y, pZ)_0$$

**Proof.** For a $\omega \in \Gamma(X, \Phi^p((p+1)Y))$, we shall show that its résidue form $r\dot{\varepsilon}s[\omega] = \{\alpha_{i|Y} \}$ (precisely speaking, a closed form representing the class $r\dot{\varepsilon}s[\omega]$ of $I^p(X, (p+1)Z)\simeq I^{p-1}(Y, pZ)_0$ is of the second kind in the sense of Picard-Lefschetz. From this the assertion of the proposition follows, since $I^{p-1}(Y, (p+1)Z) \simeq I^{p-1}(Y, pZ)_0$. As before we take an open covering $\{U_i\}_{i \in I}$ of $X$ such that there is a local coordinate system $(z_1^i, \cdots, z_{n+1}^i)$ on each $U_i$, subject to the conditions in (4.6), and take a $\varphi \in \Gamma(X, \Omega^{p-1}(pY + *Y'))$ such that $\omega - d\varphi \in \Gamma(X, \Phi^p(\log Y + (p+1)Y'))$. On each $U_i$ with $U_i \cap Y \neq \emptyset$, we write

$$\omega - d\varphi = \frac{dz_i}{z_1} \wedge \alpha_i + \beta_i$$

(4.9)

as in (4.7). We will show that for a point $x_0 \in Z \cap U_i$, $r\dot{\varepsilon}s[\omega]|_{U_i} = \alpha_{i|Y}$ is a holomorphic form modulo derived meromorphic forms in a sufficiently small neighborhood of $x_0$ in $Y$. For this end we take a generic prime hypersurface section $Y''$ which is linearly equivalent to $Y'$, which does not go through $x_0$ and intersect $Y$ and $Y'$ transversely. We think $\omega$ as an element of $\Gamma(X, \Phi^p((p+1)Y + *Y''))$. Since $I^p(X, (p+1)Y + *Y'') \simeq I^p(X, \log Y + (p+1)Y'')$ by Proposition 4.4, there exists a $\varphi' \in \Gamma(X, \Omega^{p-1}(pY + *Y''))$ with $\omega - d\varphi' \in \Gamma(X, \Phi^p(\log Y + (p+1)Y''))$. Let

$$\omega - d\varphi' = \frac{dz_i}{z_1} \wedge \alpha_i' + \beta_i'$$

(4.10)

be the expression of $\omega - d\varphi'$ as in (4.7) on each $U_i \cap Y \neq \emptyset$. If $U_i_{x_0}$ is the coordinate neighborhood with $x_0 \in U_i_{x_0} \cap Z$, since $Y''$ does not go through $x_0$, $\alpha'_{i|Y_0}$ is holomorphic in a sufficiently open neighborhood of $x_0$ in $U_i_{x_0} \cap Y$. From (4.9) and (4.10),

$$d(\varphi' - \varphi) = \frac{dz_{i_{x_0}}}{z_1} \wedge (\alpha_{i_0} - \alpha_{i_0}') + (\beta_{i_0} - \beta_{i_0}')$$

(4.11)

Since $d(\varphi' - \varphi) \in \Gamma(X, \Phi^p(\log Y + *Y'' + Y'''))$ is zero in $I^p(X, (p+1)Y + *Y'' + Y''')$, by Proposition 4.4, there exists a $\psi \in \Gamma(X, \Omega^{p-1}(\log Y + *Y'' + Y'''))$ with $d\psi = d(\varphi' - \varphi)$. On each $U_i$, we write

$$\psi = \frac{dz_i}{z_1} \wedge \gamma_i + \xi_i$$

(4.12)

as in (4.7). Then $d\psi = d(\varphi' - \varphi)$ implies

$$d\gamma_i = \alpha_i - \alpha_i'$$

Hence $d\gamma_i|Y = \alpha_{i|Y} - \alpha'_{i|Y}$ for each $i$ where $d\gamma_i$ denotes the exterior derivation on $Y$. This means $d\gamma_i|Y = \alpha_{i|Y} - \alpha'_{i|Y}$ where $r\dot{\varepsilon}s[\psi] \in \Gamma(Y, \Omega^{p-2}_X((pY' + Y''')))$. Since $r\dot{\varepsilon}s[\alpha']$ is holomorphic at $x_0$, so is $r\dot{\varepsilon}s[\alpha]$ modulo derived meromorphic forms as required. $\square$
Proof of Theorem 4.1:

We can now easily deduce Theorem 4.1 from what we have proved till now. First, by Theorem 1.15,
\[ H^n(Y, \mathbb{C})_0 = R^{n+1}(H^{n+1}(Y, \mathbb{C})) \oplus r^n(H^n(X, \mathbb{C})_0). \]

By Theorem 3.1,
\[ F_k^i(H^n(Y, \mathbb{C})_0) = R^{n+1}(F_k^{i+1}H^{n+1}(Y, \mathbb{C})) \oplus r^n(F_k^iH^n(X, \mathbb{C})_0). \]

By Theorem 3.3, (3.5),
\[ F_k^{i+1}H^{n+1}(Y, \mathbb{C}) \simeq I_{k+1}^{n+1}(X, (n+2)Y). \]

Applying Theorem 3.3, (3.6) to the pair \((X, Y')\) instead of \((X, Y)\), we have
\[ F_k^iH^n(X, \mathbb{C})_0 \simeq I_k^n(X, (n+1)Y'). \]

From (4.13), (4.14) and (4.15) it follows that
\[ F_k^iH^n(Y, \mathbb{C})_0 = R^{n+1}I_{k+1}^{n+1}(X, (n+2)Y') \oplus r^n(I_k^n(X, (n+1)Y')_0). \]

Here the map \(R^{n+1} : I^{n+1}(X, (n+2)Y) \simeq H^{n+1}(X - Y, \mathbb{C}) \to H^n(Y, \mathbb{C})\) should be interpreted in terms of \(C^\infty\) De Rham group as follows: By use of isomorphisms
\[ H^{n+1}(X - Y, \mathbb{C}) \simeq \mathbb{H}^{n+1}(X, \Omega_X(\log Y)) \simeq I^{n+1}(X, (n+2)Y) \simeq H^{n+1}(A(\log Y)), \]
(cf. Proposition 2.2 and its proof), we can take a \(\varphi \in \text{Ker}([A^{n+1}(\log Y)] \to A^{n+2}(\log Y))\) with \(\varphi = \varphi\) modulo \(d\varphi(X, \Phi^{n+1}(X, (n+2)Y))\) is defined by
\[ R^{n+1}([\varphi]) = 2\pi \sqrt{-1}[\alpha_Y]. \]

where \(\alpha = \alpha \wedge \eta + \beta\),

\[ \alpha_Y = \sum_{i} \alpha_i, \beta = \sum_{i} \beta_i. \]

where \(\eta = C^\infty\) form of type \((1, 0)\) with the property \(\overline{\partial} \eta\) represents the first Chern class \(c_1([Y])\), and \(\alpha \in A^{n-1}(X), \beta \in A^{n+1}(X)\) (cf. (4.4). \(d\alpha = 0\) implies \(dy(\alpha_Y) = 0\). Then \(R^{n+1}([\omega]) ([\omega] \in I^{n+1}(X, (n+2)Y))\)

is defined by
\[ R^{n+1}([\omega]) = 2\pi \sqrt{-1}[\alpha_Y]. \]

where \([\alpha_Y]\) denote the De Rham cohomology class represented by \(\alpha_Y\). Taking into consideration this fact, we will be done if we see
\[ R^{n+1}(I_{k+1}^{n+1}(X, (n+2)Y)) = R\epsilon s(I_{k+1}^{n+1}(X, (n+2)Y)) \]

in the De Rham cohomology. To see this, we first note that both of the right and left hand sides of (4.16) are included in \(H^n(Y, \mathbb{C})_0\), due to Theorem 1.15 and Theorem 2.4. Hence, by Proposition 1.9 and Proposition 1.10, in order to prove (4.16), it suffices to show that
\[ \int_{\tau_\gamma} \omega = \int_{\gamma} r \epsilon s[\omega] \]

for \(\omega \in \Gamma(X, \Phi^{n+1}((n+2)Y))\) and an \(n\) cycle \(\gamma\) lying in \(Y - Z\), where \(\tau_\gamma(\gamma) = \partial U_{\epsilon_{\gamma}}, \) the restriction of the boundary of a topological \(\epsilon\) tubular neighborhood \(U_{\epsilon}\) of \(Y\) in \(X\) to \(\gamma\). We are now going to prove (4.17). We take the local expression (4.7) of \(\omega\) with respect to some open covering \(\{U_i\}_{i \in I}\) of \(X\) and a local coordinate system \((x_1, \ldots, x_{n+1})\) on each \(U_i\), subject to the conditions in (4.6). Let \(\{\rho_i\}\) be a partition of unity subordinate to the covering \(\{U_i\}_{i \in I}\). Then
\[ \int_{\tau_\gamma} \omega = \int_{\tau_\gamma} \sum_i \rho_i \frac{dz_i}{z_i^1} \wedge \alpha_i + \beta_i + d\varphi \]

\[ = \int_{\tau_\gamma} \sum_i \rho_i \frac{dz_i}{z_i^1} \wedge \alpha_i + \beta_i \]

\[ = \sum \int_{\tau_\gamma} \rho_i \frac{dz_i}{z_i^1} \wedge \alpha_i + \beta_i. \]
Locally, $\tau_\varepsilon(\gamma)$ looks like $\mathbb{R}^{n+1} \times \{ |z| = \varepsilon \ | z \in \mathbb{C} \} \ (\varepsilon > 0)$. Hence

$$\sum_i \int_{\tau_\varepsilon(\gamma) \cap U_i} \rho_i (\frac{dz^i}{2\pi i} \wedge \alpha_i + \beta_i) = \lim_{\varepsilon \to 0} \sum_i \int_{\tau_\varepsilon(\gamma) \cap U_i} \rho_i (\frac{dz^i}{2\pi i} \wedge \alpha_i + \beta_i) = 2\pi \sqrt{-1} \sum_i (\rho_i \alpha_i) |_{\gamma \cap U_i}.$$

as required. This completes the proof of Theorem 4.1.

**Remark 4.1.** For $[\omega] \in I_{k+1}^+(X, (n+2)Y)$ it can be proved more directly that the Hodge type of $R^{n+1}([\omega]) = R\hat{e}s([\omega])$ is $(n, 0) + (n - 1, 1) + \cdots + (k, n - k)$. By virtue of the isomorphism

$$I_{k+1}^+(X, (n+2)Y) \simeq H^{n+1}(\mathbb{C}^{n+1}(B))$$

(cf. the proof of Theorem 3.3, (3.5)), $\omega \in \Gamma(X, \Phi_\Delta^{n+1}(X, (n+2)Y)$ is cohomologous to a closed form $\varphi$ of $\sum_{n=0}^{n-k} B_{n-k}^-(n - \ell + k + 1),$ in the De Rham cohomology. If we write $\varphi$ as

$$\varphi = \varphi^{(n+1, 0)} + \varphi^{(n, 1)} + \cdots + \varphi^{(k+1, n-k)},$$

where $\varphi^{(n-\ell+1, \ell)} \in B_{n-\ell+1}^1(n - \ell + k + 2)$ $(0 < \ell < n - k)$, then each $\varphi^{(n-\ell+1, \ell)}$ is written in each $U_i$ as

$$\varphi^{(n-\ell+1, \ell)} = \frac{\alpha_i^{(n-\ell+1, \ell)}}{(z_i^{n-k} + \beta_i^{(n-\ell+1, \ell)}},$$

where $\alpha_i^{(n-\ell, \ell)}$, $\beta_i^{(n-\ell+1, \ell)}$ are regular $C^\infty$ differential forms of types $(n - \ell, \ell)$, $(n - \ell + 1, \ell)$, respectively, not involving $z_1$, where $z_i = 0$ is the local defining equation of $Y$. This is because $(z_i)^{n-k}\varphi^{(n, \ell)}$ and $(z_i)^{n-k}\varphi^{(n-\ell+1, \ell)}$ are $C^\infty$ regular forms by the definition of $B_{n-k}^-(n - \ell + k + 1)$. Put

$$\psi^{(n-\ell, \ell)} := \frac{\alpha_i^{(n-\ell, \ell)}}{(n - \ell + k)(z_i)^{n-k}} \quad (0 \leq \ell \leq n - k - 1),$$

then

$$\eta^{(n-\ell+1, \ell) + (n, \ell+1)} = \frac{d\psi^{(n-\ell+1, \ell)}}{(n - \ell + k)(z_i)^{n-k}} + \frac{\beta_i^{(n-\ell+1, \ell)}}{(z_i)^{n-k}}$$

is a semi-meromorphic form of type $(n - \ell + 1, \ell) + (n, \ell + 1)$ and has poles of order $n - \ell - k$ along $Y$. Let $\{\rho_1\}$ be a partition of unity subordinate to the open covering $\{U_i\}_{i \in I}$ as before. We put

$$\psi^{(n-\ell, \ell)} = \sum_i \rho_i \psi_i^{(n-\ell, \ell)},$$

$$\eta^{(n-\ell+1, \ell) + (n, \ell+1)} = \sum_i \rho_i \eta_i^{(n-\ell+1, \ell) + (n, \ell+1)}$$

Now,

$$\varphi^{(n-\ell+1, \ell)} - d\psi^{(n-\ell, \ell)} = \varphi^{(n-\ell+1, \ell)} - \sum_i \rho_i \psi_i^{(n-\ell, \ell)} + \sum_i \rho_i d\psi_i^{(n-\ell, \ell)} = \sum_i \rho_i \eta_i^{(n-\ell+1, \ell) + (n, \ell+1)} - \sum_i \rho_i \psi_i^{(n-\ell, \ell)} = \eta^{(n-\ell+1, \ell) + (n, \ell+1)} - \sum_i \rho_i \psi_i^{(n-\ell, \ell)}$$
which is a semi-morphic form of type \((n - \ell + 1, \ell) + (n - \ell, \ell + 1)\) having poles of order \(n - \ell - k\) along \(Y\). Continuing this process, \(\phi^{(n-\ell+1,\ell)} (0 < \ell < n - k)\) is reduced to a semi-meromorphic form of the type \((n - \ell + 1, \ell) + \cdots + (k + 1, n - k)\), having poles of order 1 along \(Y\) modulo derived forms. Hence \(\phi\) is reduced to a closed semi-meromorphic form \(\xi\) of \(A_{n+1}^0(\log Y) + \cdots + A_{k+1}^{n-k}(\log Y)\) modulo derived forms. Hence the Hodge type of \(R^{n+1}(\omega) = R^{n+1}(\xi)\) is \((n, 0) + (n - 1, 1) + \cdots + (k, n - k)\).

References