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著者	SAKAI Koukichi, YUKINO Yuko
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Construction and Enumeration of Graphical Sequences Corresponding to Graphs Having Exact Three Vertices with the Same Degree

Koukichi SAKAI * and Yuko YUKINO †

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Abstract

The aim of this note is to construct all the graphical sequences corresponding to graphs which have exact three vertices with the same degree. This work is a continuation of the first author's paper [2] in this Reports.

Key words: Graph, Graph having exact three vertices with the same degree, Degree sequence, Graphical sequence.

1 Classification

In this note we use freely the terminology and notation concerning graphs in G.Chartrand and L.Lesniak [1]. For any positive integer n and non-negative integer m with $m < n$, we use the following notation:

$[n] := \{1, 2, 3, ..., n\} \quad [m, n] := \{m, m + 1, ..., n\}$.

A sequence $s : s_1, s_2, ..., s_n$ of non-negative integers is said to be *graphical* if there exists a simple graph G of order n whose degree sequence is s.

The purpose of this note is to determine all the graphical sequences $s : s_1, s_2, ..., s_n$ with the following property:

(*) $\quad n - 1 \geq s_1 > s_2 > ... > s_{k-1} > s_k = s_{k+1} = s_{k+2} > s_{k+3} > ... > s_n \geq 0$

for some $k \in [n - 2]$.

For the sake of brevity any sequence $s : s_1, s_2, ..., s_n$ of non-negative integers with the property $(*)$ is said to be $(n, 3)$-*admissible* and any sequences with $(*)$ are denoted by $s_n(s_1, s_n; s_k)$. For any fixed s_1 and s_n let $S_n(s_1, s_n)$ be the set of $(n, 3)$-admissible sequences given in the form $s_n(s_1, s_n; s_k)$. It is seen easily that the set of all $(n, 3)$-admissible sequence is partitioned into the five classes $S_n(n - 1, 2)$, $S_n(n - 1, 1)$, $S_n(n - 2, 1)$, $S_n(n - 2, 0)$ and $S_n(n - 3, 0)$. We note that $s_n(n - m, 3 - m; k), k \in [3 - m, n - m]$, expresses a sequence for $m = 1, 2, 3$. Further we denote by $GS(n, 3)$ and $GS_n(s_1, s_n)$ the set of all graphical $(n, 3)$-admissible sequences and the

* Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima 890-0065, Japan.

† Kagoshima Immaculate Heart University, Sendai, Kagoshima 895-0011, Japan

set of all graphical sequences in $S_n(s_1, s_n)$ respectively. Then we have

Lemma 1.1 *$GS(n,3)$ is partitioned into the five classes as follows :*
$GS_n(n-1,2) \cup GS_n(n-1,1) \cup GS_n(n-2,1) \cup GS_n(n-2,0) \cup GS_n(n-3,0)$.

Computing directly, we get the following Lemmas 1.2 – 1.5.

Lemma 1.2 *$GS(3,3) = \{(2,2,2),(0,0,0)\}$. More precisely we have*
$GS_3(2,2) = \{(2,2,2)\}$
$GS_3(2,1) = GS_3(1,1) = GS_3(1,0) =$ *empty*
$GS_3(0,0) = \{(0,0,0)\}$.

Lemma 1.3 *$GS(4,3) = \{(3,1,1,1),(2,2,2,0)\}$. More precisely we have*
$GS_4(3,2) =$ *empty*
$GS_4(3,1) = \{(3,1,1,1)\}$
$GS_4(2,1) =$ *empty*
$GS_4(2,0) = \{(2,2,2,0)\}$
$GS_4(1,0) =$ *empty.*

Lemma 1.4 *$GS(5,3)$ consists of the following five sequences :*
$GS_5(4,2) =$ *empty*
$GS_5(4,1) = \{(4,3,3,3,1)\}$
$GS_5(3,1) = \{(3,2,1,1,1),(3,2,2,2,1),(3,3,3,2,1)\}$
$GS_5(3,0) = \{(3,1,1,1,0)\}$
$GS_5(2,0) =$ *empty.*

Lemma 1.5 *$GS(6,3)$ consists of the following twelve sequences :*
$GS_6(5,2) = \{(5,4,3,2,2,2),(5,4,3,3,3,2),(5,4,4,4,3,2)\}$
$GS_6(5,1) = \{(5,4,2,2,2,1)\}$
$GS_6(4,1) = \{(4,3,2,1,1,1),(4,3,2,2,2,1),(4,3,3,3,2,1),(4,4,4,3,2,1)\}$
$GS_6(4,0) = \{(4,3,3,3,1,0)\}$
$GS_6(3,0) = \{(3,2,1,1,1,0),(3,2,2,2,1,0),(3,3,3,2,1,0)\}$.

2 Construction of $GS(n,3)$

In this section we shall construct inductively all sequences in $GS(n,3)$, $n \geq 4$. The next lemma, noted in [1,Theorem 1.4], plays the essential role in our discussion.

Lemma 2.1 *A sequence $s : s_1, s_2, ..., s_n$ of non-negative integer with $s_1 \geq s_2 \geq ... \geq s_n$, $n \geq 2$, $s_1 \geq 1$, is graphical if and only if the following sequence $h(s)$ with $n-1$ terms is graphical:*
$h(s) : s_2 - 1, s_3 - 1, ..., s_{t+1} - 1, s_{t+2}, s_{t+3}, ..., s_n$
where $t = s_1$.

Now for any sequence $s : s_1, s_2, ..., s_{n-1}, n \geq 2$, of integers with $n-1$ terms we define the sequence $p(s)$ with n terms by

$p(s) : n-1, s_1+1, s_2+1, ..., s_{n-1}+1$.

For any set F of sequences of integers, we set $p(F) = \{p(s); s \in F\}$. Some $(n-1,3)$-admissible sequences are mapped injectively to $(n,3)$-admissible ones by the map $p : s \mapsto p(s)$. More precisely we have

Lemma 2.2 *Let n be any positive integer with $n \geq 4$. Then we have*

(1) $h(p(s)) = s$ *for any* $s \in S_{n-1}(n-3,1) \cup S_{n-1}(n-3,0) \cup S_{n-1}(n-4,0)$.

(2) $S_n(n-1,2) = p(S_{n-1}(n-3,1)) \cup \{s_n(n-1,2;n-1)\}$.

(3) $S_n(n-1,1) = p(S_{n-1}(n-3,0) \cup S_{n-1}(n-4,0)) \cup \{s_n(n-1,1;n-1)\}$.

Various criteria for sequences to be graphic are shown in G. Sierksma and H. Hoogeven [3]. We use the next criterion noted in [1,Theorem 1.5].

Lemma 2.3 *A sequence $s : s_1, s_2, ..., s_n (n \geq 2)$ of non-negative integers with $s_1 \geq s_2 \geq s_3 \geq ... \geq s_n$ is graphical if and only if the following two conditions hold:*

(P) $\quad \sum_{k=1}^{n} s_k =$ *even*

and for each integer $k \in [n-1]$,

(E_k) $\quad \sum_{j=1}^{k} s_j \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, s_j\}$.

In what follows, for any sequence s as in Lemma 2.3 the left[resp. right] hand side of (E_k) is denoted by (EL_k)[resp. (ER_k)].

Lemma 2.4 *Let n be any positive integer. Then we have*

(1) $s = s_n(n-1,2;n-1)$ *is not graphical for* $n \geq 4$.

(2) *Any sequences of type $s_n(n-1,1;n-1)$ are not graphical for* $n \geq 3$.

Proof For the sequence s in (1), $(EL_3) = 3n-3 > 3n-4 = (ER_3)$. So s is not graphical by Lemma 2.3. We note that $s_3(2,2;2) = (2,2,2)$ is graphical. (2) is seen similarly. □

From Lemmas 2.1-2.4, it follows that $GS_n(n-1,2)$ and $GS_n(n-1,1)$ is constructed from $GS(n-1,3)$ by the map p.

Theorem 2.5 *Let n be any positive integer with $n \geq 4$. Then we have*

(1) $GS_n(n-1,2) = p(GS_{n-1}(n-3,1))$.

(2) $GS_n(n-1,1) = p(GS_{n-1}(n-3,0) \cup GS_{n-1}(n-4,0))$.

For any $(n,3)$-admissible sequence $s : s_1, s_2, ..., s_n$, we define a $(n,3)$-admissible sequence $c(s)$ by :

$c(s) : n-1-s_n, n-1-s_{n-1}, ..., n-1-s_2, n-1-s_1$

Considering a graph and its complement graph, we see that s is graphical if and only if so is $c(s)$. For any set F of $(n,3)$-admissible sequences, we set $c(F) = \{c(s); s \in F\}$. Then the next is seen easily

Theorem 2.6 *Let n be any positive integer with $n \geq 3$. Then we have*

(1) $GS_n(n-3,0) = c(GS_n(n-1,2))$.

(2) $GS_n(n-2,0) = c(GS_n(n-1,1))$.

Finally we determine explicitly any sequences in $GS_n(n-2,1)$.

Lemma 2.7 *Let m be any positive integer. Every sequence in $S_n(n-2,1)$ is not graphical for $n=4m-1$ and $n=4m$.*

Proof This follows from the fact that for $n=4m-1$ and $n=4m$, every sequence in $S_n(n-2,1)$ does not satisfy the condition (P) in Lemma 2.3. □

Lemma 2.8 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;t)$ is not graphical for any t, $1 \leq t < m$.*

(2) *$s_{4m+2}(4m,1;t)$ is not graphical for any t, $1 \leq t < m$.*

Proof For the sequence s in (1) we have

$(EL_{2m}) = 6m^2 - m$ and $(ER_{2m}) = 6m^2 - 3m + 2t$.

$(ER_{2m}) - (EL_{2m}) = 2(t-m) < 0$.

Hence s is not graphical. We see (2) similarly. □

Lemma 2.9 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;m)$ is graphical.*

(2) *$s_{4m+2}(4m,1;m)$ is graphical.*

Proof For the sequence s in (1) we have

$$(ER_k) - (EL_k) = \begin{cases} k & \text{if } 1 \leq k \leq m \\ 2m-k & \text{if } m < k \leq 2m \\ 2(2m-k)^2 & \text{if } 2m < k \leq 3m \\ 2\{(k-2m-1)^2 + 2m\} & \text{if } 3m < k \leq 4m. \end{cases}$$

Hence we have (1). Similarly we see (2). □

Lemma 2.10 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;t)$ is graphical for any t, $m \leq t \leq 3m$.*

(2) *$s_{4m+2}(4m,1;t)$ is graphical for any t, $m \leq t \leq 3m+1$.*

Proof We prove (1) by the induction on m. By Lemma 1.4 the assertion is true for the case $m=1$. Let $m>1$, $s(t) = s_{4m+1}(4m-1,1;t)$ and $3m > t > m$. Let us apply twice Lemma 2.1 to $s(t)$. Then we see that $h(h(s(t))) = s_{4m-3}(4m-5,1;t-2),1,1$. Obviously $h(h(s(t)))$ is graphical if and only if so is $s_{4m-3}(4m-5,1;t-2)$. Since $3(m-1) \geq t-2 \geq m-1$, $s_{4m-3}(4m-5,1;t-2)$ is graphical by the inductive hypothesis, and hence so are $h(h(s(t)))$ and $s(t)$ by Lemma 2.1. From Lemma 2.9, $s(m)$ is graphical and so is $s(3m) = c(s(m))$. Similarly we have (2). □

By virtue of Lemmas 2.7-2.10, $GS_n(n-2,1)$ is characterized explicitly as follows.

Theorem 2.11 *Let m be any positive integer. Then we have*

(1) *$GS_{4m-1}(4m-3,1)$ and $GS_{4m}(4m-2,1)$ are empty.*

(2) *$GS_{4m+1}(4m-1,1) = \{s_{4m+1}(4m-1,1;t);\ t \in [m,3m]\}$.*

(3) *$GS_{4m+2}(4m,1) = \{s_{4m+2}(4m,1;t);\ t \in [m,3m+1]\}$.*

3 Enumeration of $GS(n,3)$

Let n be any positive integer with $n \geq 3$, and let a_n, b_n, c_n, d_n, e_n and g_n be the cardinal number of $GS_n(n-1,2), GS_n(n-1,1), GS_n(n-2,1), GS_n(n-2,0), GS_n(n-3,0)$ and $GS(n,3)$ respectively. The next two lemmas are immediate consequences from Theorems 2.5, 2.6 and 2.11.

Lemma 3.1
(1) $a_n = e_n = c_{n-1}$ *and* $b_n = d_n = a_{n-1} + b_{n-1}$ *for any positive integher* $n \geq 4$.
(2) $a_3 = 1$, $b_3 = c_3 = d_3 = 0$ *and* $e_3 = 1$.

Lemma 3.2 *For any positive integer* m, *we have*

$$a_{n+1} = c_n = \begin{cases} 0 & if \ \ n = 4m-1, 4m \\ 2m+1 & if \ \ n = 4m+1 \\ 2m+2 & if \ \ n = 4m+2. \end{cases}$$

The next follows from Lemmas 3.1 and 3.2.

Lemma 3.3 *For any positive integer* m, *we have*

$$b_{4m} = b_{4m+1} = b_{4m+2} = 2m^2 + m - 2,$$

$$b_{4m-1} = \begin{cases} 2m^2 - m - 2 & if \ \ m \geq 2 \\ 0 & if \ \ m = 1. \end{cases}$$

Since $g_n = 2(a_n + b_n) + c_n$ by Lemma 3.1, we conclude the next theorem from Lemmas 3.2 and 3.3.

Theorem 3.4 *Let* m *be any positive integer. The cardinal number* g_n *of* $GS(n,3)$ *is expressed in the following form:*

$$g_n = \begin{cases} 4m^2 + 2m - 4 & if \ \ n = 4m-1, 4m \\ 4m^2 + 4m - 3 & if \ \ n = 4m+1 \\ 4m^2 + 8m & if \ \ n = 4m+2. \end{cases}$$

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