

# On Compact Open Topology in the case of Local Fields

著者	KOSHIBA Yoichi
journal or publication title	鹿児島大学理学部紀要=Reports of the Faculty of Science, Kagoshima University
volume	30
page range	13-15
URL	<a href="http://hdl.handle.net/10232/6219">http://hdl.handle.net/10232/6219</a>

# On Compact Open Topology in the case of Local Fields

著者	KOSHIBA Yoichi
journal or publication title	鹿児島大学理学部紀要=Reports of the Faculty of Science, Kagoshima University
volume	30
page range	13-15
URL	<a href="http://hdl.handle.net/10232/00006971">http://hdl.handle.net/10232/00006971</a>

## On Compact Open Topology in the case of Local Fields

Yôichi KOSHIBA\*

(Received September 10, 1997)

**Key words:** compact open topology, local field

We shall consider below an result on the topological fields and the concept of compact open topology. We mean by linear topology on linear space  $k^m$ , i.e. of product space of  $n$  copies of a topological field  $k$ .

**Theorem** *Let  $k$  be a topological field,  $M(n, k)$  be  $n \times n$  matrix space with entry in  $k$ .  $M(n, k)$  is regarded as a family of continuous maps from linear space  $k^n$  into  $k^n$  itself. Then the compact open topology on  $M(n, k)$  as family of maps, coincides to the linear topology on  $M(n, k)$ , i.e. isomorphic to  $k^{n^2}$ .*

**Proof** When we regard  $M(n, k)$  as a family of maps,  $M(n, k)$  has also the simple topology(=topology of pointwise convergence). We shall compare the simple topology, and the linear topology, the compact open topology through three steps.  $\square$

**Lemma 1** *The simple topology on the space  $M(n, k)$  is stronger than the linear topology on the space  $M(n, k)$ .*

We can assume  $n = 2$  without loss of generality.

Let  $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be convergence in simple topology on  $M(n, k)$ .

For any point  $(x, y) \in k^2$ ,

$$\lim_{n \rightarrow \infty} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

holds good. This limit is the convergence in linear topology on  $k^2$ . If we put

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then we obtain  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} c_n = c$ . Similarly by putting

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

---

\* Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima, 890-0065, Japan. e-mail:koshiba@sci.kagoshima-u.ac.jp

then we obtain  $\lim_{n \rightarrow \infty} b_n = b$ ,  $\lim_{n \rightarrow \infty} d_n = d$ . Therefore in the linear topology,

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \square$$

**Remark** Generally, the compact open topology is stronger than the simple topology. Since the set of a single point is compact, from the definition of compact open topology, this topology is stronger than simple topology.  $\square$

**Lemma 2** *The linear topology on  $M(n, k)$  is stronger than the compact open topology on  $M(n, k)$  as a family of maps.*

**Proof** By checking following elementary equations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$X = ax + by$$

$$Y = cx + dy$$

variable  $\begin{pmatrix} X \\ Y \end{pmatrix}$  is continuous in variables  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $\begin{pmatrix} x \\ y \end{pmatrix}$ . Because variable  $X, Y$  are continuous function form, i.e. plus and multiplications of variable,  $a, b, c, d, x, y$ .

Let  $X, Y$  be topological spaces.

$$Y^X \times X \Longrightarrow Y$$

$$(f, x) \xrightarrow{\varphi} f(x) \in Y$$

$$f \in Y^X \quad x \in X .$$

Above diagramme, we call the map  $\varphi$  evaluation map. In our case,  $X = Y = k^n$ , the evaluation map is linear transformation from  $k^n$  into  $k^n$  by matrix of  $M(n, k)$ . Generally, if the evaluation map is continuous for a subset  $S$  of  $Y^X$ , then the topology of a familys of maps(really in our case, the linear topology on the space  $M(n, k)$ ) is stronger than the compact open topology on the  $S$ .  $\square$

We now return the proof of our Theorem. From Lemma 1, Remark, Lemma 2, we have

**Linear Topology=Simple Topology=Compact Open topology**

for topologies on space  $M(n, k)$ . We have verified the theorem.  $\square$

Especially, let  $k$  be the real number field, the complex number field, or a local field i.e. complete field with a non-archimedean valuation individually, in matrix space  $M(n, k)$ , linear topology (if  $k$  is real, its a Euclidean topology) coincides to compact open topology as maps  $k^n \rightarrow k^n$ . Replacing  $M(n, k)$  to  $m \times n$  matrix space  $M(m, n; k)$ , similar result holds clearly.

\*\*We add an result on the evaluation map. We give an elementary example of non continuous evaluation map.

**Example** Let  $X, Y$  be closed interval  $[0, 1]$  with ordinary Euclidean topology. In mapping space  $Y^X$ , we define the points  $f_n \in Y^X$ , point  $f \in Y^X$  as follows:

$$f_n(x) = x^n \quad n = 1, 2, \dots$$

and

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

The space  $Y^X$  is Hausdorff. In the simple topology on space  $Y^X$ , points sequence  $f_n$  converges to point  $f$ .

$$\lim_{n \rightarrow \infty} f_n = f.$$

If we define the sequence  $\{x_n\}$  in  $X$  by  $x_n = 1 - \frac{1}{n}$ , then  $\lim_{n \rightarrow \infty} x_n = 1$ . For point sequence  $(f_n, x_n) \in Y^X \times X$ ,

$$\lim_{n \rightarrow \infty} (f_n, x_n) = (f, 1).$$

If the evaluation map  $\varphi$  is continuous, the equation

$$\lim_{n \rightarrow \infty} \varphi((f_n, x_n)) = \varphi((f, 1))$$

must be hold.

$$\varphi((f, 1)) = f(1) = 1,$$

$$\varphi((f_n, x_n)) = f_n(x_n) = \left(1 - \frac{1}{n}\right)^n.$$

But from elementary analysis,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}.$$

Therefore this evaluation map can not be continuous.

## References

- [1] J. L. Kelly, General Topology, Graduate Texts in Mathematics 27, Springer-Verlag, 1955, Especialy Chapter 7.