On Graphs with Maxclique Partition (Appendix: Corrections to previous author's paper)

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On Graphs with Maxclique Partition

(Appendix : Corrections to previous author's paper)

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Abstract

As well known (e.g. [4]) every graph is isomorphic to the line graph of a hypergraph. In this note, for any graph $G$ with maxclique partition, we shall characterize the hypergraph $H$ whose line graph $L(H)$ is isomorphic to $G$. We also consider the complete $r$-partite graphs $(r \geq 3)$ with maxclique partition.

Key words: graph, hypergraph, line graph, maxclique partition, complete $r$-partite graph.

1 Graphs with maxclique partition

In this note the terminology and notion concerning graphs and hypergraphs follow Chartrand and Lesniak [2] and Duchet [3] respectively unless otherwise stated. We assume always that any graphs and hypergraphs are finite, simple and connected. Let $G$ be a graph. For any subgraph $G'$ of $G$ we denote by $V(G')$ and $E(G')$ the vertex set and the edge set of $G'$ respectively. For any subset $W$ of $V(G)$, $<W>$ is the subgraph of $G$ induced by $W$. Any complete subgraph of $G$ is called a clique, and especially it is called a maxclique if it is not properly contained in another cliques. Let $MC(G)$ be the set of maxcliques of $G$. A subfamily $F$ of $MC(G)$ is called a maxclique partition of $G$ if the family \( \{E(Q); Q \in F\} \) is a partition of $E(G)$. In this case we may assume that

\[ |V(Q)| \geq 2 \text{ for any } Q \in F. \]

Moreover, by contraction of edges, we may assume that

\[ \text{Any } Q \in F \text{ has at most one vertex which does not belong to another members in } F. \]

For brevity we say that $G$ is an MCP-graph, denoted by the pair $(G, F)$, if there exists a maxclique partition $F$ of $G$ with (1.1) and (1.2).

In what follows let $(G, F)$ be an MCP-graph. Then we can define a hypergraph $\Psi(G, F) := (F, E)$ on $F$, which is called an MP-hypergraph of MCP-graph $(G, F)$ for brevity. Here the hyperedge set $E = \{\psi(v); v \in V(G)\}$, where $\psi(v)$ is the subset of $F$.

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defined by

\[(1.3) \ \psi(v) = \{Q \in F; v \in V(Q)\}.\]

By virtue of (1.2) the map \(\psi : V(G) \ni v \mapsto \psi(v) \in E\) is bijective. So we note that the hyperedge set \(E\) is identified with \(V(G)\). For any \(Q \in F\) we put

\[(1.4) \ H(Q) := \{\psi(v); Q \ni v\}.\]

Then the next Lemma follows immediately from the fact that \(F\) is a maxclique partition of \(G\) with (1.1) and (1.2).

**Lemma 1.1.**

\[(1.5) \ \text{For any distinct } u, v \in V(G), |\psi(u) \cap \psi(v)| \leq 1 \text{ and } |\psi(u) \cap \psi(v)| = 1 \text{ if and only if } u \text{ and } v \text{ are adjacent,}\]

\[(1.6) \ \psi(u) \cap \psi(v) = \{Q\} \text{ if and only if } uv \in E(Q),\]

\[(1.7) \ \text{For any } Q \in F, \text{ any hyperedge } \psi(w) \text{ belongs to } H(Q) \text{ if } \psi(w) \cap \psi(v) \neq \emptyset \text{ for all } \psi(v) \in H(Q),\]

\[(1.8) \ |H(Q)| \geq 2 \text{ for any } Q \in F.\]

The assertion (1.5) implies that the map \(\psi\) is an isomorphism from \(G\) to the line graph \(\Psi(G) := L(\Psi(G,F))\) of the \(MP\)-hypergraph \(\Psi(G,F)\). Each \(H(Q), Q \in F\), induces a maxclique \(<H(Q)>\) of \(\Psi(G)\) by (1.7). Moreover the family \(\Psi(F) := \{<H(Q)>; Q \in F\}\) is a maxclique partition of \(\Psi(G)\) by (1.6). Therefore, under these notation, we get the following

**Theorem 1.2.** Let \((G,F)\) be an \(MCP\)-graph.

(1) \(\Psi(G)\) is an \(MCP\)-graph with the maxclique partition \(\Psi(F)\),

(2) \((\Psi(G), \Psi(F))\) is isomorphic to \((G,F)\).

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### 2 Characterization of \(MP\)-hypergraphs

In this section we shall characterize the hypergraph \(H\) whose line graph \(L(H)\) becomes an \(MCP\)-graph. Let \(H = (X,E)\) be a simple and connected hypergraph with finite vertex set \(X\) and hyperedge set \(E\). For any \(x \in X\) we set

\[(2.1) \ H(x) := \{e \in E; x \in e\}.\]

A subfamily \(E'\) of \(E\) is said to be intersecting if \(e_1 \cap e_2 \neq \emptyset\) for any \(e_1, e_2 \in E'\). An intersecting family \(E'\) is said to be maximal if it is not properly contained in another intersecting family of \(E\).

**Definition 2.1.** Any hypergraph \(H = (X,E)\) is called an \(ML\)-hypergraph if it satisfies the following conditions:

\[(2.2) \ \text{Each hyperedge } e \text{ is nonempty},\]

\[(2.3) \ |H(x)| \geq 2 \text{ for any } x \in X,\]

\[(2.4) \ |e_1 \cap e_2| \leq 1 \text{ for any distinct } e_1, e_2 \in E,\]

\[(2.5) \ H(x) \text{ is a maximal intersecting subfamily of } E \text{ for any } x \in X.\]
We note that the next condition (2.6) follows from the condition (2.4):
(2.6) \(|H(x) \cap H(y)| \leq 1\) for any distinct \(x, y \in X\).
Obviously the \(MP\)-hypergraph \(\Psi(G, F)\) of any \(MCP\)-graph \((G, F)\) is an \(ML\)-hypergraph by Lemma 1.1.

Lemma 2.2. The line graph \(L(H)\) of any \(ML\)-hypergraph \(H = (X, E)\) is an \(MCP\)-
graph, and the family \(H(X) := \{< H(x) >; x \in X\}\) is a maxclique partition of \(L(H)\),
where \(< H(x) >\) is the subgraph of \(L(H)\) induced by \(H(x)\).

Proof. Since \(H(x)\) is an intersecting family in \(E\), \(< H(x) >\) is a clique of \(L(H)\),
and is maximal by (2.5). Let \(e_1, e_2 \in E\) be adjacent in \(L(H)\). Then by (2.4) there exists
an unique \(x_0 \in X\) such that \(e_1 \cap e_2 = \{x_0\}\). So the edge \(e_1 e_2\) in \(L(H)\) is in the unique
maxclique \(< H(x_0) >\). Hence \(H(X)\) is a maxclique partition of \(L(H)\). For any \(x \in X\),
\(H(x)\) contains at most one singleton and the order of \(< H(x) >\) is at least two by (2.3).
Thus \(H(X)\) satisfies (1.1) and (1.2). This completes the proof.

Combining Theorem 1.2 and Lemma 2.2 we have

Theorem 2.3. A hypergraph \(H = (X, E)\) is an \(MP\)-hypergraph of any \(MCP\)-graph
\((G, F)\) if and only if it is an \(ML\)-hypergraph. In this case \(G\) is isomorphic to the line
graph of \(H\).

For any \(ML\)-hypergraph \(H = (X, E)\) we consider the Helly condition:
(2.7) Any intersecting family of \(E\) is contained in \(H(x)\) for some \(x \in X\).
If \(H\) satisfies (2.7), it is seen easily that \(MC(L(H)) = H(X)\). Hence we have

Theorem 2.4. For any graph \(G\), \(MC(G)\) is a maxclique partition of \(G\) if and only
if \(G\) is the line graph of any \(ML\)-hypergraph with the Helly condition (2.7).

3 \(ML\)-graphs

Any 2-uniform hypergraph is identified with a simple graph. So any 2-uniform
\(MZ\)-hypergraph is called an \(ML\)-graph. For any graph \(G\), the conditions (2.2) and
(2.4) hold trivially. The condition (2.3) corresponds to the condition \(\delta(G) \geq 2\),
where \(\delta(G) = \min\{\delta(v); v \in V(G)\}\). For any \(v \in V(G)\), let \(H(v)\) be the set of edges incident
to \(v\). Evidently \(H(v)\) is a maximal intersecting family if \(\delta(G) > 2\). On the other hand
let \(\delta(v) = 2\) and \(N(v) = \{w, z\}\), where \(N(v)\) is the neighborhood of \(v\). Then \(H(v)\) is
maximal if and only if \(wz \notin E(G)\), that is, \(< \{v\} \cup N(v) >\) is the path \(P_3\). Consequently
we have

Theorem 3.1. Any graph \(G\) is an \(ML\)-graph if and only if it satisfies the following
two conditions:

(1) \(\delta(G) \geq 2\),
(2) For any \(v \in V\) with \(\delta(v) = 2\), \(< \{v\} \cup N(v) >\) is the path \(P_3\).

If a graph \(G\) with \(\delta(G) \geq 2\) contains no triangles, then it is an \(ML\)-graph satisfying
the Helly condition (2.7). So from Theorem 2.4 we have
Theorem 3.2. Let $G$ be any graph with $\delta(G) \geq 2$, and $L(G)$ be the line graph of $G$. If $G$ is triangle-free, then $MC(L(G))$ is a maxclique partition of $L(G)$. 

4 Complete r-partite graphs with maxclique partition

For any $r \geq 2$, let $G := K(n_1, n_2, \ldots, n_r)$ be the complete r-partite graph with partite sets $V_j, |V_j| = n_j (j = 1, 2, \ldots, r)$. For the case $r = 2$, each edge of $G$ is a maxclique and $G$ has the maxclique partition $\{e; e \in E(G)\}$.

Now assume that $r > 2$ and $G$ has a maxclique partition $F = \{Q_j; j = 1, 2, \ldots, m\}$. We note that each $Q_j$ is of order $r$. Let $s = \sum_{j=1}^{r} n_j$. For any $v \in V(G)$ we put

\[ E_v = \{Q \in F; v \in V(Q)\}. \]

Then for every partite set $V_j$, $F$ is partitioned into the disjoint family $\{E_v; v \in V(Q)\}$, and $|E_v| = \frac{s - n_j}{r - 1}$ for any $v \in V_j$. So we have $n_j(s - n_j) = m(r - 1)$ for any $j = 1, 2, \ldots, r$. From these relations we conclude that $n := n_1 = n_2 = \cdots = n_r, m = n^2$, and $|E_v| = n$.

Lemma 4.1. If $K(n_1, n_2, \ldots, n_r)$ has a maxclique partition $F = \{Q_j; j = 1, 2, \ldots, m\}$, then $n := n_1 = n_2 = \cdots = n_r$ and $m = n^2$. 

For any fixed positive integers $n, r$, let us denote by $K(n;r)$ the complete r-partite graph such that each partite set $V_j, j = 1, 2, \ldots, r$, is an $n$-set. Evidently $K(1;r) = K_r$ and $K(n,2)$ are MCP-graphs. So in what follows let $n > 1$ and $r > 2$. Suppose $K(n;r)$ has a maxclique partition $F$. Let $\Psi(K(n;r), F) = (F, E)$ be the MP-hypergraph of $(K(n;r), F)$, where the hyperedge set $E = \{E_v; v \in V(K(n;r))\}$. For any $Q \in F$ we put

\[ H(Q) := \{E_v; v \in V(Q)\}. \]

Then the above discussions are summarized as follows.

Lemma 4.2. $\Psi(K(n;r), F)$ has the following properties:

\begin{align*}
(4.3) & \quad |F| = n^2, \\
(4.4) & \quad |H(Q)| = r \text{ for any } Q \in F, \\
(4.5) & \quad |E_v| = n \text{ for any } v \in V(K(n;r)), \\
(4.6) & \quad \text{For any partite set } V_j, P_j := \{E_v; v \in V_j\} \text{ is a partition of } F \text{ and } |P_j| = n, \\
(4.7) & \quad \text{For any distinct partite sets } V_j, V_k, |E_v \cap E_w| = 1 \text{ for any } (v, w) \in V_j \times V_k, \\
(4.8) & \quad \text{For any } Q \in F \text{ and } E_v \in E \text{ with } Q \notin E_v, \text{ there exists an unique } E_w \in H(Q) \text{ for which } E_v \cap E_w = \emptyset. \tag{\Box}
\end{align*}

Let $Q, E_v$ be as in (4.8). Then there is an unique $P_j$ containing $E_v$. By (4.6) there exists an unique $E_w \in P_j$ with $Q \in E_w$. Hence we have (4.8).

5 AF-hypergraphs

Let $n, r$ be any fixed integers with $n > 1$ and $r > 2$. We shall characterize the MP-hypergraph of MCP-graph $(K(n;r), F)$.
Definition 5.1. Any ML-hypergraph $H = (X, E)$ is called an AF-hypergraph, denoted by $H(X, E; n, r)$, if the following three conditions hold:

(5.1) $|e| = n$ for any $e \in E$,
(5.2) $|H(x)| = r$ for any $x \in X$,
(5.3) For any $x \in X$ and $e \in E$ with $x \notin e$, there exists an unique $e_0 \in H(x)$ such that $e \cap e_0 = \emptyset$. □

In (5.3), any hyperedges in $H(x)$ except $e_0$ must intersect the hyperedge $e$. From this fact we have

(5.4) $r \leq n + 1$.

By virtue of (5.3) we can define an equivalence relation $\equiv$ in $E$ as follows:

(5.5) For any $e_1, e_2 \in E, e_1 \equiv e_2$ if $e_1 = e_2$ or $e_1 \cap e_2 = \emptyset$.

We denote by $\hat{e}$ the equivalence class containing $e \in E$ and by $\hat{E}$ the quotient set of $E$ with respect to $\equiv$. Then under the these notation the following Lemma is seen easily from (5.1)-(5.3).

Lemma 5.2.

(5.6) For any $x \in X, H(x)$ is a representative system of $\hat{E}$ and $|\hat{E}| = r$,
(5.7) For any $e \in E, \hat{e}$ induces a partition of $X$, i.e., $X = \cup \{\hat{f}; f \in \hat{e}\}$, and $|\hat{e}| = n$,
(5.8) $|X| = n^2$. □

Let $\hat{E} = \{\hat{e}_j; j = 1, 2, \ldots, r\}$. Then for any $j, k$ with $1 \leq j < k \leq r, e \cap f \neq \emptyset$ for any $(e, f) \in \hat{e}_j \times \hat{e}_k$. Therefore the line graph of $H(X, E; n, r)$ is isomorphic to the complete $r$-partite graph $K(n; r)$, with partite $n$-sets $\hat{e}_j, j = 1, 2, \ldots, r$. On the other hand, as noted in Lemma 4.2, the $MP$-hypergraph of $(K(n; r), F)$ is an $AF$-hypergraph. Hence we have

Theorem 5.3. Any hypergraph $H$ is an $MP$-hypergraph of $K(n; r)$ if and only if it is an $AF$-hypergraph $H(X, E; n, r)$. □

From Theorem 5.3 and Theorem 2.2 we have

Theorem 5.4. The complete $r$-partite graph $K(n; r)$ has a maxclique partition if and only if there exists an $AF$-hypergraph $H(X, E; n, r)$. □

We note that any $AF$-hypergraph $H(X, E; n, n + 1)$ satisfies the condition:

(5.9) $|H(x) \cap H(y)| = 1$ for any distinct $x, y \in X$.

From (5.3) and (5.9), any $AF$-hypergraph $H(X, E; n, n + 1)$ is identified with the finite Affine plane of order $n$.

Theorem 5.5. $K(n; n + 1)$ has a maxclique partition for any $n = p^m$, where $p$ is a prime and $m$ is a positive integer.
Proof. This follows from Theorem 5.4 and the fact (e.g. [1]) that there exists the finite Affine plane $H(X, E; n, n + 1)$ of order $n$ for $n = p^m$.

Remark 5.6. For any integer $n \geq 2$, let $p$ be the least prime divisor of $n$. Then we can construct an AF-hypergraph $H(X, E; n, p + 1)$. Hence $K(n; p + 1)$ has a maxclique partition. Especially $K(n; 3)$ has a maxclique partition for any $n$, and so has $K(n; 4)$ for any odd $n$.

References


Appendix

There are some errors in the author's paper:

We correct these errors as follows.

(1) In Lemma 3.2(3) and Theorem 3.3 the equal $=$ must be replaced by $\leq$. So Theorem 3.3 gives an upper estimation of the intersection number of the complete $r$-partite graph. However this estimation is not so good. For example $i(K(m_1, m_2, m_3)) = m_1m_2 < m_1(m_2 + m_3 - 1)$.

(2) In Lemma 4.4 the family of maxcliques $\{Q_j; j \in [n - 1]\}$ is not $MC(G_n)$ but a minimal maxclique edge cover of $G_n$, where $n$ is even.

(3) By the above correction, in Theorem 4.5 $i(G_n) = \theta_m(G_n)$ holds only for odd $n$. 