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to Professor Dr. Joyo Kanitani on the occasion
of his ninetieth birthday

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ON GENERALIZED BERWALD CONNECTIONS

*Dedicated to Professor Dr. Jōyō Kanitani
on the occasion of his ninetieth birthday*

By

Tadashi AIKOU* and Masao HASHIGUCHI**

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Abstract

In the present paper we discuss Finsler connections of Berwald type with surviving $(v)hv$ -torsion P^i_{jk} and consider what kind of Finsler connection should be reasonable as a generalization of the Berwald connection.

§ 0. Introduction.

In a Finsler space there are known two canonical Finsler connections, that is, the Cartan one CF and the Berwald one BF . Various generalizations are possible for these connections. For example, suggested by Wagner [8], Hashiguchi, one of the authors, introduced the notion of generalized Cartan connection GCF and defined a generalized Berwald space with respect to this GCF (Hashiguchi [1], Hashiguchi-Ichijyō [2]). As a generalization of the Berwald connection, Matsumoto [4] defined the notion of Berwald connection with torsion BFT and showed that a generalized Berwald space can be also defined with respect to this BFT .

Each of these generalizations has a surviving $(h)h$ -torsion $T_j^i{}_k$. It is noted that $T_j^i{}_k$ of BFT should satisfy the so-called BF -condition, whereas $T_j^i{}_k$ of GCF is arbitrarily given. This situation should be cleared up.

On the other hand, in his recent paper [5], Matsumoto has treated a Finsler connection introduced on a hypersurface of a Finsler space, and obtained an interesting Finsler connection of Berwald type, which we could call a Berwald connection with surviving $(v)hv$ -torsion P^i_{jk} .

The purpose of the present paper is to discuss such a Finsler connection generally. A meaning of Matsumoto's BF -condition is cleared up, and it is shown what kind of Finsler connection should be reasonable as a generalization of the Berwald connection.

Throughout the present paper the terminology and notations are referred to Matsumoto's monograph [3].

* Kagoshima-chuo High School, Kagoshima, Japan.

** Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

§ 1. Finsler connections of Berwald type with torsion P^i_{jk} .

We are concerned with an n -dimensional Finsler space $F^n=(M, L)$, where $L(x, y)$ is the fundamental function, and x denotes a point of the underlying manifold M , and y denotes a supporting element. The fundamental tensor g_{ij} is given by $g_{ij}=(\partial_i\partial_jL^2)/2$, where ∂_j denotes the partial differentiation by y^j . We shall express a Finsler connection $F\Gamma$ by $F\Gamma=(F^i_{jk}, N^i_{jk}, C^i_{jk})$ in terms of its coefficients. The Cartan connection $C\Gamma$ and the Berwald one $B\Gamma$ are uniquely determined by the following systems of axioms respectively. We line up them comparatively.

$C\Gamma$ (Matsumoto [3])

- (C1) $g_{ij|k}=0$,
- (C2) $D^i_{jk}\equiv F^i_{jk}-N^i_{jk}=0$,
- (C3) $T^i_{jk}\equiv F^i_{jk}-F^i_{kj}=0$,
- (C4) $S^i_{jk}\equiv C^i_{jk}-C^i_{kj}=0$,
- (C5) $g_{ij|k}=0$;

$B\Gamma$ (Okada [7])

- (B1) $L_{|k}=0$,
- (B2) $D^i_{jk}=0$,
- (B3) $T^i_{jk}=0$,
- (B4) $P^i_{jk}\equiv\partial_kN^i_j-F^i_{kj}=0$,
- (B5) $C^i_{jk}=0$.

If we omit some of axioms from each of the above systems, we get various Finsler connections of Cartan type or Berwald type. For example, a *generalized Cartan connection* GCF is defined as a Finsler connection satisfying the axioms of $C\Gamma$ except (C3), and a *Berwald connection with torsion* BFT is defined as a Finsler connection satisfying the axioms of $B\Gamma$ except (B3).

Both GCF and BFT are with surviving $(h)h$ -torsion T^i_{jk} . Furthermore we can consider a Finsler connection of Berwald type in which the $(v)hv$ -torsion P^i_{jk} is also surviving. In a similar way as shown in [4], we have

Theorem 1. *A Finsler connection $(F^i_{jk}, N^i_{jk}, C^i_{jk})$ satisfying (B1), (B2), (B5) can be expressed in terms of its torsions T^i_{jk}, P^i_{jk} as follows:*

$$(1.1) \quad \begin{cases} N^i_{jk} = G^i_{jk} - ((T^i_{kj} + P^i_{0k}) + \partial_k(T^i_{00} + P_{00}^i))/2, \\ F^i_{jk} = \partial_j N^i_k - P^i_{kj}, \\ C^i_{jk} = 0, \end{cases}$$

where G^i_{jk} is the non-linear connection of $B\Gamma$.

Proof. Putting $F=L^2/2$, the condition (B1) is rewritten in an equivalent form

$$(1.2) \quad \partial_i F = y_r N^r_i.$$

Differentiating (1.2) by y^j , we get $\partial_j\partial_i F = g_{rj}N^r_i + y_r\partial_j N^r_i$. By this equations and (1.2), the well-known quantities $2G_j = y^i\partial_j\partial_i F - \partial_j F$ are rewritten in the form

$$2G_j = y^i g_{rj} N^r_i + y^i y_r \partial_j N^r_i - y_r N^r_j.$$

Since $\partial_j N^r_i = F^r_{ij} + P^r_{ij}$, $N^i_j = F^i_{0j} = T^i_{0j} + F^i_{j0}$, we have

$$2G^i = y^r N^i_r + T^i_{00} + P_{00}^i.$$

By differentiating by y^k , we get the expression for N^i_{jk} . The expressions for F^i_{jk} and C^i_{jk} follow from the definition of P^i_{jk} and (B5) respectively.

Putting $P^i_{jk}=0$ in Theorem 1, we get the coefficients of a Berwald connection with torsion BFT of Matsumoto [4]. According to Miron-Hashiguchi [6], a generalized Cartan connection GCF with torsion T^i_{jk} is uniquely determined for an arbitrarily given alter-

nate tensor $T_j^i{}_\kappa$. In the case of $B\Gamma T$, however, the torsion $T_j^i{}_\kappa$ should satisfy the following $B\Gamma$ -condition (Matsumoto [4]):

$$(1.3) \quad y^r(\partial_\kappa T_j^i{}_\tau - \partial_j T_\kappa^i{}_\tau) = 0.$$

In Theorem 1, therefore, the torsions $T_j^i{}_\kappa$, $P^i{}_{j\kappa}$ should satisfy some conditions, too. We want to clear up such a different matter.

§ 2. Matsumoto's $B\Gamma$ -condition.

For a given Finsler connection $F\Gamma = (F_j^i{}_\kappa, N^i{}_\kappa, C_j^i{}_\kappa)$ we get a Finsler connection $FN\Gamma = (\partial_j N^i{}_\kappa, N^i{}_\kappa, 0)$ called the N -connection of $F\Gamma$. Since $\partial_j N^i{}_\kappa = F_j^i{}_\kappa + P^i{}_{\kappa j}$, the $(h)h$ -torsion $Q_j^i{}_\kappa$ of $FN\Gamma$ is expressed by the torsions $T_j^i{}_\kappa$, $P^i{}_{j\kappa}$ of $F\Gamma$ as

$$(2.1) \quad Q_j^i{}_\kappa = T_j^i{}_\kappa - (P^i{}_{j\kappa} - P^i{}_{\kappa j}).$$

Calculating from $Q_j^i{}_\kappa = \partial_j N^i{}_\kappa - \partial_\kappa N^i{}_j$, we have

Proposition 1. *Let $F\Gamma$ be a Finsler connection. If $N^i{}_\kappa$ of $F\Gamma$ is (1) p -homogeneous, the tensor $Q_j^i{}_\kappa$ given by (2.1) satisfies the condition*

$$(2.2) \quad y^r(\partial_\kappa Q_j^i{}_\tau - \partial_j Q_\kappa^i{}_\tau) = 0.$$

In the case of $P^i{}_{j\kappa} = 0$, the condition (2.2) is the $B\Gamma$ -condition (1.3).

In the case of $G\Gamma$ we can choose $T_j^i{}_\kappa$ arbitrarily, but the condition (2.2) is implicitly imposed on $T_j^i{}_\kappa$ together with $P^i{}_{j\kappa}$. In the case of $B\Gamma T$ the condition (2.2) is explicitly imposed on $T_j^i{}_\kappa$ as the $B\Gamma$ -condition, since we assume $P^i{}_{j\kappa} = 0$ for $B\Gamma T$. This is the reason for the difference between the arbitrariness of $T_j^i{}_\kappa$ in $G\Gamma$ and the one in $B\Gamma T$.

We shall here give remarks about the arbitrariness of $Q_j^i{}_\kappa$ satisfying the condition (2.2). By Matsumoto [4] a (0) p -homogeneous alternate tensor $Q_j^i{}_\kappa$ satisfying (2.2) is written in the form

$$(2.3) \quad Q_j^i{}_\kappa = (\partial_\kappa A^i{}_j - \partial_j A^i{}_\kappa) / 2,$$

where $A^i{}_j$ is an arbitrary (1) p -homogeneous tensor. Such a $Q_j^i{}_\kappa$ is also written in the form

$$(2.4) \quad Q_j^i{}_\kappa = A_j^i{}_\kappa + y^r(\partial_\kappa A_j^i{}_\tau - \partial_j A_\kappa^i{}_\tau) / 2,$$

where $A_j^i{}_\kappa$ is an arbitrary (0) p -homogeneous alternate tensor. Then we get

Proposition 2. *For a given (0) p -homogeneous alternate tensor $T_j^i{}_\kappa$, a tensor $P^i{}_{j\kappa}$ satisfying (2.2) is expressed in the form*

$$(2.5) \quad P^i{}_{j\kappa} = (T_j^i{}_\kappa - Q_j^i{}_\kappa) / 2 + B_j^i{}_\kappa,$$

where $Q_j^i{}_\kappa$ is a tensor given by (2.3) or (2.4), and $B_j^i{}_\kappa$ is an arbitrary (0) p -homogeneous symmetric tensor.

For a given (0) p -homogeneous tensor $P^i{}_{j\kappa}$, an alternate tensor $T_j^i{}_\kappa$ satisfying (2.2) is expressed in the form

$$(2.6) \quad T_j^i{}_\kappa = P^i{}_{j\kappa} - P^i{}_{\kappa j} + Q_j^i{}_\kappa,$$

where $Q_j^i{}_\kappa$ is a tensor given by (2.3) or (2.4).

On the other hand, in order to consider the converse problem of Theorem 1, we need another relation satisfied by $P^i{}_{j\kappa}$. If we assume the p -homogeneity of $N^i{}_\kappa$ for a Finsler connection, we have $P^i{}_{j0} = -D^i{}_\kappa$, since $P^i{}_{j\kappa} = \partial_\kappa N^i{}_j - F_\kappa^i{}_j$. Thus we have a well-known

Proposition 3. *Let $F\Gamma$ be a Finsler connection. If $N^i{}_\kappa$ of $F\Gamma$ is (1) p -homogeneous, the de-*

flection tensor D^i_{κ} vanishes if and only if

$$(2.7) \quad P^i_{j0} = 0.$$

§ 3. Generalized Berwald connections.

In this section we assume Finsler connections to be p -homogeneous. The torsions $T^i_{j\kappa}$, $P^i_{j\kappa}$ should satisfy the conditions (2.2) and (2.7). The converse is also true as follows.

Theorem 2. *Given (0) p -homogeneous tensors $T^i_{j\kappa} (= -T^i_{\kappa j})$, $P^i_{j\kappa}$, there exists a unique Finsler connection FG satisfying (B1), (B2), (B5) whose (h)h- and (v)hv-torsion tensors are the given $T^i_{j\kappa}$, $P^i_{j\kappa}$ respectively, if $T^i_{j\kappa}$ and $P^i_{j\kappa}$ satisfy the conditions (2.2), (2.7), where $Q^i_{j\kappa}$ is a tensor given by (2.1).*

The coefficients of FG are given by (1.1). N^i_{κ} of FG is also expressed in the form

$$(3.1) \quad N^i_{\kappa} = G^i_{\kappa} - (Q^i_{\kappa 0} + \partial_{\kappa} Q^i_{00})/2.$$

Proof. It is directly shown that the Finsler connection given by (1.1) satisfies the conditions for FG of Theorem 2. The uniqueness follows from Theorem 1. The p -homogeneity of N^i_{κ} yields (3.1).

We can show here that there exists a Finsler connection FG of Berwald type whose (h)h-torsion is an arbitrarily given (0) p -homogeneous alternate tensor $T^i_{j\kappa}$, if we give up to impose the axiom (B4) on FG . Let $T^i_{j\kappa}$ be a (0) p -homogeneous alternate tensor. If we take the $GCG = (F^i_{j\kappa}, N^i_{\kappa}, C^i_{j\kappa})$ whose (h)h-torsion is the given $T^i_{j\kappa}$, its torsions $T^i_{j\kappa}$ and $P^i_{j\kappa}$ satisfy the conditions (2.2) and (2.7). The Finsler connection given by Theorem 2 has the given (h)h-torsion. So we shall define

Definition 1. A Finsler connection $GBG = (F^i_{j\kappa}, N^i_{\kappa}, G^i_{j\kappa})$ satisfying (B1), (B2), (B5) is called a *generalized Berwald connection*.

We have shown a method to obtain a generalized Berwald connection GBG whose (h)h-torsion is an arbitrarily given (0) p -homogeneous alternate tensor $T^i_{j\kappa}$, by taking $GCG = (F^i_{j\kappa}, N^i_{\kappa}, C^i_{j\kappa})$ whose (h)h-torsion is the given $T^i_{j\kappa}$. It is easily seen that this GBG is nothing but the C -zero connection $(F^i_{j\kappa}, N^i_{\kappa}, 0)$ of GCG . Especially, if we take $T^i_{j\kappa} = 0$, the GCG becomes $CG = (\Gamma^{*i}_{j\kappa}, G^i_{\kappa}, g^i_{j\kappa})$ and the GBG is the Rund connection $RG = (\Gamma^{*i}_{j\kappa}, G^i_{\kappa}, 0)$. Thus we have

Theorem 3. *The C -zero connection of a generalized Cartan connection is a generalized Berwald connection. Especially, the Rund connection is a generalized Berwald connection obtained from the Cartan connection as its C -zero connection.*

The above definition of a generalized Berwald connection GBG has various advantages. First, Theorem 3 and Proposition 3 show a wide freedom of choosing $T^i_{j\kappa}$, $P^i_{j\kappa}$ to construct a GBG . Next, we can give the following comparative definitions for GCG and GBG .

Definition 2. A Finsler connection $GCG = (F^i_{j\kappa}, F^i_{0\kappa}, g^i_{j\kappa})$ satisfying $g_{ij\kappa} = 0$ is called a *generalized Cartan connection*, and a Finsler connection $GBG = (F^i_{j\kappa}, F^i_{0\kappa}, 0)$ satisfying $L_{i\kappa} = 0$ is called a *generalized Berwald connection*.

Last, we pay attention to a generalized Berwald space. A Finsler connection is cal-

led *linear*, if the coefficients $F_j^i{}_{,k}$ depend on position alone. A Finsler space is a generalized Berwald space, if we can introduce a linear *B Γ T* (Matsumoto [4]). Now, if a generalized Berwald connection *GB Γ* is linear, we have $P^i{}_{,jk} = \partial_k F_0^i{}_{,j} - F_k^i{}_{,j} = 0$. So the *GB Γ* becomes a *B Γ T*. Consequently we can also define a generalized Berwald space in terms of a *GB Γ* , even if it is defined by weaker conditions than ones for a *B Γ T*.

Theorem 4. *A Finsler space is a generalized Berwald space, if we can introduce a linear GB Γ .*

By the above considerations a Finsler connection *GB Γ* given by Definition 1 seems to be reasonable as a generalization of the Berwald connection.

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