On iteration schemes K0(n) and K1(n)

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On iteration schemes K_0(n) and K_1(n)

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Abstract.

In this paper we shall analyze behaviors of finite cellular automata CA-90 (n) with boundary conditions of 0-type and 1-type, respectively, by using iteration schemes $K_0(n)$ and $K_1(n)$, introduced in [2].

1. Introduction

In [1] Fujino has developed a theory of finite cellular automata CA-90 (n) with boundary conditions of 0-type and 1-type, respectively. In this paper we shall analyze behaviors of the automata CA-90 (n) of boundary conditions of both types, by using techiques developed in the paper [2].

Finite cellular automaton CA-90 (n) with boundary condition of 0-type is interpreted as a system $K_0(n) = <X, f>$, where X is a set of all n-tuples with elements in $Z_2 = \{0, 1\}$, that is $X = Z_2^n$, and f is a mapping from X into X defined by

$$f(x) = Ax \quad mod\ 2 \qquad (x \in X),$$

where A is a matrix

$$A = \begin{pmatrix} 01 & & & \\ 101 & & 0 & \\ \ 101 & & & \\ & \cdots\ \cdots & & \\ & \ \cdots\ \cdots & & \\ & 0 & & 101 \\ & & & \ 10 \end{pmatrix},$$

and additions and multiplications for computing Ax are mod 2-operations, respectively. And the same automata with boundary condition of 1-type is represented as a system $K_1(n) = <X, g>$, where X is the set Z_2^n, and g is a mapping from X into X defined by

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$$g(x)=Ax+d \quad mod\ 2 \qquad (x\in X)$$

where A is the same matrix used in defining the mapping f, and d is a vector

$$d=\begin{pmatrix}1\\0\\0\\\vdots\\\vdots\\0\\1\end{pmatrix}$$

mod 2-operations are also used in the definition of g.

We get immediately the following relation,

$$g(x)=f(x)+d \quad mod\ 2 \qquad (x\in X) \tag{1.1}$$

In the following we shall show existence of fixed points of f and g, and behaviors of iterated sequences $\{f^p(x)\}$ and $\{g^p(x)\}$ $(p=0, 1, 2, \cdots)$, respectively.

2. Properties of schemes

We get the following propositions about schemes $K_0(n)$ and $K_1(n)$.

Proposition 2.1. *The mapping f: $X\rightarrow X$ is additive mod 2, that is,*

$$f(x+y \quad mod\ 2)=f(x)+f(y) \qquad mod\ 2$$

for each x and y in X.

Proof It follows quickly from the definition of the mapping f.

Proposition 2.2 *The mapping $f: X\rightarrow X$ is bijective if and only if the number n is even.*

Proof First we shall show the following result: The determinant detA *mod* 2, computed by *mod* 2 operations, is not zero if and only if the number n is even. Let us write the determinant detA *mod* 2 by $\det_n$A *mod* 2 in order to clearly express the number n. So we have

$$det_nA\ mod\ 2=\begin{vmatrix}0&1&&&&&\\1&0&1&&&\mathbf{0}&\\&1&0&1&&&\\&&\cdots&&&&\\&&&\cdots&&&\\&&&&\cdots&&\\&\mathbf{0}&&&1&0&1\\&&&&&1&0\end{vmatrix}\ mod\ 2$$

$$= \begin{vmatrix} 01 & & & & & \\ 001 & & & 0 & & \\ 0101 & & & & & \\ & 101 & & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & & & & \cdots & \\ & 0 & & & & 101 \\ & & & & & 10 \end{vmatrix} \; mod\ 2$$

$$= det_{n-2}A \quad mod\ 2$$

By using the above relation successively we get

$$\begin{aligned} det_nA \quad mod\ 2 &= det_{n-2}A \quad mod\ 2 \\ &= det_{n-4}A \quad mod\ 2 \\ &\quad \vdots \\ &= \begin{cases} det_2A \quad mod\ 2 & (\text{if n is even}) \\ det_3A \quad mod\ 2 & (\text{if n is odd}) \end{cases} \end{aligned}$$

So we have

$$det_nA \quad mod\ 2 = \begin{cases} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 & (\text{if n is even}) \\ \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0 & (\text{if n is odd}) \end{cases}$$

and the above result holds.
By the definition of f, the mapping $f: X \to X$ is bijective if and only if $detA$ mod 2 is not zero. The conclusion follows from combining the above two results. □

Now we introduce a mapping, called 'code' from X into N (the set of all natural numbers including zero) by

$$\mathrm{code}(x) = \sum_{j=1}^{n} x_j\, 2^{n-j} \qquad \text{for each } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in X$$

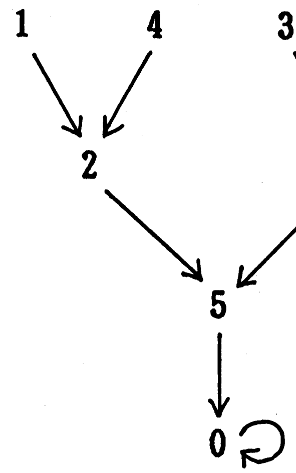
Let us denote an arrow

$$code(x) \longrightarrow code(y)$$

when $f(x)=y$ for x and y in X.

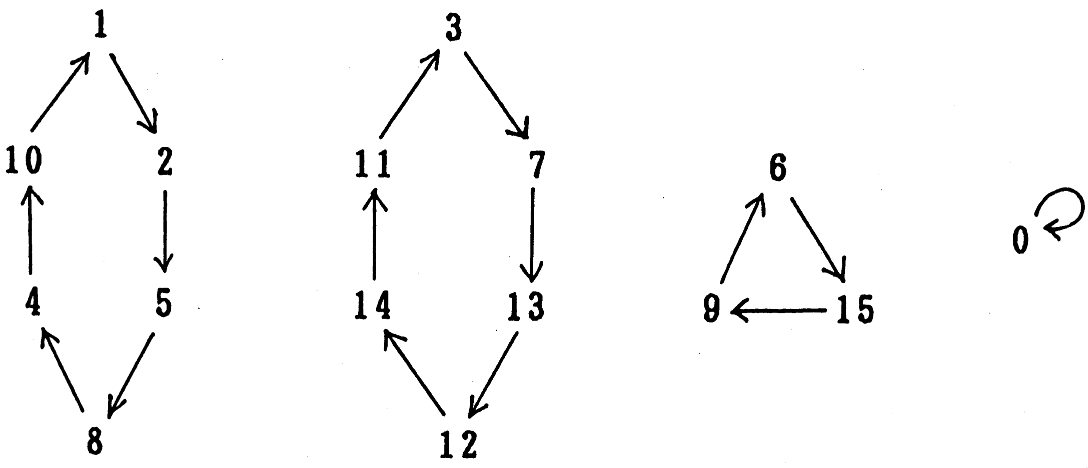
We get a diagram over $code(x)$ by writing all arrows $code(x) \longrightarrow code(y)$ for each x and y in X when $f(x)=y$ holds.

$K_0(3)$:

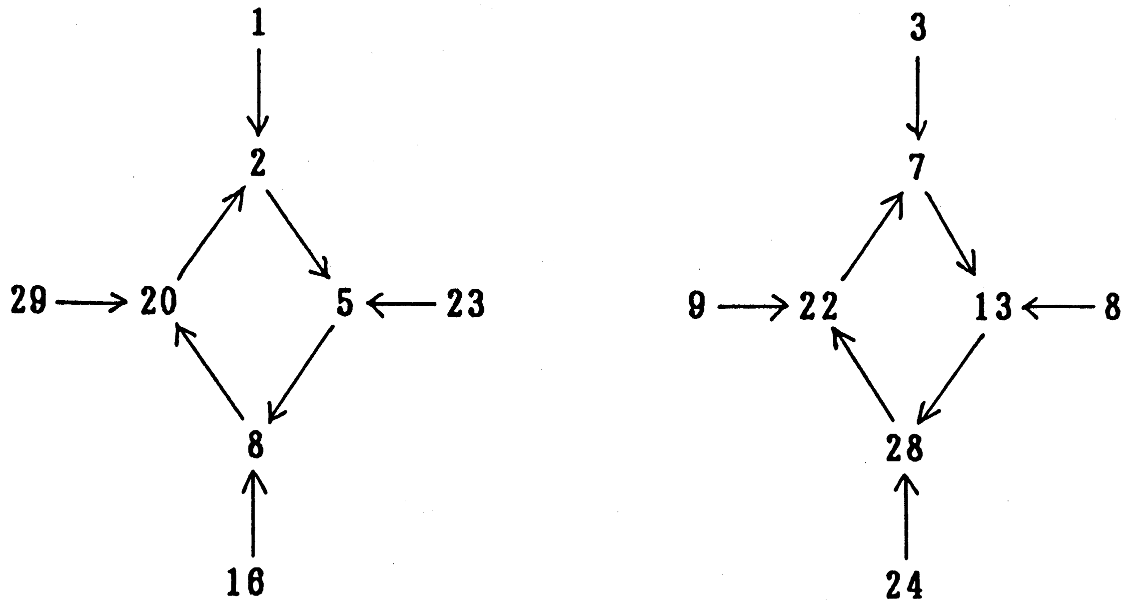


Type of scheme : $\{1_{2[2[2]]}(1)\}$

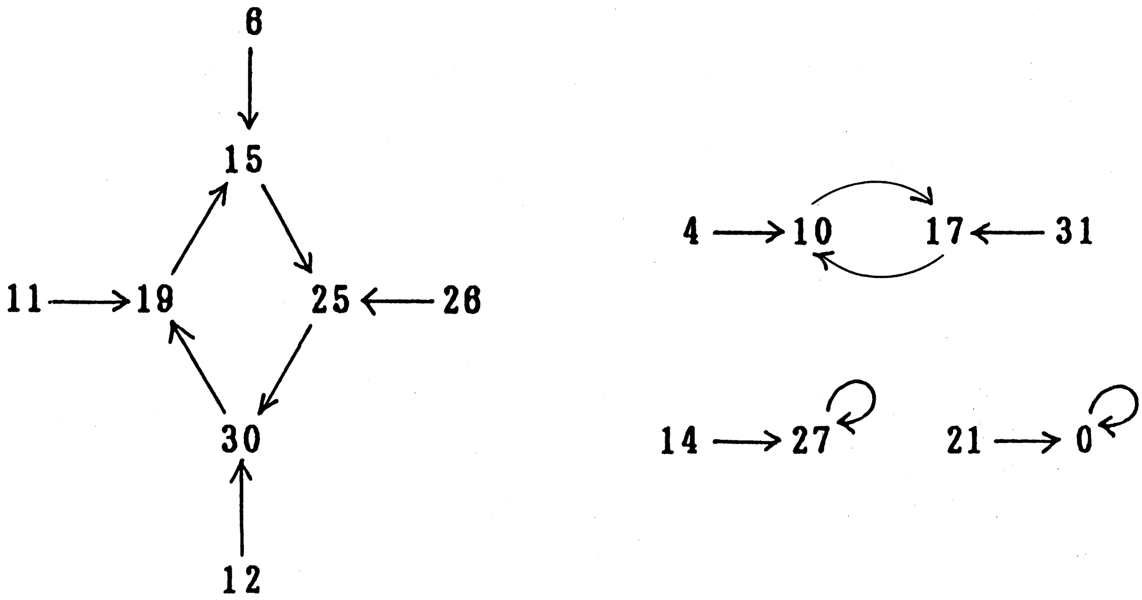
$K_0(4)$:



Type of scheme : $\{6(2), 3(1), 1(1)\}$

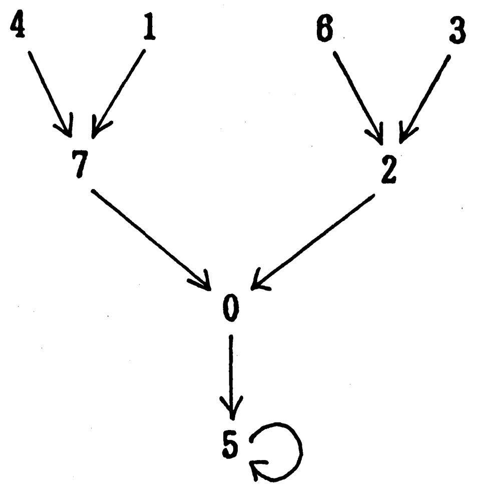


$K_0(5)$:



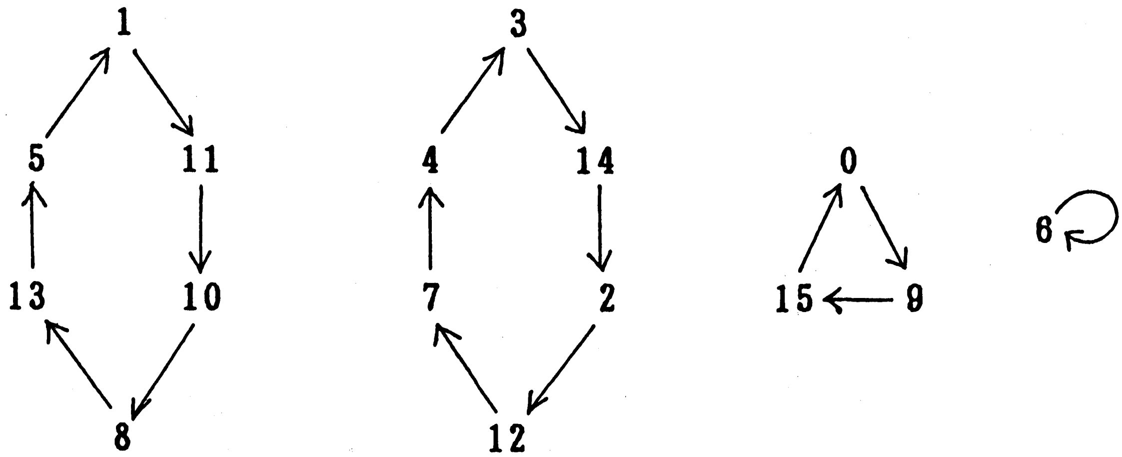
Type of scheme : $\{4_2(3),\ 2_2(1),\ 1_2(2)\}$

$K_1(3)$:



Type of scheme : $\{1_{2[2[2]]}(1)\}$

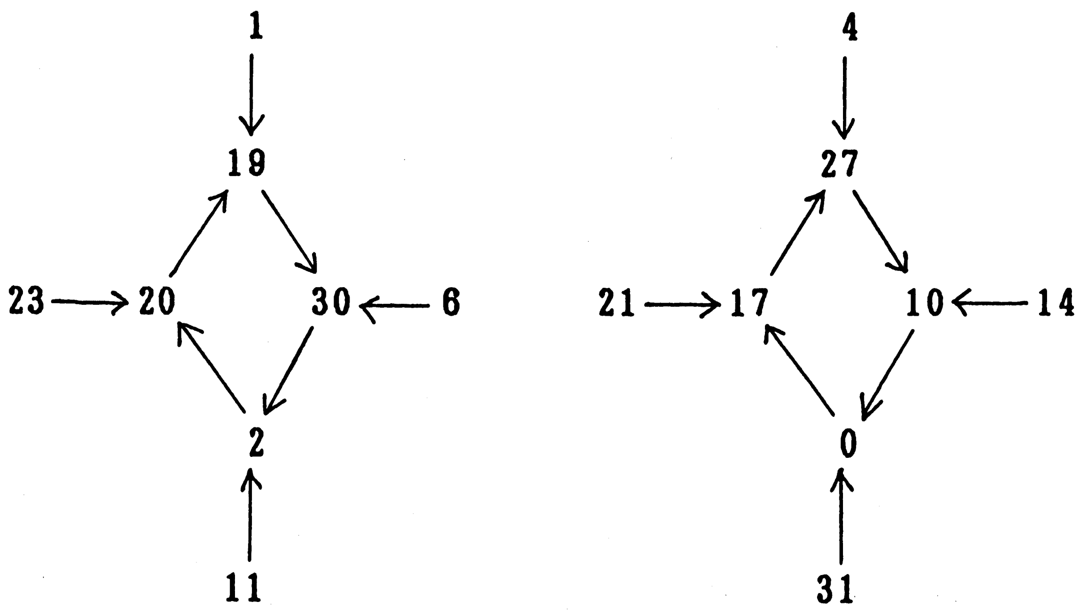
$K_1(4)$:



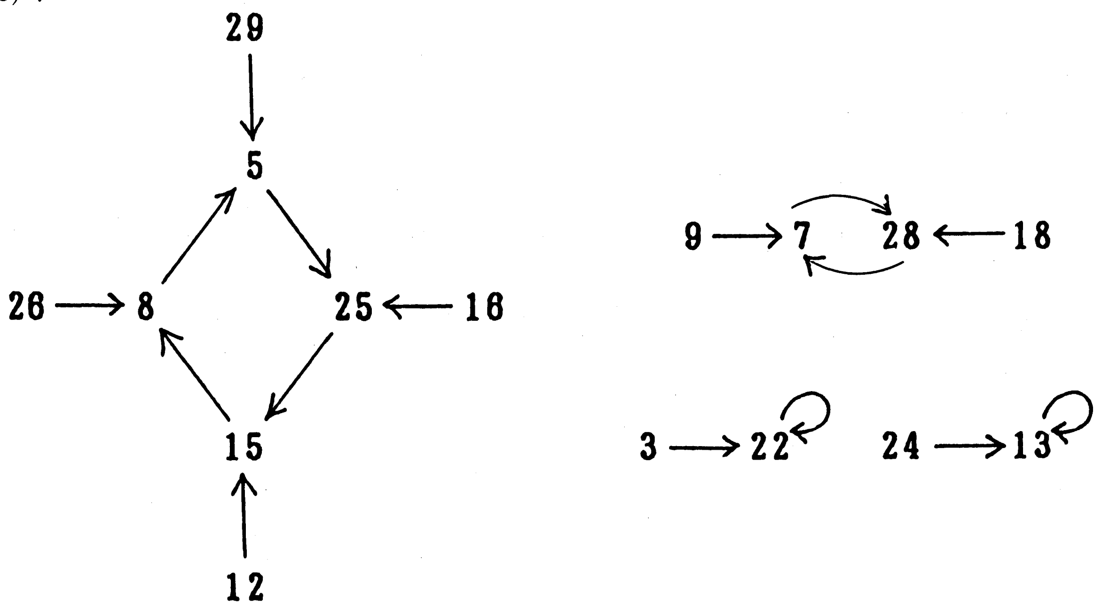
Type of scheme : $\{6(2), 3(1), 1(1)\}$

The diagram, thus constructed, is called a configuration diagram of scheme $K_0(n)$. The configuration diagram of scheme $K_1(n)$ is constructed by using the mapping g in place of the mapping f.

We shall show examples of configuration diagrams of $K_0(n)$ and $K_1(n)$ for $n=3$, 4 and 5.



$K_1(5)$:



Type of scheme : $\{4_2(3), 2_2(1), 1_2(2)\}$

In the above diagrams we show type of schemes by using notations defined in [2].

Proposition 2.3 *If n is odd, then there exists a nonzero vector y in X such that*

$$f(x+y \ mod\ 2)=f(x)$$

and

$$g(x+y \ mod\ 2)=g(x)$$

for every x in X.

Proof Let us take a vector

$$y=\begin{pmatrix}1\\0\\1\\0\\\vdots\\\vdots\\1\\0\\1\end{pmatrix}$$

Then we have $f(y)=Ay \ mod \ 2=0$, So we get, by additivity of the mapping f,

$$\begin{aligned} f(x+y \ mod \ 2) &= f(x)+f(y) \quad mod \ 2 \\ &= f(x), \end{aligned}$$

and, by the formula (1.1)

$$\begin{aligned} g(x+y \ mod \ 2) &= f(x+y \ mod \ 2)+d && mod \ 2, \\ &= f(x)+f(y)+d && mod \ 2, \\ &= f(x)+d && mod \ 2, \\ &= g(x) && mod \ 2, \end{aligned}$$

The conclusion holds. □

Next we shall study fixed points of the mapping f and g.

Proposition 2.4 *Fixed points of the mapping f, that is, the vector x in X satisfying the relation* $f(x)=x$, are as follows:

(i) 0 $(if \ n\equiv 0 \ (mod \ 3))$,

(ii) 0 $(if \ n\equiv 1 \ (mod \ 3))$,

(iii) 0 *and*

$$\begin{pmatrix}1\\1\\0\\1\\1\\0\\\vdots\\\vdots\\1\\1\\0\\1\\1\end{pmatrix} \qquad (if \ n\equiv 2 \ (mod \ 3))$$

Proof The relation $f(x)=x$ is written as follows:

$$\begin{cases} x_2=x_1, \\ x_1+x_3 \; mod \; 2=x_2, \\ x_2+x_4 \; mod \; 2=x_3, \\ \vdots \\ x_{i-1}+x_{i+1} \; mod \; 2=x_i, \\ \vdots \\ x_{n-2}+x_n \; mod \; 2=x_{n-1}, \\ x_{n-1}=x_n \end{cases} \tag{2.1}$$

By the equation (2.1), the zero vector is a fixed point of the mapping f for every n, and the fixed point the first element of which is 1 is a vector in which the second element is 1, the third element is 0, and the last two elements are equal.
Such a fixed point appears only if $n\equiv 2\,(mod\ 3)$, and the form of the fixed point is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Proposition 2.5 *Fixed points of the mapping g are as follows:*

$$\text{(i)}\qquad \left\{\begin{array}{c} \left.\begin{array}{c}1\\0\\1\end{array}\right] \\ \left.\begin{array}{c}1\\0\\1\end{array}\right] \\ \vdots \\ \left.\begin{array}{c}1\\0\\1\end{array}\right] \end{array}\right\} \qquad \text{if } n\equiv 0 \ (mod\ 3),$$

$$\text{(ii)}\qquad \left\{\begin{array}{c} \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ \vdots \\ \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ 0 \end{array}\right\} \qquad \text{if } n\equiv 1 \ (mod\ 3),$$

and

$$\text{(iii)}\qquad \left\{\begin{array}{c} \left.\begin{array}{c}1\\0\\1\end{array}\right] \\ \left.\begin{array}{c}1\\0\\1\end{array}\right] \\ \vdots \\ \left.\begin{array}{c}1\\0\\1\end{array}\right] \\ 1 \\ 0 \end{array}\right\} \qquad \textit{and} \qquad \left\{\begin{array}{c} \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ \vdots \\ \left.\begin{array}{c}0\\1\\1\end{array}\right] \\ 0 \\ 1 \end{array}\right\} \qquad \text{if } n\equiv 2 \ (mod\ 3)$$

Proof The relation $g(x)=x$ is written as follows:

$$\begin{cases} 1+x_2 \; mod \; 2=x_1, \\ x_1+x_3 \; mod \; 2=x_2, \\ x_2+x_4 \; mod \; 2=x_3, \\ \vdots \\ x_{i-1}+x_{i+1} \; mod \; 2=x_i, \\ \vdots \\ x_{n-2}+x_n \; mod \; 2=x_{n-1}, \\ x_{n-1}+1 \; mod \; 2=x_n \end{cases} \quad (2.2)$$

From the equation (2.2), we get

$$x_1=\bar{x}_2,$$

where $\bar{x}_2$ is a *mod* 2-complement of x_2, that is

$$\bar{x}_2=\begin{cases} 1 & (if \;\; x_2=0), \\ 0 & (if \;\; x_2=1). \end{cases}$$

Finally we get

$$x_{n-1}=\bar{x}_n,$$

and

$$x_{n-2}=\begin{cases} x_n & (if \;\; x_{n-1}=0, \text{ that is, } x_n=1) \\ \bar{x}_n & (if \;\; x_{n-1}=1, \text{ that is, } x_n=0) \end{cases}$$

Combining these results, it follows that

(i) Fixed point the first and the last three elements of which are both 1, 0 and 1 appears only if $n\equiv 0\,(mod\ 3)$.

(ii) Fixed point the first and the last three elements of which are 0, 1 and 1, and 1, 1 and 0, respectively, appears only if $n\equiv 1\ (mod\ 3)$.

(iii) Fixed point the first and the last three elements of which are 1, 0 and 1, and 1, 1 and 0, or, 0, 1 and 1 and 1, 0 and 1, respectively, appears only if $n\equiv 2\ (mod\ 3)$.

So we get the conclusion. □

3. Characterization number h(n)

The mappings f and g have the following property.

Proposition 3.1

$$g^p(x)=f^p(x)+g^p(0) \quad for \; each \; x\in X\,(p=1, 2,\cdots)$$

Proof (by induction)

When $p=1$, $g(x)=f(x)+d \quad mod\ 2$ (by definition)

$=f(x)+g(0) \quad mod\ 2 \quad (x \in X)$

Assume that the relation holds for p-1, that is,

$$g^{p-1}(x)=f^{p-1}(x)+g^{p-1}(0) \qquad (x \in X)$$

Then, for each x in X we have

$$g^{p}(x)=g(g^{p-1}(x))$$
$$=f(g^{p-1}(x))+d \qquad mod\ 2$$
$$=f(f^{p-1}(x)+g^{p-1}(0))+d \qquad mod\ 2$$

By additivity of the mapping f,

n	h(n)
1	1
2	2
3	x
4	3
5	4
6	7
7	x
8	14
9	6
10	31
11	x
12	63
13	14
14	30
15	x
16	15
17	28
18	511
19	x
20	126
21	62
22	2047
23	x
24	1023
25	126
26	1022
27	x

n	h(n)
28	16383
29	61
30	31
31	x
32	62
33	30
34	4095
35	x
36	87381
37	1022
38	8190
39	x
40	1023
41	252
42	127
43	x
44	8190
45	4094
46	8388607
47	x
48	2097151
49	2046
50	510
51	x
52	67108863
53	2044
54	1048575
55	x

$$=f^p(x)+f(g^{p-1}(0))+d \qquad mod\ 2$$
$$=f^p(x)+g(g^{p-1}(0)) \qquad mod\ 2$$
$$=f^p(x)+g^p(0)$$

The induction holds. □

So we immediately get the following theorem.

Theorem 3.2. *$f^p=g^p$ if and only if $g^p(0)=0$.*

Let us denote the least positive number p such that $g^p(0)=0$, if exists, by $h(n)$. We call the number $h(n)$, if exists, the characterization number of schemes $K_0(n)$ and $K_1(n)$. By using computer we get the following table of $h(n)$ (see p. 122.).

The table shows the following relation

(1) The number $h(n)$ exists if and only if $n \not\equiv 3 \ (mod\ 4)$.

(2) If n is even, then we have

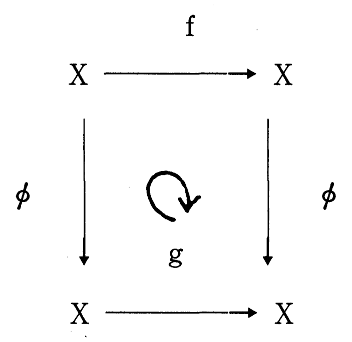
$$h(2n+1)=2h(n)$$

the proofs of which are obtained in [3], using slightly changed form of CA-$90(n)$.

4. Isomorphism between schemes $K_0(n)$ and $K_1(n)$.

The schemes $K_0(n)$ and $K_1(n)$ have the following relation.

Theorem 4.1 *For two schemes $K_0(n)=<X, f>$ and $K_1(n)=<X, g>$, there exists a bijection ϕ from X onto X such that the following diagram commutes, and, if p is a fixed point of the mapping f, then $\phi(p)$ is a fixed point of the mapping g*



Proof Let us take a mapping ϕ as follows:

$$\phi(x)=x+q\ mod\ 2 \quad (x \in X)$$

where q is a fixed point of the mapping g, which exists for each n (proposition 2.5). So we get for each x in X

$$(\phi \cdot f)(x)=f(x)+q \qquad mod\ 2,$$
$$=f(x)+g(q) \qquad mod\ 2,$$
$$=f(x)+f(q)+d \qquad mod\ 2,$$
$$=f(x+q\ mod\ 2)+d \qquad mod\ 2,$$
$$=g(x+q\ mod\ 2) \qquad mod\ 2,$$
$$=g(\phi(x))$$
$$=(g \cdot \phi)(x)$$

And, if p is a fixed point of the mapping f, that is $f(p)=p$, then we have

$$g(\phi(p))=\phi(f(p)) \qquad \text{(the commutavity)},$$
$$=\phi(p)$$

So the vector $\phi(p)$ is a fixed point of the mapping g. □

We can get similar results for schemes representing finite cellular automata CA-150(n) with both boundary conditions of 0-type and 1-type, respectively.

References

[1] S. FUJINO: *On the behavior of linear finite cellular automata CA-90(m)*, *RMC* **63**-03*J* (1988), *pp.* 25

[2] T. KITAGAWA and S. FUJINO: *On modular type iteration scheme*, *RMC* **64**-09*J* (1989), *pp.* 171

[3] Y. KAWAHARA: *On the existence of Fujino's characterization number H(m) associated with finite cellular automata CA-90(m-1)*, *RMC* **63**-08*J* (1988), *pp.* 17