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Yoshihiro ICHIJYŌ* and Masao HASHIGUCHI†

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Abstract

In the present paper we discuss the condition that an $(a,\ b,\ f)$-manifold be a Berwald space and give an example of a nearly Kaehlerian Finsler manifold.

Key words: Berwald space, Nearly Kaehlerian Finsler manifold, Rizza manifold, Generalized Randers space.

1 Introduction

In our previous paper [8] we discussed $(a,\ b,\ f)$-manifolds and gave an example of a Kaehlerian Finsler manifold (cf. Ichijyō [3, 4, 5]). Let $(M,\ \alpha)$ be a Riemannian manifold of even dimension $m = 2n$ $(n \geq 2)$. We denote a point of M and a tangent vector at the point by $x = (x^i)$ and $y = (y^i)$ respectively. Given a non-zero covariant vector field $b_i(x)$ and an almost Hermitian structure $f^i{}_j(x)$ on $(M,\ \alpha)$:

(1.1) $$f^i{}_r f^r{}_j = -\delta^i{}_j,\quad a_{rs} f^r{}_i f^s{}_j = a_{ij},$$

where $\alpha(x,\ y) = (a_{ij}(x) y^i y^j)^{1/2}$, we put

(1.2) $$b_{ij} = b_i b_j + f_i f_j,$$

where

(1.3) $$f_i = b_r f^r{}_i.$$

Since rank $(b_{ij}) = 2$, we have a singular Riemannian metric $\beta(x,\ y) = (b_{ij}(x) y^i y^j)^{1/2}$ on M. Thus, putting $L = \alpha + \beta$, we have a generalized Randers space $(M,\ L)$ which was named an $(a,\ b,\ f)$-*manifold* by Ichijyō [3] (cf. Ichijyō-Hashiguchi [7]).

* Department of Mathematical Science, Tokushima Bunri University, Yamashiro-cho, Tokushima 770, Japan.
† 2–24–7 Kinkodai, Kagoshima 891–01, Japan.

With respect to the Levi-Civita connection $\Gamma = (\{_j{}^i{}_k\})$ of the Riemannian manifold (M, α) we denote the covariant differentiation by ∇_k. We have shown

Theorem 1.1 ([8], Theorem 2.1) *An (a, b, f)-manifold $(M, \alpha+\beta)$ is a Berwald space if and only if $\nabla_k b_{ij} = 0$.*

It is noted that in an (a, b, f)-manifold satisfying $\nabla_k b_{ij} = 0$ the linear Finsler connection $B\Gamma = (\{_j{}^i{}_k\}, y^r\{_r{}^i{}_k\}, 0)$ is just the Berwald connection.

If an (a, b, f)-manifold is *normal*:

$$\nabla_k b_i = 0, \ \nabla_k f^i{}_j = 0, \tag{1.4}$$

we have $\nabla_k f_i = 0$, and so $\nabla_k b_{ij} = 0$. Thus we have

Theorem 1.2 ([8], Theorem 3.1) *A normal (a, b, f)-manifold is a Berwald space.*

In the second section we shall consider the converse problem and give the condition that an (a, b, f)-manifold be a Berwald space in terms of the given structures $a_{ij}, b_i, f^i{}_j$ (Theorem 2.1).

Now, corresponding to a Riemannian manifold with an almost Hermitian structure, we have a Finsler manifold (M, L, f) with an almost complex structure f such that the metric function L satisfies

$$L(x, \phi_\theta y) = L(x, y) \ \ (0 \leq \theta \leq 2\pi), \tag{1.5}$$

where $\phi_\theta{}^i{}_j = (\cos\theta)\,\delta^i{}_j + (\sin\theta)\, f^i{}_j$. This was named a *Rizza manifold* by Ichijyō [4]. An (a, b, f)-manifold gives a typical example of a Rizza manifold.

A Rizza manifold (M, L, f) is called a *Kaehlerian Finsler manifold* if

$$\nabla^*{}_k f^i{}_j = 0 \tag{1.6}$$

is satisfied, where $\nabla^*{}_k$ denotes the h-covariant differentiation with respect to the Cartan connection $C\Gamma$. This is a Finsler manifold corresponding to a Kaehlerian manifold in Riemannian geometry (cf. Ichijyō [4, 5]). We have shown

Theorem 1.3 ([8], Theorem 3.3) *A normal (a, b, f)-manifold is a Kaehlerian Finsler manifold.*

In his recent paper [9], H. S. Park generalized the notion of Kaehlerian Finsler manifold and discussed the Finsler manifold called the nearly Kaehlerian Finsler manifold. In the third section, based on the discussion of the second section, we shall give an example of a nearly Kaehlerian Finsler manifold (Theorem 3.1).

The present paper continues from the series of our recent papers [6, 7, 8] and the terminology and notation are also followed.

2 Condition that an (a, b, f)-manifold be Berwald

We shall obtain the condition that an (a, b, f)-manifold $(M, \alpha+\beta)$ be a Berwald space. We put $b^i = a^{ir}b_r$, $f^i = a^{ir}f_r$, and $b = (b_r b^r)^{1/2}$. Then we have

$$b^r f_r = 0, \ f_r f^r{}_i = -b_i, \ f_r f^r = b^2. \tag{2.1}$$

It is noted that at each point the vector fields b_i and f_i are orthogonal and have the same length.

Now, let us assume that $\nabla_k b_{ij} = 0$. Then we have

$$(\nabla_k b_i)b_j + b_i(\nabla_k b_j) + (\nabla_k f_i)f_j + f_i(\nabla_k f_j) = 0. \tag{2.2}$$

Contracting (2.2) by b^j and b^i successively, we have $(\nabla_k b_r)b^r = 0$, from which we have

$$b^2(\nabla_k b_i) + (\nabla_k f_r)b^r f_i = 0. \tag{2.3}$$

Putting $\lambda_k = -(1/b^2)(\nabla_k f_r)b^r$ we have $\nabla_k b_i = \lambda_k f_i$.

Then from $f_i = b_r f^r{}_i$ we have

$$\nabla_k f_i = -\lambda_k b_i + b_r(\nabla_k f^r{}_i). \tag{2.4}$$

Substituting from (2.4) into (2.2) we have

$$b_r(\nabla_k f^r{}_i)f_j + b_r(\nabla_k f^r{}_j)f_i = 0. \tag{2.5}$$

Contracting (2.5) by f^j and f^i successively, we have $b_r(\nabla_k f^r{}_i) = 0$. Thus from (2.4) we have $\nabla_k f_i = -\lambda_k b_i$.

Conversely, it is directly shown that $\nabla_k b_i = \lambda_k f_i$ and $\nabla_k f_i = -\lambda_k b_i$ imply $\nabla_k b_{ij} = 0$. Thus we have been lead to the following definition and theorem.

Definition 2.1 Two vector fields b_i and f_i given on a Riemannian manifold (M, α) are called *cross-recurrent* if there exists a vector field λ_k satisfying

$$\nabla_k b_i = \lambda_k f_i, \ \nabla_k f_i = -\lambda_k b_i. \tag{2.6}$$

Theorem 2.1 *An (a, b, f)-manifold $(M, \alpha+\beta)$ is a Berwald space if and only if the vector fields b_i and f_i are cross-recurrent. Then b_i and f_i are orthogonal and have the same constant length.*

It is noted that in an (a, b, f)-manifold the condition (2.6) is equivalent to

$$\nabla_k b_i = \lambda_k f_i, \ b_r(\nabla_k f^r{}_i) = 0, \tag{2.7}$$

and as a special case we have the condition

$$\nabla_k b_i = 0, \ \nabla_k f_i = 0, \tag{2.8}$$

or equivalently

$$\nabla_k b_i = 0, \ b_r(\nabla_k f^r{}_i) = 0. \tag{2.9}$$

3 Nearly normal (a, b, f)-metrics

A Rizza manifold (M, L, f) is called a *nearly Kaehlerian Finsler manifold* if

(3.1) $$\nabla^*_k f^i{}_j + \nabla^*_j f^i{}_k = 0$$

is satisfied. This is a Finsler manifold corresponding to a Riemannian manifold called a nearly Kaehlerian manifold (an almost Tachibana manifold or a K-space) (cf. Park [9], Yano-Kon [11]). By using Theorem 2.1 we shall give an example of a nearly Kaehlerian Finsler manifold.

Definition 3.1 An (a, b, f)-manifold $(M, \alpha+\beta)$ is called *nearly normal* if the vector fields b_i and f_i are cross-recurrent and the condition

(3.2) $$\nabla_k f^i{}_j + \nabla_j f^i{}_k = 0$$

is satisfied.

Since a nearly normal (a, b, f)-manifold $(M, \alpha+\beta)$ is a Berwald space, the h-covariant differentiaton with respect to $C\Gamma$ coincides with the one with respect to $B\Gamma = (\{_j{}^i{}_k\}, y^r\{_r{}^i{}_k\}, 0)$, and the h-covariant derivative of $f^i{}_j(x)$ with respect to $B\Gamma$ becomes the covariant derivative with respect to $\Gamma = (\{_j{}^i{}_k\})$ of the associated Riemannian manifold (M, α), so we have $\nabla^*_k f^i{}_j = \nabla_k f^i{}_j$. Thus we have

Theorem 3.1 *A nearly normal (a, b, f)-manifold is a nearly Kaehlerian Finsler manifold.*

Now, let (M, α, f) be a nearly Kaehlerian Riemannian manifold. Can we find a vector field b_i such that the corresponding (a, b, f)-manifold $(M, \alpha+\beta)$ becomes a nearly Kaehlerian Finsler manifold? This is an interesting open problem. It is noted that a 4-dimensional nearly Kaehlerian Riemannian manifold is nothing but a Kaehlerian manifold (cf. Gray [2], Takamatsu [10]) and there exists a concrete example of a 6-dimensional non-Kaehlerian nearly Kaehlerian Riemannian manifold (cf. Fukami-Ishihara [1]).

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