Design of Recursive Wiener Fixed-Point Smoother based on Innovations Approach in Linear Discrete-Time Stochastic Systems

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Abstract. This paper proposes a new design for the recursive Wiener fixed-point smoother and filter based on the innovations approach in linear discrete-time stochastic systems. The estimators require the information of the observation matrix, the system matrix for the state variable, related with the signal, the crossvariance function of the state variable with the observed value and the variance of the white observation noise. It is assumed that the signal is observed with additive white noise.

Keywords. Wiener-Hopf equation, Linear discrete-time systems, Recursive Wiener filter, Covariance information, Fixed-point smoother

1. Introduction

The estimation problem given the covariance information has been investigated extensively in the area of the detection and estimation problems for communication systems [1], [2], [3]. Here, it is assumed that the autocovariance function of the signal is expressed in the semi-degenerate kernel form. The semi-degenerate kernel is the function suitable for expressing a general kind of autocovariance function by a finite sum of nonrandom functions. The recursive Wiener fixed-point smoother [4] and filter [5] using the covariance information are designed in linear discrete-time stochastic systems. The estimators require the information of the observation matrix, the system matrix for the state variable, related with the signal, and the crossvariance function of the state variable with the observed value. The information can be obtained from the covariance function of the signal [4]. Also, it is assumed that the variance of the white observation noise is known. In [1], based on the innovations theory, the recursive fixed-point smoother using the covariance information is designed in linear continuous-time systems.

This paper examines to design the recursive Wiener fixed-point smoother and filter, using the same information as those in [4], based on the innovations approach, in linear discrete-time stochastic systems. Also, as in [4], it is assumed that the variance of the white observation noise is known. It is assumed that the signal is observed with additive white observation noise. The algorithms are derived based on the invariant imbedding method [6].

The fixed-point smoothing error variance function of the signal is also derived. From this function, it is shown that the proposed fixed-point smoother is more accurate than the filter.

2. Fixed-point smoothing problem

Let an observation equation be given by

$$y(k) = z(k) + v(k), z(k) = Hx(k), \tag{1}$$

in discrete-time stochastic systems, where $z(k)$ is an $m \times 1$ signal vector, $x(k)$ is an $n \times 1$ state variable, H is an $m \times n$ observation matrix and $v(k)$ is white observation noise. It is assumed that the signal and the observation noise are mutually independent and that $z(k)$ and $v(k)$ are zero mean. Let the autocovariance function of $v(k)$ be given by

$$E\left[v(k)v^T(s)\right] = R(k)\delta_K(k-s), R(k) > 0 \tag{2}$$

Here, $\delta_K(\cdot)$ denotes the Kronecker δ function.

Let a fixed-point smoothing estimate $\hat{x}(k,L)$ of $x(k)$ be given by

$$\hat{x}(k,L) = \sum_{i=1}^{L} g(k,i)\upsilon(i) \tag{3}$$

as a linear transformation of the innovations process $\{\upsilon(i),\ 1 \le i \le L\}$, where $g(k,s)$ and k are referred to be an impulse response function and the fixed point. The innovations process is expressed by $\upsilon(k) = y(k) - H\Phi\hat{x}(k-1,k-1)$, where $\hat{x}(k-1,k-1)$ is a filtering estimate of $x(k\text{-}1)$ and Φ represents the system matrix in the state equation for $x(k)$. The autocovariance function of the innovations process is given by [7]

$$E\left[\upsilon(k)\upsilon^T(s)\right]=\Pi(k)\delta_K(k-s). \tag{4}$$

The impulse response function, which minimizes the mean-square value of the fixed-point smoothing error,

$$J=E\left[\left\|x(k)-\hat{x}(k,L)\right\|^2\right], \tag{5}$$

satisfies

$$E\left[x(k)\upsilon^T(s)\right]=\sum_{i=1}^{L}g(k,i)E\left[\upsilon(i)\upsilon^T(s)\right] \tag{6}$$

by an orthogonal projection lemma [7]:

$$x(k)-\sum_{i=1}^{L}g(k,i)\upsilon(i)\perp\upsilon(s),\ 0\le s,k\le L. \tag{7}$$

Here, '$\perp$' denotes the notation of the orthogonality. From (4) and (6), the impulse response function is given by

$$g(k,s)=E\left[x(k)\upsilon^T(s)\right]\Pi^{-1}(s),\ \Pi(s)=E\left[\upsilon(s)\upsilon^T(s)\right]. \tag{8}$$

Let $K(k,s)$ denote the autocovariance function of the state variable $x(k)$ and let $K(k,s)$ be expressed in the form as

$$K(k,s)=\begin{cases}A(k)B^T(s), & 0\le s\le k,\\ B(k)A^T(s), & 0\le k\le s,\end{cases}$$

$$A(k)=\Phi^k, B^T(s)=\Phi^{-s}K_{xy}(s,s). \tag{9}$$

Here, $K_{xy}(s,s)$ represents the crossvariance function between the state variable $x(s)$ and the observed value $y(s)$.

The fixed-point smoothing estimate in (3) can be written as

$$\hat{x}(k,L)=\sum_{i=1}^{k}g(k,i)\upsilon(i)+\sum_{i=k+1}^{L}g(k,i)\upsilon(i). \tag{10}$$

The first term on the right hand side of (10) represents the filtering estimate $\hat{x}(k,k)$ of $x(k)$ and the second term is the correction quantity for the fixed-point smoothing estimate [7]. Hence, the fixed-point smoothing estimate $\hat{x}(k,L)$ of $x(k)$ is represented as

$$\hat{x}(k,L)=\hat{x}(k,k)+\sum_{i=k+1}^{L} g(k,i)\upsilon(i),\ \hat{x}(k,k)=\sum_{i=1}^{k} g(k,i)\upsilon(i), \tag{11}$$

in terms of the set of the innovations process $\{\upsilon(s),\ 1\le s\le L\}$.

Let us note that the linear filtering estimate $\hat{x}(k,k)$ is also given by

$$\hat{x}(k,k)=\sum_{i=1}^{k} h(k,i)y(i) \tag{12}$$

as a linear transformation of the observation set $\{y(s),\ 1\le s\le k\}$ [2],[6]. The impulse response function $h(k,s)$, which minimizes the mean-square value of the filtering error $x(k)-\hat{x}(k,k)$,

$$J=E\left[\|x(k)-\hat{x}(k,k)\|^2\right], \tag{13}$$

satisfies the Wiener-Hopf equation

$$E\left[x(k)y^T(s)\right]=\sum_{i=1}^{k} h(k,i)E\left[y(i)y^T(s)\right] \tag{14}$$

by an orthogonal projection lemma [7]:

$$x(k)-\sum_{i=1}^{k} h(k,i)y(i)\perp y(s),\ 0\le s\le k. \tag{15}$$

Substituting (1) into (14), and using the statistical properties for the signal and the observation noise, we obtain the equation

$$h(k,s)R(s)=K_{xy}(k,s)-\sum_{i=1}^{k} h(k,i)HK_{xy}(i,s),\ K_{xy}(i,s)=K(i,s)H^T,\ 0\le s\le k, \tag{16}$$

which the optimal impulse response function $h(k,s)$ satisfies. Here, $K_{xy}(i,s)$ is the crosscovariance function of $x(i)$ with $y(s)$.

For $k<s$, by substituting $\upsilon(s)=y(s)-H\Phi\hat{x}(s,s-1)$ into (8) and using (9) and (12), we have

$$g(k,s)=\left\{E\left[x(k)x^T(s)\right]H^T-\sum_{j=1}^{s-1}E\left[x(k)y^T(j)\right]h^T(s-1,j)\Phi^T H^T\right\}\Pi^{-1}(s)$$

$$=\left\{E\left[x(k)x^T(s)\right]H^T-\sum_{j=1}^{k}E\left[x(k)y^T(j)\right]h^T(s-1,j)\Phi^T H^T-\right.$$

$$\left.\sum_{j=k+1}^{s-1}E\left[x(k)y^T(j)\right]h^T(s-1,j)\Phi^T H^T\right\}\Pi^{-1}(s)$$

$$=\left\{B(k)A^T(s)H^T-\sum_{j=1}^{k}A(k)B^T(j)H^T J^T(s-1,j)(\Phi^T)^{s-1}\Phi^T H^T-\right.$$

$$\left.\sum_{j=k+1}^{s-1}B(k)A^T(j)H^T J^T(s-1,j)(\Phi^T)^{s-1}\Phi^T H^T\right\}\Pi^{-1}(s). \tag{17}$$

Introducing functions

$$D(s-1,k)=\sum_{j=1}^{k}J(s-1,j)HB(j) \tag{18}$$

and

$$E(s-1,k)=\sum_{j=k+1}^{s-1}J(s-1,j)HA(j), \tag{19}$$

we rewrite (17) as

$$g(k,s)=\left\{B(k)A^T(s)H^T-A(k)D^T(s-1,k)(\Phi^T)^{s-1}\Phi^T H^T-\right.$$
$$\left.B(k)E^T(s-1,k)(\Phi^T)^{s-1}\Phi^T H^T\right\}\Pi^{-1}(s). \tag{20}$$

By introducing the functions

$$q_1(s-1,k)=\Phi^{s-1}D(s-1,k) \tag{21}$$

and

$$q_2(s-1,k)=\Phi^{s-1}E(s-1,k), \tag{22}$$

we also rewrite (20) as

$$g(k,s)=\{B(k)A^T(s)H^T - A(k)q_1^T(s-1,k)\Phi^T H^T - B(k)q_2^T(s-1,k)\Phi^T H^T\}\Pi^{-1}(s). \tag{23}$$

3. RLS algorithm for the fixed-point smoothing estimate

Let us derive the algorithms for the fixed-point smoothing and filtering estimates. The algorithms are derived based on the invariant imbedding method [6].

Introducing an auxiliary function, which satisfies

$$J(k,s)R(s)=\Phi^{-s}K_{xy}(s,s)-\sum_{i=1}^{k}J(k,i)HK_{xy}(i,s), \tag{24}$$

we find from (16) and (24) that

$$h(k,s)=\Phi^k J(k,s),\ 0\le s\le k. \tag{25}$$

Subtracting the equation obtained by putting $k\to k-1$ in (24) from (24), we have

$$(J(k,s)-J(k-1,s))R(s)=-J(k,k)HK_{xy}(k,s)-\sum_{i=1}^{k-1}(J(k,i)-J(k-1,i))HK_{xy}(i,s). \tag{26}$$

From (24) and (26) with (9), the equation updating *J(k,s)* is given by

$$J(k,s)=J(k-1,s)-J(k,k)H\Phi^k J(k-1,s). \tag{27}$$

From (24), the function *J(k,k)* in (26) satisfies

$$J(k,k)R(k)=\Phi^{-k}K_{xy}(k,k)-\sum_{i=1}^{k}J(k,i)HK_{xy}(i,k). \tag{28}$$

Substituting (9) into (28) and introducing

$$r(k)=\sum_{i=1}^{k} J(k,i)K_{xy}^{T}(i,i)(\Phi^{T})^{-i}, \tag{29}$$

we obtain

$$J(k,k)=(\Phi^{-k}K_{xy}(k,k)-r(k)(\Phi^{T})^{k}H^{T})R^{-1}(k). \tag{30}$$

Subtracting the equation obtained by putting $k\rightarrow k-1$ in (29) from (29), we have

$$r(k)-r(k-1)=J(k,k)K_{xy}^{T}(k,k)(\Phi^{T})^{-k}+\sum_{i=1}^{k-1}(J(k,i)-J(k-1,i))K_{xy}^{T}(i,i)(\Phi^{T})^{-i}. \tag{31}$$

Substituting (27) into (31) and using (29), we obtain

$$r(k)=r(k-1)+J(k,k)(K_{xy}^{T}(k,k)(\Phi^{T})^{-k}-H\Phi^{k}r(k-1)). \tag{32}$$

The initial condition on the difference equation (32) at k=0 is $r(0)$=0 from (29).

From (30) and (32), $J(k,k)$ is expressed as

$$J(k,k)=(\Phi^{-k}K_{xy}(k,k)-r(k-1)(\Phi^{T})^{k}H^{T})(R(k)+K_{xy}^{T}(k,k)H^{T}-H\Phi^{k}r(k-1)(\Phi^{T})^{k}H^{T})^{-1}. \tag{33}$$

Putting $S(k)=\Phi^{k}r(k)(\Phi^{k})^{T}$, from (32) and (33), we obtain

$$\begin{aligned}&S(k)=\Phi S(k-1)\Phi^{T}+(K_{xy}(k,k)-\Phi S(k-1)\Phi^{T}H^{T})\\&(R(k)+K_{xy}^{T}(k,k)H^{T}-H\Phi S(k-1)\Phi^{T}H^{T})^{-1}(K_{xy}^{T}(k,k)-H\Phi S(k-1)\Phi^{T}),\\&S(0)=0.\end{aligned} \tag{34}$$

From (12) and (25), the filtering estimate $\hat{x}(k,k)$ is written as

$$\hat{x}(k,k)=\Phi^{k}\sum_{i=1}^{k} J(k,i)y(i). \tag{35}$$

Introducing a function

$$e(k)=\sum_{i=1}^{k}J(k,i)y(i), \tag{36}$$

we obtain

$$\hat{x}(k,k)=\Phi^{k}e(k). \tag{37}$$

Subtracting the equation obtained by putting $k\rightarrow k-1$ in (36) from (36) and using (27), we obtain

$$e(k)-e(k-1)=J(k,k)(y(k)-H\Phi^{k}e(k-1)). \tag{38}$$

The initial condition on the difference equation (38) at k=0 is $e(0)$ from (36). Substituting (38) into (37) and using (33), we obtain

$$\begin{aligned}&\hat{x}(k,k)=\Phi\hat{x}(k-1,k-1)+(K_{xy}(k,k)-\Phi S(k-1)\Phi^{T}H^{T})\\&(R(k)+K_{xy}^{T}(k,k)H^{T}-H\Phi S(k-1)\Phi^{T}H^{T})^{-1}(y(k)-H\Phi\hat{x}(k-1,k-1)),\\&\hat{x}(0,0)=0.\end{aligned} \tag{39}$$

In (39), it is clear that $S(k)$ represents the autovariance function of the filtering estimate

$\hat{x}(k,k), i.e.,\ S(k)=E\left[\hat{x}(k,k)\hat{x}^{T}(k,k)\right]$.

Substituting (23) into (11), we have

$$\begin{aligned}\hat{x}(k,L)=\hat{x}(k,k)+\sum_{i=k+1}^{L}&(B(k)A^{T}(i)H^{T}-A(k)q_{1}^{T}(i-1,k)\Phi^{T}H^{T}\\&-B(k)q_{2}^{T}(i-1,k)\Phi^{T}H^{T})\Pi^{-1}(i)\upsilon(i).\end{aligned} \tag{40}$$

Subtracting $\hat{x}(k,L-1)$ from $\hat{x}(k,L)$ we have

$$\begin{aligned}\hat{x}(k,L)-\hat{x}(k,L-1)=&(B(k)A^{T}(L)H^{T}-A(k)q_{1}^{T}(L-1,k)\Phi^{T}H^{T}\\&-B(k)q_{2}^{T}(L-1,k)\Phi^{T}H^{T})\Pi^{-1}(L)\upsilon(L).\end{aligned} \tag{41}$$

From (18) and (27), we see that

$$D(L,k)-D(L-1,k)=\sum_{j=1}^{k}(J(L,j)-J(L-1,j))HB(j)$$
$$=-J(L,L)H\Phi^{L}D(L-1,k). \tag{42}$$

From (42), we have

$$\Phi^{L}D(L,k)-\Phi^{L}D(L-1,k)=-\Phi^{L}J(L,L)H\Phi^{L}D(L-1,k). \tag{43}$$

From (21) and (25), (43) is rewritten as

$$q_{1}(L,k)=\Phi q_{1}(L-1,k)-h(L,L)H\Phi q_{1}(L-1,k). \tag{44}$$

From (19) and (27), we see that

$$E(L,k)-E(L-1,k)=J(L,L)HA(L)+\sum_{j=k+1}^{L-1}(J(L,j)-J(L-1,j))HA(j)$$
$$=J(L,L)HA(L)-J(L,L)H\Phi^{L}\sum_{j=k+1}^{L-1}J(L-1,j)HA(j)$$
$$=J(L,L)H(A(L)-\Phi^{L}E(L-1,k)). \tag{45}$$

From (45), we have

$$\Phi^{L}E(L,k)=\Phi\Phi^{L-1}E(L-1,k)+\Phi^{L}J(L,L)H(A(L)-\Phi\Phi^{L-1}E(L-1,k)). \tag{46}$$

From (22) and (25), (46) is rewritten as

$$q_{2}(L,k)=\Phi q_{2}(L-1,k)+h(L,L)H(A(L)-\Phi q_{2}(L-1,k)). \tag{47}$$

From (19) and (22), it is noted that $q_{2}(k,k)=\Phi^{k}E(k,k)=0$.

Now, from (9), (18) and (29), let us note that

$$D(k,k)=\sum_{j=1}^{k}J(k,j)HB(j)=r(k). \tag{48}$$

r(k) satisfies the difference equation (32). Hence, Substituting (21) into

$$\Phi^k D(k,k) = \Phi\Phi^{k-1}D(k-1,k-1) + h(k,k)H(B(k) - \Phi^k D(k-1,k-1)), \tag{49}$$

we obtain

$$q_1(k,k) = \Phi q_1(k-1,k-1) + h(k,k)H(B(k) - \Phi q_1(k-1,k-1)),\ q_1(0,0) = 0. \tag{50}$$

$q_1(k,k)$ provides the initial condition for the difference equation (44).

Let us put $P_1(L,k)$ and $P_2(L,k)$ as

$$P_1(L,k) = \Phi^k q_1^T(L,k) \tag{51}$$

and

$$P_2(L,k) = B(k)q_2^T(L,k). \tag{52}$$

Hence, (41) is rewritten as

$$\hat{x}(k,L) = \hat{x}(k,L-1) + (K(k,k)(\Phi^T)^{L-k}H^T - P_1(L-1,k)\Phi^T H^T - P_2(L-1,k)\Phi^T H^T)\Pi^{-1}(L)\upsilon(L). \tag{53}$$

From (44), we have

$$A(k)q_1^T(L,k) = A(k)q_1^T(L-1,k)\Phi^T - A(k)q_1^T(L-1,k)\Phi^T H^T h^T(L,L). \tag{54}$$

From (51) and (54), we obtain

$$P_1(L,k) = P_1(L-1,k)\Phi^T - P_1(L-1,k)\Phi^T H^T h^T(L,L). \tag{55}$$

From (25), (33) and $S(k) = \Phi^k r(k)(\Phi^k)^T$, *h(k,k)* is expressed as

$$h(k,k) = (K_{xy}(k,k) - \Phi S(k-1)\Phi^T H^T)(R(k) + K_{xy}^T(k,k)H^T - H\Phi S(k-1)\Phi^T H^T)^{-1}. \tag{56}$$

By the way, from (21), (48) and (51),we find that

$$P_1(k,k) = A(k)q_1^T(k,k) = A(k)(\Phi^k D(k,k))^T = \Phi^k r(k)(\Phi^T)^k = S(k). \tag{57}$$

From (47), we have

$$B(k)q_2^T(L,k) = B(k)q_2^T(L-1,k)\Phi^T + (B(k)A^T(L) - B(k)q_2^T(L-1,k)\Phi^T)H^T h^T(L,L). \tag{58}$$

From (52) and (58), we obtain

$$P_2(L,k) = P_2(L-1,k)\Phi^T + (K(k,k)(\Phi^T)^{L-k} - P_2(L-1,k)\Phi^T)H^T (R(L) + K_{xy}^T(L,L)H^T - H\Phi S(L-1)\Phi^T H^T)^{-1}(K_{xy}^T(L,L) - H\Phi S(L-1)\Phi^T),$$
$$P_2(k,k) = 0. \tag{59}$$

Now, let us summarize the above equations in Theorem 1.

Theorem 1. Let the system matrix Φ, the crossvariance function of $x(k)$ with the observed value $y(k)$ and the variance $R(k)$ of the white observation noise be given. Then the recursive Wiener fixed-point smoothing and filtering equations consist of (60)-(66) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \tag{60}$$

Fixed-point smoothing estimate of the state variable $x(k)$: $\hat{x}(k,L)$

$$\hat{x}(k,L) = \hat{x}(k,L-1) + (K(k,k)(\Phi^T)^{L-k}H^T - P_1(L-1,k)\Phi^T H^T - P_2(L-1,k)\Phi^T H^T)\Pi^{-1}(L)\upsilon(L) \tag{61}$$

$$P_1(L,k) = P_1(L-1,k)\Phi^T - P_1(L-1,k)\Phi^T H^T (R(L) + K_{xy}^T(L,L)H^T - H\Phi S(L-1)\Phi^T H^T)^{-1}(K_{xy}^T(L,L) - H\Phi S(L-1)\Phi^T),$$
$$P_1(k,k) = S(k) \tag{62}$$

$$P_2(L,k) = P_2(L-1,k)\Phi^T + (K(k,k)(\Phi^T)^{L-k} - P_2(L-1,k)\Phi^T H^T \Pi^{-1}(L)(K_{xy}^T(L,L) - H\Phi S(L-1)\Phi^T),$$
$$\Pi(L) = R(L) + K_{xy}^T(L,L)H^T - H\Phi S(L-1)\Phi^T H^T,$$
$$P_2(k,k) = 0 \tag{63}$$

$$S(k)=\Phi S(k-1)\Phi^T+(K_{xy}(k,k)-\Phi S(k-1)\Phi^T H^T)$$
$$(R(k)+K_{xy}^T(k,k)H^T-H\Phi S(k-1)\Phi^T H^T)^{-1}(K_{xy}^T(k,k)-H\Phi S(k-1)\Phi^T),$$
$$S(0)=0 \quad (64)$$

Filtering estimate of $z(k):\hat{z}(k,k)$

$$\hat{z}(k,k)=H\hat{x}(k,k) \quad (65)$$

Filtering estimate of $x(k):\hat{x}(k,k)$

$$\hat{x}(k,k)=\Phi\hat{x}(k-1,k-1)+(K_{xy}(k,k)-\Phi S(k-1)\Phi^T H^T)$$
$$(R(k)+K_{xy}^T(k,k)H^T-H\Phi S(k-1)\Phi^T H^T)^{-1}(y(k)-H\Phi\hat{x}(k-1,k-1)),$$
$$\hat{x}(0,0)=0 \quad (66)$$

4. Fixed-point smoothing error variance function of signal

Let *P(k,L)* represent the fixed-point smoothing error variance function of the signal.

$$P(k,L)=E\left[(z(k)-\hat{z}(k,L))(z(k)-\hat{z}(k,L))^T\right] \quad (67)$$

Let the second term on the right hand side of (40) be *f(k,L)* as follows.

$$\hat{x}(k,L)=\hat{x}(k,k)+f(k,L) \quad (68)$$

$$f(k,L)=\sum_{i=k+1}^{L}(B(k)A^T(i)H^T-A(k)q_1^T(i-1,k)\Phi^T H^T-B(k)q_2^T(i-1,k)\Phi^T H^T)\Pi^{-1}(i)\upsilon(i) \quad (69)$$

From the orthogonal projection lemmas $z(k)-\hat{z}(k,L)\perp\hat{z}(k,L)$ and $x(k)-\hat{x}(k,L)\perp\upsilon(i), i=k+1,k+2,\cdots,L$, (67) is written as, in terms of (3), (8) and (68),

$$P(k,L)=HK_{xy}(k,k)-H\sum_{i=1}^{k}g(k,i)E\left[\upsilon(i)\hat{z}^T(k,L)\right]-H\sum_{i=k+1}^{L}g(k,i)E\left[\upsilon(i)\hat{z}^T(k,L)\right]$$

$$= HK_{xy}(k,k) - HE\left[\hat{x}(k,k)\hat{z}^T(k,L)\right] - H\sum_{i=k+1}^{L} g(k,i)\Pi(i)g^T(k,i)H^T$$

$$= HK_{xy}(k,k) - HE\left[\hat{x}(k,k)\hat{x}^T(k,k)\right]H^T - HE\left[\hat{x}(k,k)f^T(k,k)\right]H^T$$

$$-H\sum_{i=k+1}^{L} g(k,i)\Pi(i)g^T(k,i)H^T. \qquad (70)$$

From (69), it is seen that the third term on the right hand side of (70) equals the zero matrix.

The variance function *p(k,k)* of the filtering error $z(k)-\hat{z}(k,k)$ is expressed as. $P(k,k) = HK_{xy}(k,k) - HS(k)H^T$. Hence,

$$0 \le P(k,L) \le P(k,k). \qquad (71)$$

This shows that the fixed-point smoothing estimate might be superior in estimation accuracy to the filter.

5. A Numerical simulation example

Let a scalar observation equation be given by

$$y(k)=z(k)+v(k). \qquad (72)$$

Let the observation noise *v(k)* be zero-mean white Gaussian process with the variance, *R*, *N*(0,*R*). Let the autocovariance function of the signal *z(k)* be given by

$$K(0) = \sigma^2,$$
$$K(m) = \sigma^2\left\{\alpha_1\left(\alpha_2^2-1\right)\alpha_1^m / \left[\left(\alpha_2-\alpha_1\right)\left(\alpha_2\alpha_1+1\right)\right]\right.$$
$$\left. -\alpha_2\left(\alpha_1^2-1\right)\alpha_2^m / \left[\left(\alpha_2-\alpha_1\right)\left(\alpha_1\alpha_2+1\right)\right]\right\},\ 0<m,$$
$$\alpha_1,\alpha_2 = \left(-a_1 \pm \sqrt{a_1^2-4a_2}\right)/2, a_1 = -0.1, a_2 = -0.8, \sigma = 0.5. \qquad (73)$$

Based on the method in [4],[8], the observation vector *H,* the crossvariance $K_{xy}(k,k)$ and the system matrix Φ in the state equation for the state variable *x(k)* are as follows:

$$H = [1 \quad 0], K_{xy}(k,k) = \begin{bmatrix} K(0) \\ K(1) \end{bmatrix}, \Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},$$

$$K(0) = 0.25, K(1) = 0.125. \quad (74)$$

If we substitute (74) into the estimation algorithms of Theorem 1, we can calculate the fixed-point smoothing estimate $\hat{z}(k,L)$ and the filtering estimate $\hat{z}(k,k)$ of the signal recursively.

Fig.1 illustrates the signal *z(k),* the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,k+5)$ vs. k for the white Gaussian observation noise. $N(0,0.5^2)$. Fig.2 illustrates the mean-square values (MSVs) of the fixed-point and filtering errors by the proposed recursive Wiener estimators for the observation noises, $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$ vs. L, $0 \le L \le 10$. For L=0, the MSV of the filtering error is shown. The MSVs of the fixed-point smoothing and filtering errors are evaluated by $\sum_{i=1}^{500}\sum_{k=1}^{L}(z(i)-\hat{z}(i,i+k))^2/(500\cdot L)$ and $\sum_{i=1}^{500}(z(i)-\hat{z}(i,i))^2/500$.

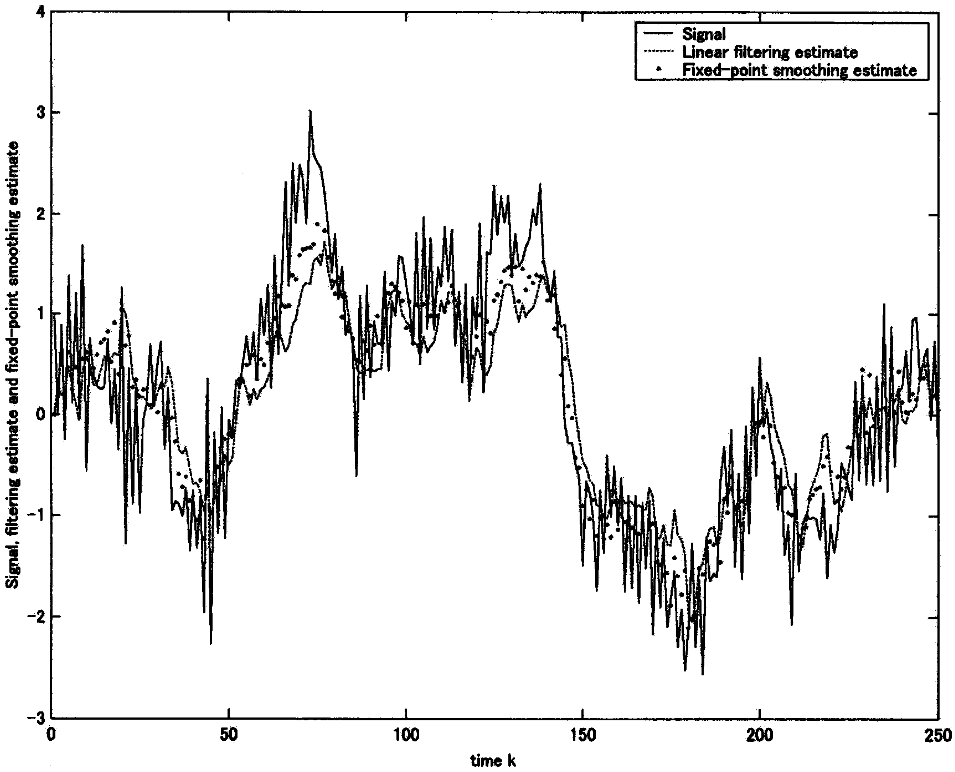

Fig.1 Signal *z(k),* the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,k+5)$ by the proposed recursive Wiener fixed-point smoother vs. k for the white Gaussian observation noise $N(0,0.5^2)$.

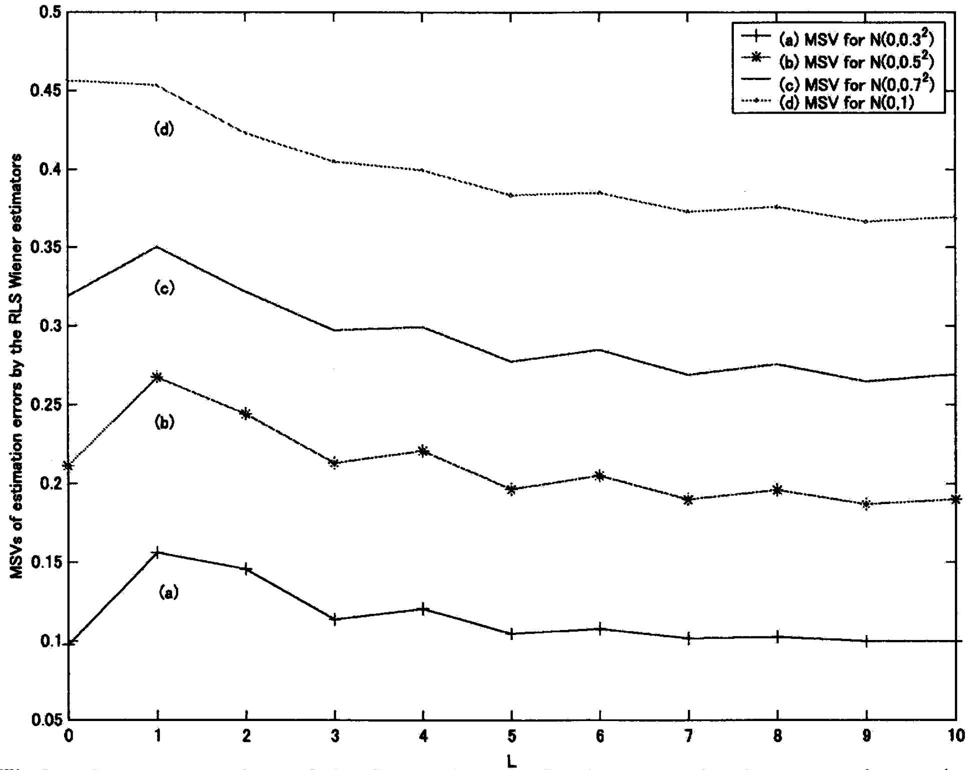

Fig.2 Mean-square values of the fixed-point and filtering errors by the proposed recursive Wiener estimators for the observation noises $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,0.7^2)$ and $N(0,1)$ *vs*. L, $0 \le L \le 10$.

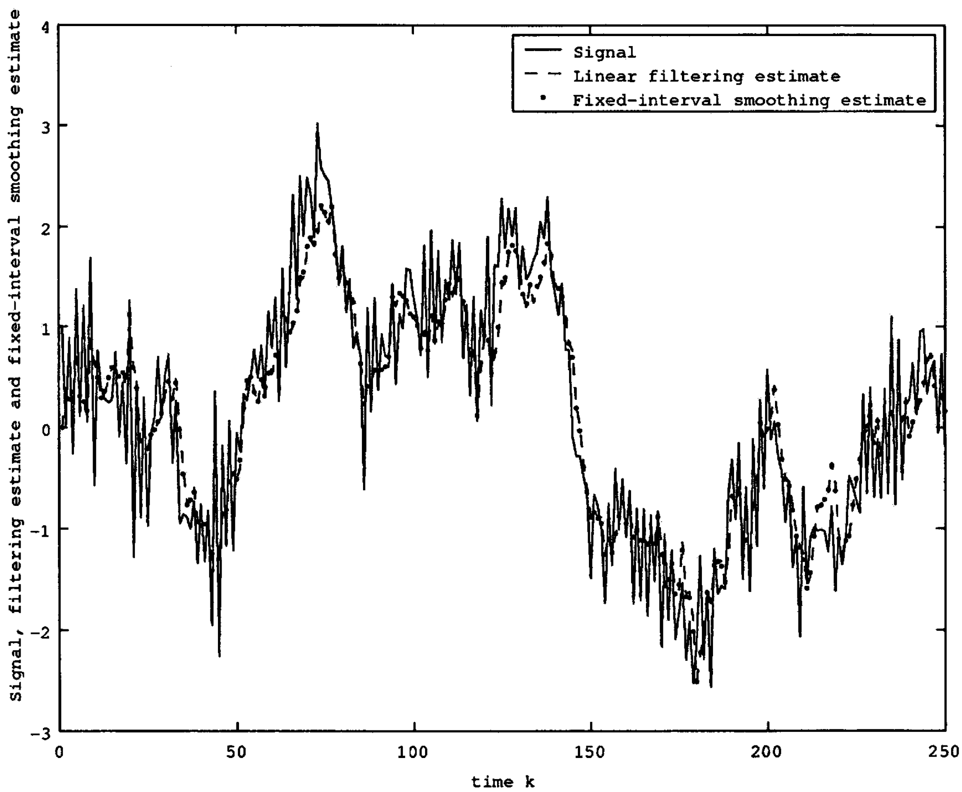

Fig.3 The signal $z(k)$, the filtering estimate $\hat{z}(k,k)$ and the fixed-interval smoothing estimate $\hat{z}(k,250)$ vs. k for the white Gaussian observation noise $N(0,0.5^2)$

In Fig.1, the fixed-point smoothing estimate is superior in estimation accuracy to the filtering estimate. Fig.2 shows that the estimation accuracy of the fixed-point smoother is improved in comparison with the filter. As the noise variance becomes large, the estimation accuracies of the smoother and the filter are degraded. In the fixed-point smoother, for the fixed value of L, the fixed-interval smoothing estimate can also be calculated. Fig.3 illustrates the signal $z(k)$, the filtering estimate $\hat{z}(k,k)$ and the fixed-interval smoothing estimate $\hat{z}(k,L)$ for $L=250$ vs. k under the white Gaussian observation noise $N(0,0.5^2)$.

It should also be mentioned that the MSV of the fixed-point smoothing error is precisely equals that of the recursive Wiener fixed-point smoother in [4].

For references, the autoregressive (AR) model, which generates the signal process, is given by

$$z(k+1) = -a_1 z(k) - a_2 z(k-1) + w(k+1), \ E[w(k)w(s)] = \sigma^2 \delta_K(t-s). \tag{75}$$

6. Conclusions

In this paper, based on the innovations approach, the recursive Wiener fixed-point smoother and filter have been designed in linear discrete-time stochastic systems. It is assumed that the covariance information of the signal and the observation noise is known.

From the simulation results, it has been shown that the proposed estimation algorithms are feasible and the estimation accuracy of the fixed-point smoother equals that of the recursive Wiener fixed-point smoother in [4]. The fixed-point smoothing algorithm in [4] is somewhat simple in comparison with the current smoothing algorithm.

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