ON FINITE TOPOLOGICAL SPACES

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By

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§ 1. Introduction

V. KRISHNAMURTHY [1] has intended to estimate the number of the possible topologies in a given finite set, in which he has mentioned that a topology of a finite set can be represented by a suitable matrix. It seems that such matrix is a powerful tool to study the structure of finite topological spaces.

After defining a topogenous matrix in section 2, we shall consider some elementary properties of finite spaces in section 3. Especially, it is shown that the topogenous matrix for the product space $X\times Y$ is the direct product of the topogenous matrices of X and Y. In section 4, a concept of a dual space of a finite space is introduced. In section 5, it is shown that the topogenous matrix of a finite T_0-space is equivalent to a triangular matrix. In section 6, a topological invariant which we call be the degree of connection is defined.

§ 2. Topogenous matrix

Let X be a finite set $\{a_1, a_2, \cdots, a_n\}$ and let τ be a topology on X, where τ is the family of all open subsets of X.

Since X is finite, each point a_i of X has a unique minimal neighborhood U_i, which is the intersection of open neighborhoods of a_i. Hence if X is the set $\{a_1, a_2, \cdots, a_n\}$, then $\{U_1, U_2, \cdots, U_n\}$ is an open neighborhood basis of the space (X, τ).

Now, we shall introduce the topogenous matrices which play an important role in our investigation of finite topological spaces.

DEFINITION 1. In a finite space (X, τ), let a (n, n) matrix $A=[a_{ij}]$ be defined as follows:

$$a_{ij}=1 \quad \text{if} \quad a_j \in U_i$$
$$\quad =0 \quad \text{otherwise} \qquad (i, j=1, 2, \cdots, n).$$

Then, the matrix A is said to be a *topogenous matrix* of the space (X, τ).

A topogenous matrix has the following important properties.

THEOREM 1. (V. KRISHNAMURTHY). *Let A be the topogenous matrix of a finite space X, then the matrix A satisfies the following three conditions*

(1) $$a_{ij}=0 \quad or \quad 1,$$

(2) $$a_{ii}=1,$$

(3) $$a_{ik}=a_{kj}=1 \Rightarrow a_{ij}=1,$$

where $i, j, k=1, 2, \cdots, n$.

Conversely, if a matrix $A=[a_{ij}]$ *satisfies these three conditions, then* A *induces a topology in* X.

DEFINITION 2. We define the *permutation matrix*, which corresponds to the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix}$, by $T=[\delta_{ip(j)}]$ where $\delta_{ip(j)}$ is the Kronecker's delta.

DEFINITION 3. Given two topogenous matrices A and B, if there exists a permutation matrix T such that

$$B=T'AT,$$

then A and B are said to be *equivalent* to each other, and are noted as

$$A \sim B.$$

THEOREM 2. *Let* τ_1 *and* τ_2 *be two topologies in a finite set* X, *and let* A_1 *and* A_2 *be the topogenous matrices of* (X, τ_1) *and* (X, τ_2) *respectively. Then* (X, τ_1) *and* (X, τ_2) *are homeomorphic if and only if* A_1 *and* A_2 *are equivalent.*

PROOF. Let $\boldsymbol{B}_1=\{U_1, U_2, \cdots, U_n\}$ and $\boldsymbol{B}_2=\{V_1, V_2, \cdots, V_n\}$ be the minimal basic neighborhood systems of (X, τ_1) and (X, τ_2), respectively, and let $f:(X, \tau_1)\rightarrow(X, \tau_2)$ be a homeomorphism such that

$$f(a_i)=a_{p(i)} \qquad (i=1, 2, \cdots, n).$$

f induces a mapping

$$f(U_i)=V_{p(i)} \qquad (i=1, 2, \cdots, n),$$

which preserves the inclusion relation in $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$. If A_1 is noted as $[a_{ij}]$, then we have

$$a_{ij}=1 \Leftrightarrow U_j \subset U_i \Leftrightarrow V_{p(j)} \subset V_{p(i)}$$
$$\Leftrightarrow a_{p(i)p(j)}=1,$$

and

$$a_{ij}=a_{p(i)p(j)} \qquad (i, j=1, 2, \cdots, n).$$

Set

$$T=[\delta_{ip(j)}],$$

where $\delta_{ip(j)}$ is the Kronecker's delta.

Then we have

$$A_2 = T' A_1 T,$$

i. e.

$$A_1 \sim A_2.$$

Conversely assume $A_1 \sim A_2$, then there exists a permutation matrix $T=[\delta_{ip(j)}]$ such that $A_2 = T' A_1 T$. Define $f : (X, \tau_1) \rightarrow (X, \tau_2)$ by

$$f(a_i) = a_{p(i)},$$

then f is a homeomorphism.

From the above proof we also obtain:

COROLLARY. *A matrix which is equivalent to a certain topogenous matrix is a topogenous matrix.*

§ 3. Some elementary properties

We consider a topological space (X, τ) with a finite set $X = \{a_1, a_2, \cdots, a_n\}$ and a family $\{U_1, U_2, \cdots, U_n\}$ of the corresponding minimal basic neighborhoods. And between elements of (X, τ), we define the following order:

$$a_i \leq a_j \Leftrightarrow U_i \subset U_j \qquad (\text{or } a_i \in U_j),$$

then this relation $\leq$ is transitive and reflexive. This means that $(X, \leq)$ is a quasi ordered set.

The following is the known result.

THEOREM 3. (ALEXANDROFF [2]). *A finite space (X, τ) is a T_0-space if and only if $(X, \leq)$ is a partially ordered set.*

REMARK 1. Let $A = [a_{ij}]$ be the topogenous matrix of a finite space (X, τ), then

$$a_{ij} = 1 \Leftrightarrow a_j \leq a_i.$$

REMARK 2. Given two finite spaces (X, τ) and (X, σ), mapping $f : X \rightarrow Y$ is continuous if and only if the following is satisfied:

$$a \leq b (a, b \in X) \Rightarrow f(a) \leq f(b).$$

Let A be the topogenous matrix of a finite space (X, τ), where $X = \{a_1, a_2, \cdots, a_n\}$. For a subset $Y = \{b_1, b_2, \cdots, b_k\}$ of X, we consider the subspace (Y, τ_Y) and its topogenous matrix A_Y.

We have the following theorem.

THEOREM 4. *(Y, τ_Y) is closed in (X, τ) if and only if the following holds:*

$$A \sim \left[\begin{array}{c|c} A_Y & * \\ \hline 0 & A_{Y^c} \end{array}\right] \quad \text{or} \quad A \sim \left[\begin{array}{c|c} A_{Y^c} & 0 \\ \hline * & A_Y \end{array}\right].$$

Proof. Y is closed in (X, τ) if and only if $U_x \cap Y = \phi$ for $x \in Y^c$ and for the minimal neighborhood U_x of x. Hence from the definition of the topogenous matrix, the theorem follows immediately.

Corollary 1. *Y is open and closed if and only if the following holds:*

$$A \sim \left[\begin{array}{c|c} A_Y & 0 \\ \hline 0 & A_{Y^c} \end{array}\right].$$

Corollary 2. *Let A be the topogenous matrix of a finite space (X, τ). Then we have*

$$A \sim \begin{bmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_k} \end{bmatrix}$$

where A_i is a topogenous matrix of a component of (X, τ).

For finite spaces (X, τ) and (Y, σ), let A_X and A_Y be the topogenous matrices of (X, τ) and (Y, σ), respectively, and let $A_{X \times Y}$ be the topogenous matrix of the product space $(X \times Y, \tau \times \sigma)$. As for the relations between the matrices A_X, A_Y and the matrix $A_{X \times Y}$. We have the following theorem.

Theorem 5. *Let A_X, A_Y and $A_{X \times Y}$ be the topogenous matrices of finite spaces (X, τ), (Y, σ) and the product space $(X \times Y, \tau \times \sigma)$, respectively. Then $A_{X \times Y}$ is equivalent to the direct product of A_X and A_Y, i. e.*

$$A_{X \times Y} \sim A_X \dot{\times} A_Y.$$

Proof. Let $X = \{a_1, a_2, \cdots, a_m\}$ and $Y = \{b_1, b_2, \cdots, b_n\}$, $\{U_1, U_2, \cdots, U_m\}$ and $\{V_1, V_2, \cdots, V_n\}$ be families of the minimal basic neighborhoods, respectively. And let $A_X = [a_{ij}]$ and $A_Y = [b_{ij}]$ be the topogenous matrices of these spaces.

Now, consider

$$X \times Y = \{(a_i, b_j) \mid i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n\}.$$

If U_i and V_j are minimal neighborhoods of a_i in X and b_j in Y, then $U_i \times V_j$ is the minimal neighborhood of (a_i, b_j) in $X \times Y$. Then the topogenous matrix $A_{X \times Y}$ is noted in the form

$$A_{X\times Y}=\begin{array}{c} \\ U_1\times V_1 \\ \vdots \\ U_i\times V_j \\ \vdots \\ U_m\times V_n \end{array}\begin{array}{c} (a_1,\ b_1)\cdots(a_h,\ b_k)\cdots(a_m b_n) \\ \left[\begin{array}{c} \vdots \\ \cdots\cdots\cdots c_{(i,j)(h,k)}\cdots\cdots\cdots \\ \vdots \end{array}\right] \end{array},$$

where

$$c_{(i,j)(h,\,k)}=1 \Leftrightarrow (a_h,\ b_k)\in U_i\times V_j$$

$$\Leftrightarrow a_h\in U_i \quad \text{and} \quad b_k\in V_j$$

$$\Leftrightarrow a_{ih}=1 \quad \text{and} \quad b_{jk}=1.$$

Hence, we have $A=A_X \dot{\times} A_Y$.

§ 4. Dual spaces

A finite space is a quasi ordered set $(X, \geq)$. Then there exists a quasi ordered set $(X, <)$ in which the ordering $a<b$ means $a\geq b$ in $(X, \geq)$. The *dual space* of a finite topological space (X, τ) is a topological space corresponding to $(X, <)$. The following is evident.

LEMMA 1. *Let M be the (n, n) topogenous matrix of a finite space (X, τ). Then a space $(X, \tilde{\tau})$ is a dual space of (X, τ) if and only if the topogenous matrix N of $(X, \tilde{\tau})$ is equivalent to the transposed matrix M' of M.*

Let $M=[a_{ij}]$ be a (n, n) topogenous matrix. Then we define a matrix $M^*=[a^*_{ij}]$ by

$$a^*_{ij}=a_{n-j+1\ n-i+1}.$$

The matrix M^* is obtained from mapping reflectively all elements of M in the diagonal which is not principal. We have the following.

THEOREM 6. *Let M be the (n, n) topogenous matrix of a finite space (X, τ). Then*

$$M'\sim M^*$$

is satisfied.

PROOF. If we put $M=[a_{ij}]$, $M'=[a'_{ij}]$, $M^*=[a^*_{ij}]$, then we have

$$a^*_{ij}=a_{n-j+1\ n-i+1}=a'_{n-i+1\ n-j+1}.$$

Now let $T(i, j)$ be the permutation matrix which corresponds to a transposition (i, j), and consider

$$T = T(n, 1)T(n-1, 2) \cdots T(n-k, k+1) \cdots \quad \left(k \leq \frac{n-1}{2}\right).$$

Then we have

$$T'M^*T = M',$$

i. e.

$$M^* \sim M'.$$

§ 5. Finite T_0-spaces

In the present section, we shall find a condition for the topogenous matrix that a given finite topological space is a T_0-space.

THEOREM 7. *A finite space (X, τ) is a T_0-space if and only if the topogenous matrix A of (X, τ) is equivalent to a certain triangular matrix.*

PROOF. Assume that (X, τ) is a finite T_0-space with a topogenous matrix A. Let $X = \{a_1, a_2, \cdots, a_n\}$, and let U_i be the minimal neighborhood of a_i. We note N_i the number of the elements of U_i, and rearrange X as $X = \{a_{p_1}, a_{p_2}, \cdots, a_{p_n}\}$ such that if $i \leq j$, then $N_{p_i} \leq N_{p_j}$. We consider the topogenous matrix $B = [b_{ij}]$ which corresponds to the new basis $(a_{p_1}, a_{p_2}, \cdots, a_{p_n})$ of X. If $i < j$, then we have $a_{p_i} \notin U_{p_j}$, and $b_{ij} = 0$. Therefore, B is a triangular matrix which is equivalent to A.

Conversely, assume that for a topogenous matrix A of (X, τ), there exists a triangular matrix $A_1 = [a_{ij}]$ such that $A \sim A_1$. If $a_i, a_j \in X$ and $a_i < a_j$, then $a_{ij} = 0$ since A_1 is a triangular matrix. Hence $a_j \notin U_i$, that is, (X, τ) is a T_0-space.

§ 6. Degree of connection

Let (X, τ) be a finite T_0-space and let $A = [a_{ij}]$ be the topogenous matrix of X, and E be the unit matrix.

Now, set

$$dA = A - E,$$

and define

$$d^p A = (dA)^p \qquad (p = 1, 2, \cdots),$$

$$d^0 A = E.$$

DEFINITION 4. Let (X, τ) be a finite T_0-space and let A be the topogenous matrix of X. If

$$d^n A \neq 0 \quad \text{and} \quad d^{n+1} A = 0,$$

then we say that the *degree of connection* of X is n, and we denote

$$degX=n.$$

Let $(X, \geq)$ be a partially ordered set, and x be an element of X. Consider the family of chains in X such that

$$x_0<x_1<x_2<\cdots<x_h=x.$$

The maximum number of h defined for all the above chains is denoted by $l[x]$, and is called the *length of* x. We also define *the length* $l[X]$ *of* X by

$$l[X]=max\{l[x] \mid x \in X\}.$$

THEOREM 8. *If* (X, τ) *is a finite* T_0*-space, then*

$$degX=l[X].$$

PROOF. Let $A=[a_{ij}]$ be the triangular topogenous matrix.
Set

$$d^pA=[a_{ij}(p)] \qquad (p=1, 2, \cdots),$$

and let $\mu\{(a_j, a_1', a_2', \cdots, a_{p-1}', a_i) \mid a_j<a_1'<a_2'<\cdots<a_{p-1}'<a_i\}$ be the number of the different chains of the form $a_j<a_1'<a_2'<\cdots<a_{p-1}'<a_i$.

Then we shall prove by the induction that

(1) $$a_{ij}(p)=\mu\{(a_j, a_1', a_2', \cdots, a_{p-1}', a_i) \mid a_j<a_1'<a_2'<\cdots<a_{p-1}'<a_i\}.$$

In the case that $p=1$, we have evidently

$$a_{ij}(1)=\begin{cases} 1 & \text{for } a_j<a_i \\ 0 & \text{otherwise} \end{cases},$$

on the other hand,

(2) $$\mu\{(a_j, a_i) \mid a_j<a_i\}=\begin{cases} 1 & \text{if } a_j<a_i \\ 0 & \text{otherwise} \end{cases},$$

Hence, we obtain (1) in the case that $p=1$.

Second, we assume that (1) is valid for p, and consider the case of $p+1$.

Remarking that

(3) $$(dA)^{p+1}=(dA)^p(dA),$$

we have

$$a_{ij}(p+1)=\sum_{k+1}^{n} a_{ik}(p)a_{kj}(1).$$

$$=\sum_{k=1}^{n} [\mu\{(a_k, a_1', \cdots, a_{p-1}', a_i) \mid a_k<a_1'<\cdots<a_{p-1}'<a_i\}\cdot\mu\{(a_j, a_k) \mid a_j<a_k\}]$$

$$=\mu\{(a_j, a_k, a'_1, \cdots, a'_{p-1}, a_i) \mid a_j < a_k < a'_1 < \cdots < a'_{p-1} < a_i\}.$$

So, we obtain (1) in the case of $p+1$.

Hence, by the induction, (1) is established.

Now, let $degX=m$, then

$$d^m A \neq 0 \quad \text{and} \quad d^{m+1} A = 0.$$

Hence by (1) there exists at least a chain of the length m, but no chains of the length $m+1$.

Therefore

$$degX = l[X].$$

Reference

[1] V. Krishnamurthy: On the number of topologies on a finite set. Amer. Math. Monthly 73, (1966), 154-157.

[2] P. Alexandroff: Diskrete Räume. Mat. Sb. (N.S) Vol. **2**, (1937), 501-518.